Current Natural Sciences

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Autofrettage Technology and Its Applications in Pressured Apparatuses

SCIENCE PRESS
edp sciences

Printed in France

EDP Sciences – ISBN(print): 978-2-7598-3111-1 – ISBN(ebook): 978-2-7598-3112-8
DOI: 10.1051/978-2-7598-3111-1

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Foreword

Pressure vessels are one of the pressured apparatuses, which are widely used in many industrial fields, such as mechanical engineering, chemical engineering, petrochemical engineering, pharmaceutical factory, energy industry, material industry, food industry, metallurgical industry, oil industry, architectural engineering, aeronautical engineering, aerospace industry, weaponry industry, etc. This book focuses on the study of pressure vessels as one of the pressured apparatuses. The basic requirement of pressure vessels is safety and economy. Security is the core issue. It is necessary to maximize the economy under the premise of fully guaranteed security. The strength and structure of pressure vessels are the most important factors that influence safety. The stresses in the wall of a pressure vessel must be reduced and limited in a safe range in structural design so as to ensure the strength of the pressure vessel.

The circumferential stress on the inner surface is quite great and uneven, using high or ultra-high-strength material or increasing the thickness cannot solve the problem, introducing pre-stresses into the wall is an excellent way to reduce the stresses in the wall, and raising the load-bearing capacity of a pressure vessel. The autofrettage technology is an effectual measure to introduce pre-stresses into the wall. After a pressure vessel is processed with autofrettage technology, the circumferential stress on the inner surface is lowered and the distributions of the stresses in the wall become even because of the superposition of the operation stresses and the pre-stresses (residual stresses or thermal stresses), and all stresses are within the elastic range, the elastic operating range enlarged and the elastic load-bearing capacity raised. Especially, when the operating pressure is greater than the yield pressure of the inner surface or initial yield pressure, processing with autofrettage technology for a pressure vessel becomes more necessary.

At present, there is no monograph on autofrettage technology in the world. This book is written based on our research work and absorbs our latest research results. In this book, the systematic theory of autofrettage technology of pressure vessels is established and studied, and a large number of application examples are presented.

DOI: 10.1051/978-2-7598-3111-1.c901
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The whole book consists of seven chapters. Chapter 1 develops the theories of elastic-plastic stresses of pressure vessels based on the maximum shear stress theory (Tresca yield criterion) and the maximum distortion strain energy theory (Mises yield criterion). Chapter 2 discusses the mechanical autofrettage technology according to the maximum shear stress theory (Tresca yield criterion). Chapter 3 studies the mechanical autofrettage technology according to the maximum distortion strain energy theory (Mises yield criterion). Chapter 4 investigates the mechanical autofrettage technology by limiting circumferential residual stress based on maximum distortion strain energy theory (Mises yield criterion). Chapter 5 studies the mechanical autofrettage technology when the operating pressure is several times of the initial yield pressure according to the maximum shear stress theory and the maximum distortion strain energy theory. In consideration of that when there is a temperature difference between the inner and outer surfaces, thermal stresses will appear in the wall of a pressure vessel and the thermal stresses can be used as pre-stresses, chapter 6 studies thermal autofrettage technology according to the maximum shear stress theory (Tresca criterion), and chapter 7 studies thermal autofrettage technology according to the maximum distortion strain energy theory (Mises criterion). This book tries to clarify the ideas, introduce the principle, and show the effect of autofrettage technology.

In this book, interesting and useful results for engineering applications are presented, the laws contained in the autofrettage theory are revealed, the essential cause and reason for the obtained results are analyzed, the inherent and meaningful relations between various parameters in autofrettage theory are brought to light, and the safe depth of plastic region (overstrain) and the conditions of loading or the optimum operation conditions are found out.

This book pays attention to the rigor of the theoretical deduction and the maneuverability of the engineering practice, it serves three purposes. The first purpose is to provide a reference book for the analysis and design of pressured apparatuses that are widely used in industries. The second purpose is to help students and graduate students have a good command of the theory and applications of autofrettage technology of pressure vessels. The third purpose is to aid engineers in understanding the background and cause and effect of some of the design equations in related Codes.

Thanks to the authors of the references.

The authors are aware that due to the limited level of the authors, there must be some mistakes in the book. Please do not hesitate to point them out.

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Contents

Foreword III

CHAPTER 1
Study on Elastic-Plastic Mechanical Stresses in Cylindrical Pressure Vessels . 1

1.1 Introduction 1
1.2 Studies of Elastic Stresses 2
1.3 Analysis of Elastic-Plastic Stresses 7
1.4 Chapter Summary 17
References 20

CHAPTER 2
Mechanical Autofrettage Technology Based on Tresca Yield Criterion 21

2.1 General Study on Mechanical Autofrettage Technology 21
    2.1.1 In General Forms 22
    2.1.2 The Critical Radius Ratio 33
    2.1.3 The Optimum Plastic Depth $k_{\mathrm{j}*}$ (k_{j} is Written as $k_{\mathrm{j}*}$) 34
    2.1.4 The Results When $k_{\mathrm{j}} = k_{\mathrm{j}*}$ 35
2.2 Mechanical Autofrettage Technology Under Entire Yield State 41
2.3 Mechanical Autofrettage Technology with Radius of Elastic-Plastic Juncture Being Arithmetic Mean Radius of Inside Radius and Outside Radius 46
2.4 Mechanical Autofrettage Technology with Radius of Elastic-Plastic Juncture Being Geometrical Mean Radius of Inside Radius and Outside Radius 52
2.5 Mechanical Autofrettage Technology with Minimum Equivalent Total Stress on Elastic-Plastic Juncture 56
2.6 Comparison Between Three Cases 70
2.7 Chapter Summary 77
References 81

CHAPTER 3
Mechanical Autofrettage Technology Based on Mises Yield Criterion 83

3.1 General Study on Mechanical Autofrettage Technology 83
    3.1.1 In General Forms 83

3.1.2 The Critical Radius Ratio 98
3.1.3 The Optimum Plastic Depth k_{j*} (k_j is Written as k_{j*}) 99
3.1.4 The Results When $k_j = k_{j^*}$ or $k^2 \ln k_{j^*}^2 - k^2 - k_{j^*}^2 + 2 = 0$ 100
3.2 Mechanical Autofrettage Technology Under Entire Yield State 109
3.2.1 The Residual Stresses 109
3.2.2 The Total Stresses 111
3.3 Mechanical Autofrettage Technology with Radius of Elastic-Plastic Juncture Being Arithmetic Mean Radius of Inside Radius and Outside Radius 117
3.4 The Solutions with Radius of Elastic-Plastic Juncture Being Geometrical Mean Radius of Inside Radius and Outside Radius 127
3.5 The Solutions with Minimum Equivalent Total Stress on Elastic-Plastic Juncture 133
3.6 Comparison Between the Three Cases 146
3.6.1 $k = 2.5 > k_c$ 147
3.6.2 $k = 2 < k_c$ 150
3.7 Chapter Summary 153
References 157

CHAPTER 4

Mechanical Autofrettage Technology by Limiting Circumferential Residual Stress Based on Mises Yield Criterion 159

4.1 The Optimum Plastic Depth When Circumferential Residual Stress on the Inside Surface Controlled 159
4.2 The Distribution of Residual Stresses When Circumferential Residual Stress on the Inside Surface Controlled 166
4.2.1 General Discussion 166
4.2.2 The Residual Stresses for Entire Yield 176
4.2.3 The Residual Stresses with Radius of Elastic-Plastic Juncture Being Arithmetic Mean Radius of Inside Radius and Outside Radius 177
4.2.4 The Residual Stresses with Radius of Elastic-Plastic Juncture Being Geometrical Mean Radius of Inside Radius and Outside Radius 179
4.2.5 The Residual Stresses When Equivalent Total Stress on Elastic-Plastic Juncture is the Minimum 180
4.3 The Total Stresses and the Load-Bearing Capacity When Circumferential Residual Stress on the Inside Surface Controlled 186
4.4 Control Circumferential Total Stress Directly 196
4.5 Chapter Summary 198
References 202

CHAPTER 5

Mechanical Autofrettage Technology Under Low Load 203

5.1 Introduction 203
5.2 The Optimum Plastic Depth 204
5.3 Analysis of Residual Stresses Under the Optimum Plastic Depth 208
5.4 Analysis of the Stresses Caused by Internal Pressure and Total Stresses 219
5.5 Analysis of the Effect of Load Ratio (λ) and Plastic Depth ($k_{i\lambda}$) 228
5.6 Analysis of Load-Bearing Capacity 234
  5.6.1 Based on Tresca Criterion 234
  5.6.2 Based on Mises Criterion 237
5.7 Chapter Summary 240
References 244

CHAPTER 6

Summary of Implement Methods and Their Characteristics of Mechanical and Thermal Autofrettage Technology 245

6.1 Implement Methods and Characteristics of Mechanical Autofrettage Technology 245
6.2 Implement Methods and Characteristics of Thermal Autofrettage Technology 248
References 255

CHAPTER 7

Thermal Autofrettage Technology Based on Tresca Yield Criterion 257

7.1 Introduction 257
7.2 Derivation of Thermal Stresses 258
7.3 The Characteristics of the Thermal Stresses 265
7.4 The Analysis of Total Stresses and Investigation of Optimum Operation Conditions 273
7.5 Examples 284
7.6 The Total Stresses Under Optimum Operation Conditions 296
7.7 Chapter Summary 302
References 306

CHAPTER 8

Thermal Autofrettage Technology Based on Mises Yield Criterion 307

8.1 The Analysis of Thermal Stresses 307
8.2 The Analysis of Total Stresses and Investigation of Optimum Operation Conditions 309
8.3 Examples 331
8.4 Chapter Summary 345
References 348

Nomenclature 349

Chapter 1

Study on Elastic-Plastic Mechanical Stresses in Cylindrical Pressure Vessels

1.1 Introduction

Cylindrical pressure vessels are widely used in industries. Since the distributions of stresses caused by operation pressure in the wall of a cylinder under an elastic state are very uneven, the load-bearing capacity is thus limited seriously. The autofrettage technique is an effective and important measure to even distributions of stresses, improve safety and raise the load-bearing capacity, and extend operating life for pressure vessels. Usually, in the most commonly employed autofrettage process, a pressure vessel is pressurized to a quite high internal hydraulic pressure; as a result, the portion of the cylinder from the inner radius to some intermediate radius becomes plastic, while the remaining portion remains elastic. After releasing the pressure subjected to the pressure vessel during the autofrettage process (called autofrettage pressure, p_{a}), the residual stresses are set up in the wall of the pressure vessel. When the pressure vessel is put into production, the operation stresses caused by the operation load are superimposed on the residual stresses so that advantageous distributions of total stresses in the wall of the pressure vessel are achieved, thus the pressure vessel can endure a higher pressure. We call these methods mechanical autofrettage because the pre-stresses are caused by mechanical forces. Mechanical autofrettage technology has been widely applied in industries, such as high and ultra-pressure reactors, high and ultra-pressure compressors, pipings, chemical equipment, petrochemical equipment, high-pressure pump, battleship, and tank cannon barrels, and fuel injection systems for diesel engines, etc.

Since temperature differences in the wall of an apparatus can cause pre-stresses, heating and/or cooling the thick apparatuses to obtain pre-stresses is a promising approach for the autofrettage of thick apparatuses. We call this autofrettage technology thermal autofrettage technology, which will be investigated in the last two chapters.

In autofrettage technology of pressure vessels, the variation law of the residual stresses and total stresses (the residual stresses plus operation stresses), the reasonable and the optimum plastic depth or overstrain, the reasonable and the

DOI: 10.1051/978-2-7598-3111-1.c001
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optimum load-bearing capability, the relationship between some key parameters, the optimum temperature difference, etc., are important to the design of high-pressure apparatus. In order to discover the general law in autofrettage theory for the design of high and ultrahigh-pressure vessels, it is important to investigate the autofrettage technology deeply, systematically, and in detail.

Engineering practice shows that the mechanical behaviors of the engineering materials tally with the maximum shear stress theory (Tresca's yield criterion) and the maximum distortion strain energy theory (Mises's yield criterion), thus the research work is based on these two theories[1].

Engineering conditions vary in thousands of ways. We bypass specific engineering conditions and carry out the research according to the following ideal conditions: (1) the materials of pressure vessels are perfectly elastic-plastic and the Bauschinger effect is temporarily neglected, the compressive yield limit of the material is equal to the tensile yield limit; (2) the strain hardening is ignored; (3) there is no any defect in the materials. The problems with autofrettage under specific engineering conditions can be resolved by reference to the results of this study on the basis of the specific engineering conditions.

Elastoplastic stresses are the foundation of autofrettage technology, in this chapter; we study the elastoplastic mechanical stresses of cylindrical pressure vessels.

1.2 Studies of Elastic Stresses

When a cylindrical pressure vessel is subjected to an internal pressure p, in an elastic state, Lamé equations express the elastic stresses in three directions, *i.e.*, the axial elastic stress $\sigma_{\mathrm{z}}^{\mathrm{p}}$, the radial elastic stress $\sigma_{\mathrm{r}}^{\mathrm{p}}$, and the circumferential elastic stress $\sigma_{\theta}^{\mathrm{p}}$, that are as follows[2, 3]:

$$\begin{cases} \dfrac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \dfrac{1}{k^2-1}\dfrac{p}{\sigma_{\mathrm{y}}} \\ \dfrac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \left(1-\dfrac{k^2}{x^2}\right)\dfrac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = -\dfrac{k^2-x^2}{x^2(k^2-1)}\dfrac{p}{\sigma_{\mathrm{y}}} \\ \dfrac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \left(1+\dfrac{k^2}{x^2}\right)\dfrac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = -\dfrac{k^2+x^2}{x^2(k^2-1)}\dfrac{p}{\sigma_{\mathrm{y}}} \end{cases} \quad \text{or} \quad \begin{cases} \dfrac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{p} = \dfrac{1}{k^2-1} \\ \dfrac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{p} = \left(1-\dfrac{k^2}{x^2}\right)\dfrac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{p} \\ \dfrac{\sigma_{\theta}^{\mathrm{p}}}{p} = \left(1+\dfrac{k^2}{x^2}\right)\dfrac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{p} \end{cases} \tag{1.1}$$

In order to show the stress characteristics and facilitate the follow-up research, by taking $k = 4$, Lamé equations are plotted in figure 1.1.

When it's not necessary, we sometimes omit the superscript p, s, and d.

From equation (1.1) and figure 1.1, it is seen that the distributions of the elastic stresses along with radius r or x ($x = r/r_{\mathrm{i}}$) are seriously uneven, with $\sigma_{\theta}^{\mathrm{p}}$ being the

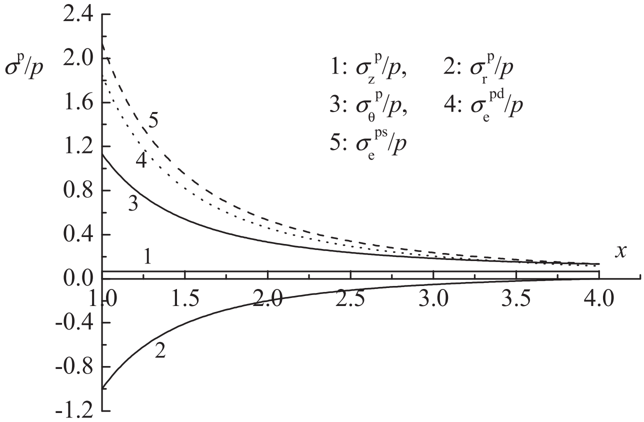


FIG. 1.1 – Distributions of stresses in the wall of a cylindrical pressure vessel.

greatest and σ_r^p being the smallest (algebraic value, not absolute value). Therefore, with the internal pressure p increasing, whether based on the maximum shear stress theory (Tresca criterion) or based on the maximum distortion strain energy theory (Mises's yield criterion), it is certain that the inside surface of the pressure vessel becomes yielded first, and then the plastic region develops towards the outside surface and retains a plastic cylinder concentric with the elastic cylinder.

The equivalent stress based on the maximum distortion strain energy theory (Mises's yield criterion) is[1]

$$\sigma_e^d = \sqrt{\frac{1}{2}\left[(\sigma_\theta - \sigma_r)^2 + (\sigma_r - \sigma_z)^2 + (\sigma_z - \sigma_\theta)^2\right]} \tag{1.2}$$

For a cylindrical pressure vessel, the axial stress σ_z is the arithmetic mean value of circumferential stress σ_θ and radial stress σ_r, or

$$\sigma_z = \frac{\sigma_\theta + \sigma_r}{2} \tag{1.3}$$

Equation (1.3) is satisfied under both elastic and plastic states.

Substituting equation (1.3) into equation (1.2) leads to

$$\sigma_e^d = \frac{\sqrt{3}}{2}(\sigma_\theta - \sigma_r) \quad \text{or} \quad \sigma_\theta - \sigma_r = \frac{2}{\sqrt{3}}\sigma_e^d \tag{1.4}$$

Normally σ_θ^p is the first principal stress (σ_1) and σ_r^p is the third principal stress (σ_3). Therefore, the equivalent stress based on the maximum shear stress theory (Tresca criterion) is

$$\sigma_e^s = \sigma_1 - \sigma_3 = \sigma_\theta - \sigma_r \tag{1.5}$$

From equations (1.4) and (1.5), the relationship between the equivalent stress based on the maximum shear stress theory $\sigma_{\rm e}^{\rm s}$, and the equivalent stress based on the maximum distortion strain energy theory $\sigma_{\rm e}^{\rm d}$ is

$$\sigma_{\rm e}^{\rm d} = \frac{\sqrt{3}}{2}\sigma_{\rm e}^{\rm s} \tag{1.6}$$

When the equivalent stress reaches the yield limit $\sigma_{\rm y}$, or $\sigma_{\rm e} = \sigma_{\rm y}$, the material of the pressure vessel becomes yielded. Therefore, the yield condition based on the maximum distortion strain energy theory is

$$\sigma_\theta - \sigma_{\rm r} = \frac{2}{\sqrt{3}}\sigma_{\rm y} \tag{1.7}$$

and the yield condition based on the maximum shear stress theory is

$$\sigma_\theta - \sigma_{\rm r} = \sigma_{\rm y} \tag{1.8}$$

By comparing equations (1.7) and (1.8), it is known that the results based on the maximum distortion strain energy theory can be obtained by replacing $\sigma_{\rm y}$ in the results based on the maximum shear stress theory with $\frac{2}{\sqrt{3}}\sigma_{\rm y}$, and the results based on the maximum shear stress theory can be obtained by replacing $\sigma_{\rm y}$ in the results based on the maximum distortion strain energy theory with $\sqrt{3}\sigma_{\rm y}/2$. In fact, this is the right way to transit from the Tresca criterion to the Mises criterion or from the Mises criterion to the Tresca criterion, as long as there is the functional relation as equation (1.3) between the stresses in three directions (axial, radial and circumferential directions). We call the way DS transition rule.

According to the maximum shear stress theory, the equivalent stress of the stresses caused by operation pressure p (called operation stresses) $\sigma_{\rm e}^{\rm ps}$ is

$$\frac{\sigma_{\rm e}^{\rm ps}}{\sigma_{\rm y}} = \frac{\sigma_\theta^{\rm p}}{\sigma_{\rm y}} - \frac{\sigma_{\rm r}^{\rm p}}{\sigma_{\rm y}} = \frac{2k^2}{x^2}\frac{\sigma_{\rm z}^{\rm p}}{\sigma_{\rm y}} = \frac{2k^2}{x^2(k^2-1)}\frac{p}{\sigma_{\rm y}} \quad \text{or} \quad \frac{\sigma_{\rm e}^{\rm ps}}{p} = \frac{2k^2}{x^2(k^2-1)} \tag{1.9}$$

In accordance with the above DS transition rules, the equivalent operation stresses based on the maximum distortion strain energy theory (Mises yield criterion) $\sigma_{\rm e}^{\rm pd}$ is

$$\frac{\sigma_{\rm e}^{\rm pd}}{\sigma_{\rm y}} = \frac{\sqrt{3}}{2}\left(\frac{\sigma_\theta^{\rm p}}{\sigma_{\rm y}} - \frac{\sigma_{\rm r}^{\rm p}}{\sigma_{\rm y}}\right) = \frac{\sqrt{3}k^2}{x^2(k^2-1)}\frac{p}{\sigma_{\rm y}} \quad \text{or} \quad \frac{\sigma_{\rm e}^{\rm pd}}{p} = \frac{\sqrt{3}k^2}{x^2(k^2-1)} \tag{1.10}$$

Equation (1.9) or equation (1.10) shows that when $x = 1$ or on the inside surface $\sigma_{\rm ei}^{\rm ps}$ or $\sigma_{\rm ei}^{\rm pd}$ is the greatest, which is $\sigma_{\rm ei}^{\rm ps} = \frac{2k^2}{k^2-1}p$ or $\sigma_{\rm ei}^{\rm pd} = \frac{\sqrt{3}k^2}{k^2-1}p$. When $\sigma_{\rm ei}^{\rm p} = \sigma_{\rm y}$, the pressure vessel fails to work. Then, letting $\sigma_{\rm ei}^{\rm ps} = \sigma_{\rm y}$ or $\sigma_{\rm ei}^{\rm pd} = \sigma_{\rm y}$, we obtain the maximum elastic load-bearing capability or initial yield pressure of an unautofrettaged cylinder, $p_{\rm e}$

$$\frac{p_{\rm e}}{\sigma_{\rm y}} = \frac{k^2-1}{2k^2} \quad \text{(Tresca yield criterion)} \tag{1.11}$$

$$\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}=\frac{k^2-1}{\sqrt{3}k^2} \quad \text{(Mises yield criterion)} \tag{1.12}$$

Although increasing the thickness or increasing k can raise the maximum elastic load-bearing capability, the increase in p_{e} is restricted greatly, especially when k is quite great, even if k is increased rapidly, p_{e} rises very slowly, as is intuitively shown in figure 1.2.

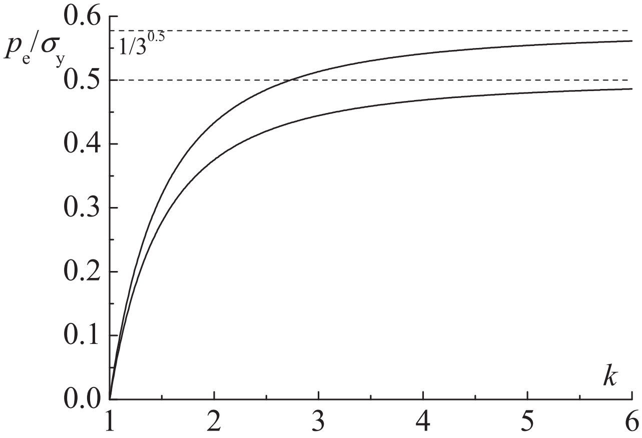


FIG. 1.2 – The maximum elastic load-bearing capability of an unautofrettaged cylinder.

From equations (1.11), (1.12), and figure 1.2, the limit of $\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$ is $\frac{1}{2}$ or $\frac{1}{\sqrt{3}}$, in other words, no matter how thick a pressure vessel is, even if it is infinitely thick, the pressure it can contain is only $0.5\sigma_{\mathrm{y}}$ or $\frac{\sigma_{\mathrm{y}}}{\sqrt{3}}$.

From the above, it is thus evident that measures must be taken to raise the load-bearing capability of pressure vessel.

Autofrettage technology is an effective measure to even the distributions of stresses in the wall of the pressure vessel and to raise the load-bearing capability of the pressure vessel. The key to autofrettage technology is the pre-stresses in the wall of the pressure vessel. The pre-stresses are usually achieved either by applying hydraulic pressure to the bore of the thick-walled pressure vessel or by pushing an oversized mandrel through the bore or by detonating an explosive charge inside the pressure vessel to deform it plastically, thus the compressive residual stresses on the inner surface of the pressure vessel produced, enabling it to withstand higher working pressure. We call these methods mechanical autofrettage since the pre-stresses are caused by mechanical forces. We first study mechanical autofrettage. We will study mechanical autofrettage technology in chapters 2–5.

For equations (1.1) and (1.9), setting $\frac{\sigma_{\mathrm{e}}^{\mathrm{ps}}}{\sigma_{\mathrm{y}}}=\frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}$ obtains $x=k$. When $x<k$, $\sigma_{\mathrm{e}}^{\mathrm{ps}}>\sigma_{\theta}^{\mathrm{p}}$, or within the whole wall, $\sigma_{\mathrm{e}}^{\mathrm{ps}}$ is always larger than $\sigma_{\theta}^{\mathrm{p}}$.

For equations (1.1) and (1.10), the setting $\frac{\sigma_e^{pd}}{\sigma_y} = \frac{\sigma_\theta^p}{\sigma_y}$ obtains

$$x = \sqrt{\sqrt{3} - 1}k = x_a < k \tag{1.13}$$

When $x < x_a$, $\sigma_e^{pd} > \sigma_\theta^p$; When $x > x_a$, $\sigma_e^{pd} < \sigma_\theta^p$. There is a crossover point of curve $\frac{\sigma_e^{pd}}{\sigma_y}$ and curve $\frac{\sigma_\theta^p}{\sigma_y}$ at $x = x_a$. When $x = x_a$, $\frac{\sigma_e^{pd}}{\sigma_y} = \frac{\sigma_\theta^p}{\sigma_y} = \frac{\sqrt{3}}{(\sqrt{3}-1)(k^2-1)} \frac{p}{\sigma_y}$.

For $k = 4$ and $\frac{p}{\sigma_y} = 0.8$, Lamé equations in another form are plotted in figures 1.3 and 1.4.

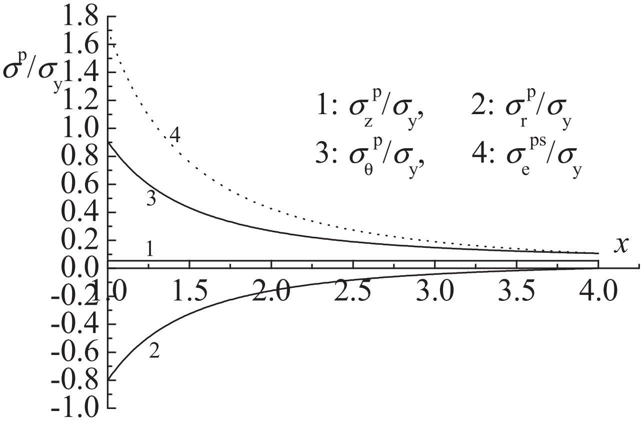


FIG. 1.3 – The distributions of operation stresses based on the Tresca criterion.

In figure 1.3, at $x = k$, $\sigma_e^{ps} = \sigma_\theta^p$.

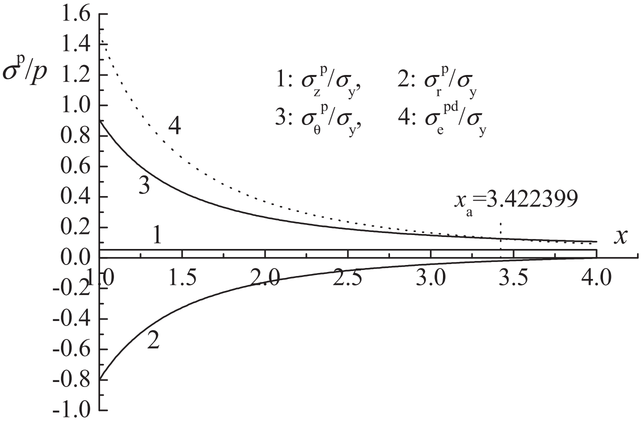


FIG. 1.4 – The distributions of operation stresses based on the Mises criterion.

In figure 1.4, at $x = x_a = 3.4$, $\sigma_e^{pd} = \sigma_\theta^p$; When $x < x_a$, $\sigma_e^{pd} > \sigma_\theta^p$; When $x > x_a$, $\sigma_e^{pd} < \sigma_\theta^p$.

1.3 Analysis of Elastic-Plastic Stresses

Elastoplastic stresses are the basis of autofrettage technology.

In the case to not cause confusion, we omit some superscripts or subscripts.

When a pressure vessel is subjected to an internal pressure p, as the internal pressure p increases, after the yielding of the inside surface and before the entire yielding of the whole wall, the inner layer of the pressure vessel is a plastic cylinder (or a plastic region) and the outside layer of the pressure vessel is a homocentric elastic cylinder (or an elastic region) as shown in figure 1.5, where r_i is the inner radius, r_o is the outer radius and r_j is the radius of elastic-plastic juncture. The elastic cylinder is separated from the plastic cylinder, as is shown in figure 1.6 which is the elastic cylinder, and figure 1.7 which is the plastic cylinder, where p_j is the interaction pressure between the plastic layer and the elastic layer. Then, the elastic region is a cylinder with inside radius r_j and outside radius r_o, subjected to internal pressure p_j; and the plastic region is a cylinder with inside radius r_i and outside radius r_j, subjected to internal pressure p and external pressure p_j. We now do mechanical analysis on these two cylinders.

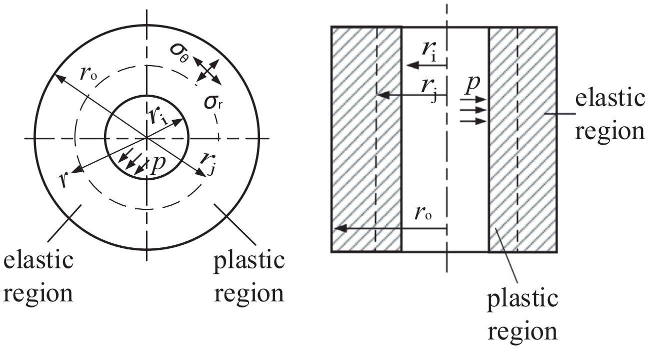


FIG. 1.5 – A section of an elastic-plastic cylinder.

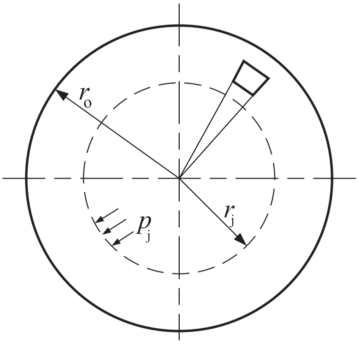


FIG. 1.6 – The elastic layer of the elastic-plastic cylinder.

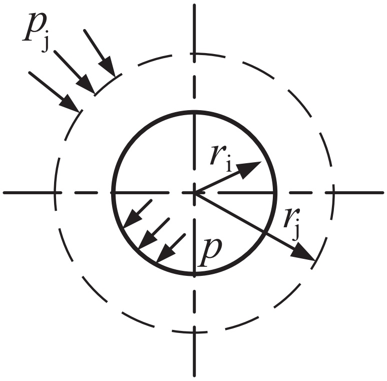


FIG. 1.7 – The plastic layer of the elastic-plastic cylinder.

In whether the elastic region or plastic region, at an arbitrary radius r, cutting the cylinder with two concentric circles which are at a distance of dr, two planes with included angle $d\theta$, and two parallel planes at a distance of dz, we obtain an element as shown in figure 1.8 as a free-body diagram. The equilibrium equation in the radial direction for the differential element is as follows:

$$(\sigma_\mathrm{r} + d\sigma_\mathrm{r})(r + dr)\,d\theta - \sigma_\mathrm{r} r d\theta - 2\sigma_\theta\, dr \sin\frac{d\theta}{2} = 0$$

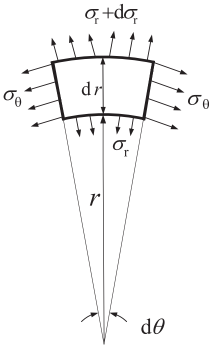


FIG. 1.8 – A differential element from the wall of the cylinder.

Ignoring higher order infinitesimal and taking $\sin\frac{d\theta}{2} \approx \frac{d\theta}{2}$ into consideration, we then have

$$\sigma_\theta - \sigma_\mathrm{r} = r\frac{d\sigma_\mathrm{r}}{dr} \tag{1.14}$$

Equation (1.14) is tenable in both the elastic state and plastic state.

In the plastic region, according to the maximum distortion strain energy theory, substituting equation (1.7) into equation (1.14) obtains

$$d\sigma_{\mathrm{r}} = \frac{2}{\sqrt{3}}\sigma_{\mathrm{y}}\frac{dr}{r} \tag{1.15}$$

Integrating both sides of equation (1.15) gives[4]

$$\sigma_{\mathrm{r}} = \frac{2}{\sqrt{3}}\sigma_{\mathrm{y}}\ln r + C \tag{1.16}$$

where C is an integration constant, to be determined by the following boundary conditions

$$\begin{cases}\sigma_{\mathrm{r}} = -p, & r = r_{\mathrm{i}}\\ \sigma_{\mathrm{r}} = -p_{\mathrm{j}}, & r = r_{\mathrm{j}}\end{cases} \tag{1.17}$$

Substituting the first one of equation (1.17) into equation (1.16) obtains the integration constant C, then substituting C back into equation (1.16) results in

$$\sigma_{\mathrm{r}} = \frac{2}{\sqrt{3}}\sigma_{\mathrm{y}}\ln\frac{r}{r_{\mathrm{i}}} - p \quad \text{or} \quad \frac{\sigma_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\ln x - \frac{p}{\sigma_{\mathrm{y}}} \tag{1.18}$$

Substituting equation (1.18) into equation (1.7) obtains

$$\sigma_{\theta} = \frac{2}{\sqrt{3}}\sigma_{\mathrm{y}}\left(\ln\frac{r}{r_{\mathrm{i}}} + 1\right) - p \quad \text{or} \quad \frac{\sigma_{\theta}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}(\ln x + 1) - \frac{p}{\sigma_{\mathrm{y}}} \tag{1.19}$$

The axial stress σ_{z} can be obtained by equation (1.3). Therefore, the stresses within the plastic region are summarized as follows:

$$\begin{cases}\dfrac{\sigma_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = \dfrac{2}{\sqrt{3}}\left(\dfrac{1}{2} + \ln\dfrac{r}{r_{\mathrm{i}}}\right) - \dfrac{p}{\sigma_{\mathrm{y}}} = \dfrac{2}{\sqrt{3}}\left(\dfrac{1}{2} + \ln x\right) - \dfrac{p}{\sigma_{\mathrm{y}}}\\ \dfrac{\sigma_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = \dfrac{2}{\sqrt{3}}\ln x - \dfrac{p}{\sigma_{\mathrm{y}}}\\ \dfrac{\sigma_{\theta}}{\sigma_{\mathrm{y}}} = \dfrac{2}{\sqrt{3}}(1 + \ln x) - \dfrac{p}{\sigma_{\mathrm{y}}}\\ \dfrac{\sigma_{\mathrm{e}}}{\sigma_{\mathrm{y}}} \equiv 1\end{cases} \tag{1.20}$$

Seemingly, the elastic-plastic stresses within the plastic region are not related to the plastic depth k_{j}, but in fact, k_{j} affects the elastic-plastic stresses within the plastic region.

Substituting the second one of equation (1.17) into the second one of equation (1.20) obtains

$$p_{\mathrm{j}} = -\frac{2}{\sqrt{3}}\sigma_{\mathrm{y}}\ln\frac{r_{\mathrm{j}}}{r_{\mathrm{i}}} + p \quad \text{or} \quad \frac{p_{\mathrm{j}}}{\sigma_{\mathrm{y}}} = -\frac{2}{\sqrt{3}}\ln k_{\mathrm{j}} + \frac{p}{\sigma_{\mathrm{y}}} \tag{1.21}$$

The inner surface of the elastic layer is just the outer surface of the plastic layer (see figures 1.5–1.7). So, on the inner surface of the elastic layer, there is also $\sigma_\theta - \sigma_\mathrm{r} = \frac{2}{\sqrt{3}}\sigma_\mathrm{y}$. According to Lamé formula, on the inner surface of the elastic layer

$$\sigma_\theta = p_\mathrm{j}\frac{\left(\frac{r_\mathrm{o}}{r_\mathrm{j}}\right)^2+1}{\left(\frac{r_\mathrm{o}}{r_\mathrm{j}}\right)^2-1} \quad \text{and} \quad \sigma_\mathrm{r} = -p_j \tag{1.22}$$

Substituting equation (1.22) into equation (1.7) obtains

$$p_\mathrm{j} = \frac{\sigma_\mathrm{y}}{\sqrt{3}}\frac{r_\mathrm{o}^2 - r_\mathrm{j}^2}{r_\mathrm{o}^2} \tag{1.23}$$

Letting equation (1.23) equal equation (1.21) obtains the relationship between the internal pressure p, the depth of plastic region k_j ($k_\mathrm{j} = r_\mathrm{j}/r_\mathrm{i}$), the radius ratio k ($k = r_\mathrm{o}/r_\mathrm{i}$) and yield strength σ_y:

$$\frac{p}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}\ln k_\mathrm{j} + \frac{1}{\sqrt{3}} - \frac{k_\mathrm{j}^2}{\sqrt{3}k^2} = \frac{2}{\sqrt{3}}\ln k_\mathrm{j} + \frac{k^2 - k_\mathrm{j}^2}{\sqrt{3}k^2} \tag{1.24}$$

In the plastic region, letting $\frac{\sigma_\mathrm{e}}{\sigma_\mathrm{y}} \geq \frac{\sigma_\theta}{\sigma_\mathrm{y}}$ gives

$$\frac{p}{\sigma_\mathrm{y}} - \frac{2}{\sqrt{3}}\ln x \geq \frac{2-\sqrt{3}}{\sqrt{3}} \tag{1.25}$$

When $x = k_\mathrm{j}$, equation (1.25) becomes $\frac{p}{\sigma_\mathrm{y}} - \frac{2}{\sqrt{3}}\ln k_\mathrm{j} \geq \frac{2-\sqrt{3}}{\sqrt{3}}$. From equation (1.24), $\frac{p}{\sigma_\mathrm{y}} - \frac{2}{\sqrt{3}}\ln k_\mathrm{j} = \frac{k^2-k_\mathrm{j}^2}{\sqrt{3}k^2}$. Letting $\frac{k^2-k_\mathrm{j}^2}{\sqrt{3}k^2} \geq \frac{2-\sqrt{3}}{\sqrt{3}}$ results in

$$k_\mathrm{j} \leq \sqrt{\sqrt{3}-1}k = x_\mathrm{a} \tag{1.26}$$

Therefore, when the load is determined by equation (1.24), in the plastic region, if $k_\mathrm{j} \leq x_\mathrm{a}$, $\frac{\sigma_\mathrm{e}}{\sigma_\mathrm{y}} \geq \frac{\sigma_\theta}{\sigma_\mathrm{y}}$; if $k_\mathrm{j} \geq x_\mathrm{a}$, $\frac{\sigma_\mathrm{e}}{\sigma_\mathrm{y}} \leq \frac{\sigma_\theta}{\sigma_\mathrm{y}}$. On the other hand, from equation (1.25) we obtain

$$x \leq \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_\mathrm{y}} - \frac{2-\sqrt{3}}{2}\right) = x_\mathrm{z} \tag{1.27}$$

When $x \leq x_\mathrm{z}$, $\frac{\sigma_\mathrm{e}}{\sigma_\mathrm{y}} \geq \frac{\sigma_\theta}{\sigma_\mathrm{y}}$; when $x \geq x_\mathrm{z}$, $\frac{\sigma_\mathrm{e}}{\sigma_\mathrm{y}} \leq \frac{\sigma_\theta}{\sigma_\mathrm{y}}$. If $k_\mathrm{j} \leq \sqrt{\sqrt{3}-1}k = x_\mathrm{a}$, x_z does not exist within $1 \leq x \leq k_\mathrm{j}$. x_z and x_a do not exist at the same time.

The elastic layer has an inner radius r_i, outer radius r_o, and is subjected to internal pressure p_j. Therefore, according to the Lamé formula, the elastic stresses within the elastic region are as follows:

$$\begin{cases} \dfrac{\sigma_{\mathrm{z}}}{\sigma_{\mathrm{y}}}=\dfrac{k_{\mathrm{j}}^2}{\sqrt{3}k^2} \\ \dfrac{\sigma_{\mathrm{r}}}{\sigma_{\mathrm{y}}}=\dfrac{k_{\mathrm{j}}^2}{\sqrt{3}k^2}\left(1-\dfrac{k^2}{x^2}\right)=\dfrac{\sigma_{\mathrm{z}}}{\sigma_{\mathrm{y}}}\left(1-\dfrac{k^2}{x^2}\right) \\ \dfrac{\sigma_{\theta}}{\sigma_{\mathrm{y}}}=\dfrac{k_{\mathrm{j}}^2}{\sqrt{3}k^2}\left(1+\dfrac{k^2}{x^2}\right)=\dfrac{\sigma_{\mathrm{z}}}{\sigma_{\mathrm{y}}}\left(1+\dfrac{k^2}{x^2}\right) \\ \dfrac{\sigma_{\mathrm{e}}}{\sigma_{\mathrm{y}}}=\dfrac{k_{\mathrm{j}}^2}{x^2} \end{cases} \tag{1.28}$$

In the elastic region, setting $\frac{\sigma_{\mathrm{e}}}{\sigma_{\mathrm{y}}} \geq \frac{\sigma_{\theta}}{\sigma_{\mathrm{y}}}$ leads to $x \leq \sqrt{\sqrt{3}-1}k = x_{\mathrm{a}} = 0.8556k < k$. Then, when $x \leq x_{\mathrm{a}}$, $\frac{\sigma_{\mathrm{e}}}{\sigma_{\mathrm{y}}} \geq \frac{\sigma_{\theta}}{\sigma_{\mathrm{y}}}$; when $x \geq x_{\mathrm{a}}$, $\frac{\sigma_{\mathrm{e}}}{\sigma_{\mathrm{y}}} \leq \frac{\sigma_{\theta}}{\sigma_{\mathrm{y}}}$.

Since $x_{\mathrm{a}} = \sqrt{\sqrt{3}-1}k = 0.8556k < k$, $\frac{\sigma_{\theta}}{\sigma_{\mathrm{y}}} \geq \frac{\sigma_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$ certainly occurs at some point in the elastic region. In the plastic region, $\frac{\sigma_{\theta}}{\sigma_{\mathrm{y}}} \geq \frac{\sigma_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$ is conditional, that is $k_{\mathrm{j}} \leq \sqrt{\sqrt{3}-1}k = x_{\mathrm{a}}$ or equation (1.26). If there exists $\frac{\sigma_{\theta}}{\sigma_{\mathrm{y}}} \geq \frac{\sigma_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$ in the plastic region, there certainly exists $\frac{\sigma_{\theta}}{\sigma_{\mathrm{y}}} \geq \frac{\sigma_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$ in the whole elastic region.

If based on the maximum shear stress theory, according to the DS transition rule, the corresponding results are listed as follows.

The plastic stresses within the plastic region

$$\begin{cases} \dfrac{\sigma_{\mathrm{z}}}{\sigma_{\mathrm{y}}}=0.5+\ln\dfrac{r}{r_{\mathrm{i}}}-\dfrac{p}{\sigma_{\mathrm{y}}}=0.5+\ln x-\dfrac{p}{\sigma_{\mathrm{y}}} \\ \dfrac{\sigma_{\mathrm{r}}}{\sigma_{\mathrm{y}}}=\ln x-\dfrac{p}{\sigma_{\mathrm{y}}} \\ \dfrac{\sigma_{\theta}}{\sigma_{\mathrm{y}}}=1+\ln x-\dfrac{p}{\sigma_{\mathrm{y}}} \\ \dfrac{\sigma_{\mathrm{e}}}{\sigma_{\mathrm{y}}}\equiv 1 \end{cases} \tag{1.29}$$

The relationship between the internal pressure p, the depth of plastic region k_{j} ($k_{\mathrm{j}} = r_{\mathrm{j}}/r_{\mathrm{i}}$), the radius ratio k ($k = r_{\mathrm{o}}/r_{\mathrm{i}}$), and yield strength σ_{y} is as follows[3].

$$\frac{p}{\sigma_{\mathrm{y}}}=\ln k_{\mathrm{j}}+\frac{1}{2}-\frac{k_{\mathrm{j}}^2}{2k^2}=\ln k_{\mathrm{j}}+\frac{k^2-k_{\mathrm{j}}^2}{2k^2} \tag{1.30}$$

In fact, the first term of equation (1.24) or equation (1.30) is the pressure to cause the plastic layer to become entirely yielded, and the second term is the initial yield pressure of the elastic layer.

When based on the maximum shear stress theory, in the plastic region, letting $\frac{\sigma_{\mathrm{e}}}{\sigma_{\mathrm{y}}} \geq \frac{\sigma_{\theta}}{\sigma_{\mathrm{y}}}$ results in $\frac{p}{\sigma_{\mathrm{y}}} - \ln x \geq 0$. When $x = k_{\mathrm{j}}$, $\frac{p}{\sigma_{\mathrm{y}}} - \ln x \geq 0$ becomes $\frac{p}{\sigma_{\mathrm{y}}} - \ln k_{\mathrm{j}} \geq 0$. From equation (1.30), $\frac{p}{\sigma_{\mathrm{y}}} - \ln k_{\mathrm{j}} = \frac{k^2-k_{\mathrm{j}}^2}{2k^2} \geq 0$. Therefore, when based on the maximum shear stress theory, in the plastic region, we always have $\frac{\sigma_{\mathrm{e}}}{\sigma_{\mathrm{y}}} \geq \frac{\sigma_{\theta}}{\sigma_{\mathrm{y}}}$. This is because that

$\sigma_{\mathrm{e}}^{\mathrm{s}} = \sigma_\theta - \sigma_{\mathrm{r}}$, and $\sigma_{\mathrm{r}} < 0$, then, $\sigma_{\mathrm{e}}^{\mathrm{s}} > \sigma_\theta$ whether in elastic or plastic zone. However, since $\sigma_{\mathrm{e}}^{\mathrm{d}} = \frac{\sqrt{3}}{2}(\sigma_\theta - \sigma_{\mathrm{r}})$, obviously, it is possible that $\sigma_{\mathrm{e}}^{\mathrm{d}} < \sigma_\theta$.

The elastic stresses within the elastic region

$$\begin{cases} \dfrac{\sigma_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = \dfrac{k_{\mathrm{j}}^2}{2k^2} \\ \dfrac{\sigma_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = \dfrac{k_{\mathrm{j}}^2}{2k^2}\left(1 - \dfrac{k^2}{x^2}\right) = \dfrac{\sigma_{\mathrm{z}}}{\sigma_{\mathrm{y}}}\left(1 - \dfrac{k^2}{x^2}\right) \\ \dfrac{\sigma_\theta}{\sigma_{\mathrm{y}}} = \dfrac{k_{\mathrm{j}}^2}{2k^2}\left(1 + \dfrac{k^2}{x^2}\right) = \dfrac{\sigma_{\mathrm{z}}}{\sigma_{\mathrm{y}}}\left(1 + \dfrac{k^2}{x^2}\right) \\ \dfrac{\sigma_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \dfrac{k_{\mathrm{j}}^2}{x^2} \end{cases} \tag{1.31}$$

In the elastic region, we always have $\frac{\sigma_{\mathrm{e}}}{\sigma_{\mathrm{y}}} \geq \frac{\sigma_\theta}{\sigma_{\mathrm{y}}}$.

In equations (1.24) and (1.30), letting $k_{\mathrm{j}} = 1$ obtains the initial yield pressure of the unautofrettaged pressure vessel, *i.e.* equations (1.11) and (1.12) respectively. When $k_{\mathrm{j}} = 1$, the terms expressing the entire yield pressure in equations (1.24) and (1.30) disappear.

In equation (1.24), letting $k_{\mathrm{j}} = k$ obtains the entire yield pressure, or equation (1.28). When $k_{\mathrm{j}} = k$, the term expressing initial yield pressure in equation (1.24) disappears.

$$\frac{p_{\mathrm{y}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}} \ln k \tag{1.32}$$

When $k = \infty$, equation (1.24) becomes

$$\frac{p_\infty}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}} \ln k_{\mathrm{j}} + \frac{1}{\sqrt{3}} \tag{1.33}$$

When $k = \infty$, if $k_{\mathrm{j}} = \mathrm{e}^{0.5}$, then, $\dfrac{p}{\sigma_{\mathrm{y}}} = \dfrac{p_\infty}{\sigma_{\mathrm{y}}} = \dfrac{1}{\sqrt{3}}$.

Letting $k = \infty$ in equation (1.20) results in

$$\begin{cases} \dfrac{\sigma_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = \dfrac{2}{\sqrt{3}} \ln x - \dfrac{2}{\sqrt{3}} \ln k_{\mathrm{j}} = \dfrac{2}{\sqrt{3}} \ln \dfrac{x}{k_{\mathrm{j}}} \\ \dfrac{\sigma_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = \dfrac{2}{\sqrt{3}} \ln x - \dfrac{2}{\sqrt{3}} \ln k_{\mathrm{j}} - \dfrac{1}{\sqrt{3}} = \dfrac{2}{\sqrt{3}} \ln \dfrac{x}{k_{\mathrm{j}}} - \dfrac{1}{\sqrt{3}} \\ \dfrac{\sigma_\theta}{\sigma_{\mathrm{y}}} = \dfrac{2}{\sqrt{3}} \ln x - \dfrac{2}{\sqrt{3}} \ln k_{\mathrm{j}} + \dfrac{1}{\sqrt{3}} = \dfrac{2}{\sqrt{3}} \ln \dfrac{x}{k_{\mathrm{j}}} + \dfrac{1}{\sqrt{3}} \\ \dfrac{\sigma_{\mathrm{e}}}{\sigma_{\mathrm{y}}} \equiv 1 \end{cases} \quad \text{(plastic region)} \tag{1.34}$$

Letting $k = \infty$ in equation (1.28) results in

$$\begin{cases} \dfrac{\sigma_\mathrm{z}}{\sigma_\mathrm{y}} = 0 \\ \dfrac{\sigma_\mathrm{r}}{\sigma_\mathrm{y}} = -\dfrac{k_\mathrm{j}^2}{\sqrt{3}x^2} \\ \dfrac{\sigma_\theta}{\sigma_\mathrm{y}} = \dfrac{k_\mathrm{j}^2}{\sqrt{3}x^2} = -\dfrac{\sigma_\mathrm{r}}{\sigma_\mathrm{y}} \\ \dfrac{\sigma_\mathrm{e}}{\sigma_\mathrm{y}} = \dfrac{k_\mathrm{j}^2}{x^2} \end{cases} \quad \text{(elastic region)} \tag{1.35}$$

If based on the maximum shear stress theory, in view of the DS transition rule, the corresponding results are listed as follows

$$\frac{p_\mathrm{y}}{\sigma_\mathrm{y}} = \ln k \tag{1.36}$$

$$\frac{p_\infty}{\sigma_\mathrm{y}} = \ln k_\mathrm{j} + \frac{1}{2} \tag{1.37}$$

$$\begin{cases} \dfrac{\sigma_\mathrm{z}}{\sigma_\mathrm{y}} = \ln x - \ln k_\mathrm{j} = \ln \dfrac{x}{k_\mathrm{j}} \\ \dfrac{\sigma_\mathrm{r}}{\sigma_\mathrm{y}} = \ln x - \ln k_\mathrm{j} - \dfrac{1}{2} = \ln \dfrac{x}{k_\mathrm{j}} - \dfrac{1}{2} \\ \dfrac{\sigma_\theta}{\sigma_\mathrm{y}} = \ln x - \ln k_\mathrm{j} + \dfrac{1}{2} = \ln \dfrac{x}{k_\mathrm{j}} + \dfrac{1}{2} \\ \dfrac{\sigma_\mathrm{e}}{\sigma_\mathrm{y}} \equiv 1 \end{cases} \quad \text{(plastic region)} \tag{1.38}$$

$$\begin{cases} \dfrac{\sigma_\mathrm{z}}{\sigma_\mathrm{y}} = 0 \\ \dfrac{\sigma_\mathrm{r}}{\sigma_\mathrm{y}} = -\dfrac{k_\mathrm{j}^2}{2x^2} \\ \dfrac{\sigma_\theta}{\sigma_\mathrm{y}} = \dfrac{k_\mathrm{j}^2}{2x^2} = -\dfrac{\sigma_\mathrm{r}}{\sigma_\mathrm{y}} \\ \dfrac{\sigma_\mathrm{e}}{\sigma_\mathrm{y}} = \dfrac{k_\mathrm{j}^2}{x^2} \end{cases} \quad \text{(elastic region)} \tag{1.39}$$

It is noted that no matter within the elastic region or plastic region, the equivalent stress remains unchanged when based on the maximum shear stress theory and the maximum distortion strain energy theory. The reason for this is as follows:

Under the same operation pressure p, the relationship between the equivalent stress based on the maximum shear stress theory $\sigma_\mathrm{e}^\mathrm{s}$ and the equivalent stress based on the maximum distortion strain energy theory $\sigma_\mathrm{e}^\mathrm{d}$ is $\sigma_\mathrm{e}^\mathrm{d} = \frac{\sqrt{3}}{2}\sigma_\mathrm{e}^\mathrm{s}$ (equation (1.6)), but the stress components based on the maximum distortion strain energy theory are $\frac{2}{\sqrt{3}}$ times as much as those based on the maximum shear stress theory, therefore, $\sigma_{\mathrm{e}}^{\mathrm{d}} = \sigma_{\mathrm{e}}^{\mathrm{s}}$.

To intuitively display the variation of elastoplastic stresses, figure 1.9a–d shows the distributions of elastoplastic stresses based on the maximum distortion strain energy theory, where

Figure 1.9a is for $k = 4$, $k_{\mathrm{j}} = 1.5$, $x_{\mathrm{a}} = 3.422399 > k_{\mathrm{j},}$ and $p/\sigma_{\mathrm{y}} = 0.964351\ldots$ which is calculated from equation (1.24).

Figure 1.9b is for $k = 4$, $k_{\mathrm{j}} = 1.694172\ldots$, $x_{\mathrm{a}} = 3.422399 > k_{\mathrm{j}}$ and $p/\sigma_{\mathrm{y}} = 1.082532\ldots$ which is calculated from equation (1.24).

Figure 1.9c is for $k = k_{\mathrm{j}} = k_{\mathrm{c}} = 2.2184574899167\ldots$ (entirely yielded), $p/\sigma_{\mathrm{y}} = 0.920081\ldots$ which is calculated from equation (1.24) or equation (1.32), $x_{\mathrm{z}} = 1.94029$ from equation (1.27). $x_{\mathrm{a}} = 1.898112 < k_{\mathrm{j}}$, so, x_{a} does not exist and x_{z} appears.

Figure 1.9d is for $k = 4$, $k_{\mathrm{j}} = 3.7$, $x_{\mathrm{z}} = 3.478307$ from equation (1.27), $x_{\mathrm{a}} = 3.422399 < k_{\mathrm{j}}$ and $p/\sigma_{\mathrm{y}} = 1.594088\ldots$ which is calculated from equation (1.24).

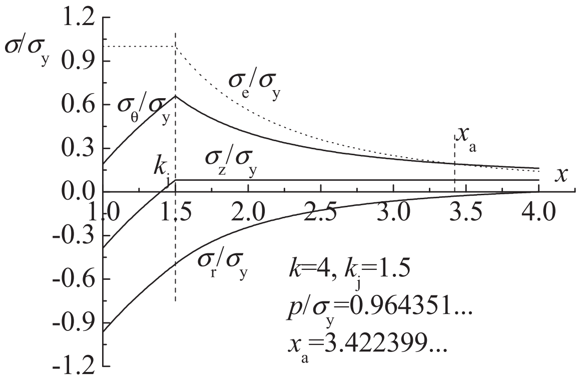
Figure 1.10a–d shows the distributions of elastoplastic stresses based on the maximum shear stress theory, where

Figure 1.10a is for $k = 4$, $k_{\mathrm{j}} = 1.5$, $p/\sigma_{\mathrm{y}} = 0.835153\ldots$ which is determined from equation (1.30).

Figure 1.10b is for $k = 4$, $k_{\mathrm{j}} = 1.694172\ldots$, $p/\sigma_{\mathrm{y}} = 0.9375\ldots$ which is determined from equation (1.30).

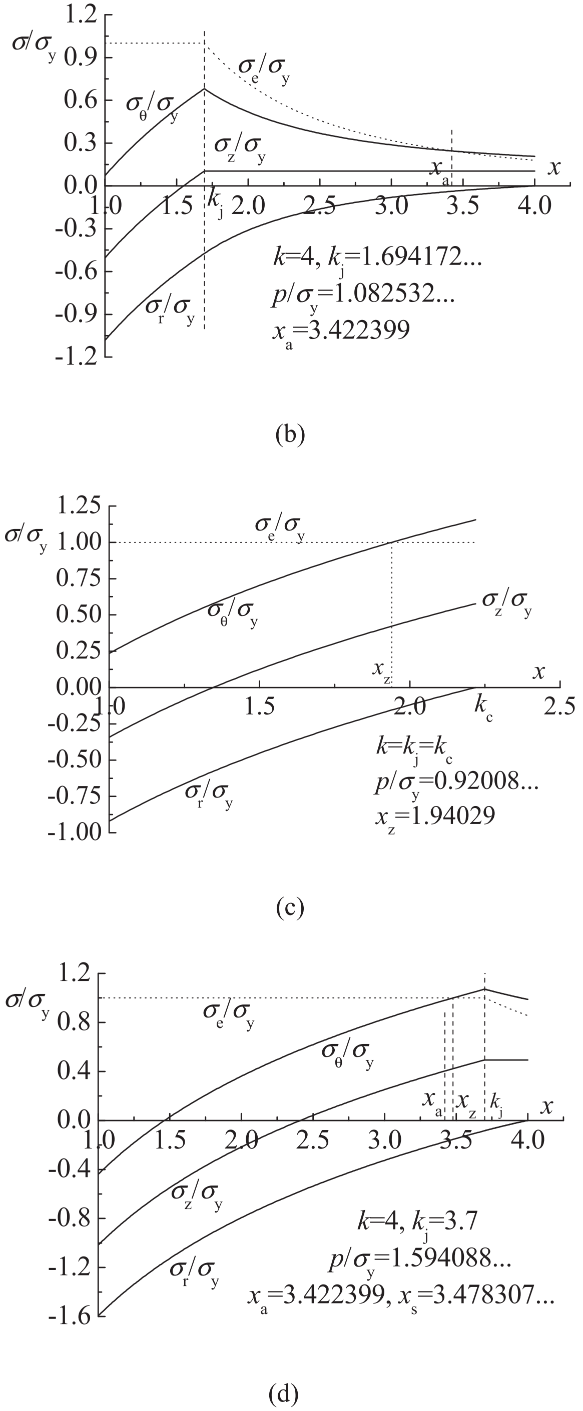
Figure 1.10c is for $k = k_{\mathrm{j}} = k_{\mathrm{c}} = 2.2184574899167\ldots$ (entirely yielded), $p/\sigma_{\mathrm{y}} = 0.796812\ldots$ which is determined from equation (1.30) or equation (1.36).

Figure 1.10d is for $k = 4$, $k_{\mathrm{j}} = 3.7$, $p/\sigma_{\mathrm{y}} = 1.38052\ldots$ which is calculated from equation (1.30).



(a)

FIG. 1.9 – The distributions of elastoplastic stresses based on the Mises yield criterion.

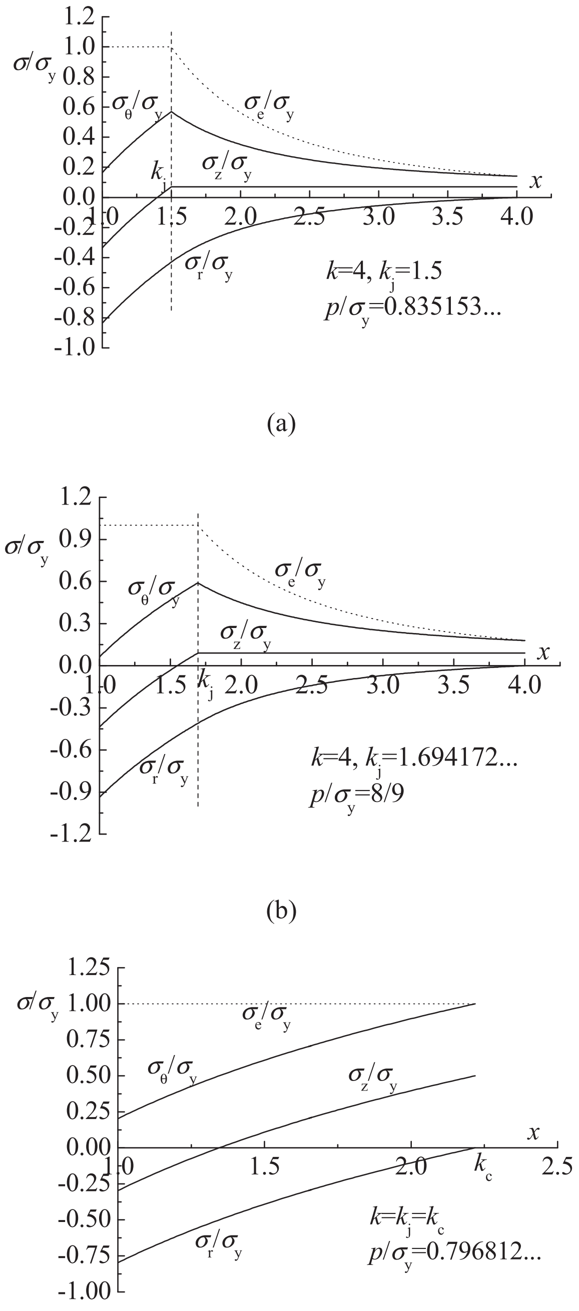


(b)

(c)

(d)

FIG. 1.9 – (continued).

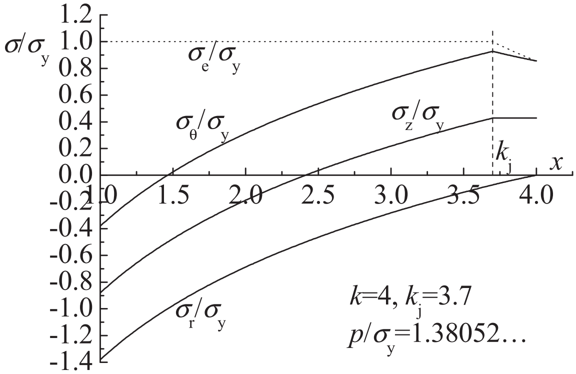


(a)

(b)

(c)

FIG. 1.10 – The distributions of elastoplastic stresses based on the Tresca yield criterion.



(d)

Fig. 1.10 – (continued).

Pressure vessels play a key role in the pillar industries in the national economy, emerging industries of strategic importance, aviation, military space technology, and so on. In recent years, with the rapid change of the world economic situation, resource quality deterioration, and energy structure adjustment, in addition to the large-scale, harsh medium environment, extreme service temperature, etc. development direction, pressure vessels are also showing a development situation of lightweight of heavy pressure vessels. Autofrettage is one of the effective ways of lightweight heavy-pressure vessels. The lightweighting, green design, and manufacturing of pressure vessels under the premise of ensuring their inherent safety, and the reduction of environmental pollution caused by the failure of pressure vessels subjected to disasters, are outstanding problems that need to be solved focally. It is hoped that autofrettage will play an important role in the lightweighting, green design, and manufacturing of pressure vessels[5–16].

1.4 Chapter Summary

The main equations and conclusion are listed in table 1.1.

TAB. 1.1 – Collection of the main equations conclusion.

Criterion		Within plastic region	Within elastic region
Tresca criterion	Equivalent stress σ_e	$\sigma_\mathrm{e}^\mathrm{s} = \sigma_\theta - \sigma_\mathrm{r}$	
Mises criterion		$\sigma_\mathrm{e}^\mathrm{d} = \dfrac{\sqrt{3}}{2}(\sigma_\theta - \sigma_\mathrm{r}) = \dfrac{\sqrt{3}}{2}\sigma_\mathrm{e}^\mathrm{s}$	
Tresca criterion	Elastoplastic stresses	$\dfrac{\sigma_\mathrm{z}}{\sigma_\mathrm{y}} = 0.5 + \ln x - \dfrac{p}{\sigma_\mathrm{y}}$ $\dfrac{\sigma_\mathrm{r}}{\sigma_\mathrm{y}} = \ln x - \dfrac{p}{\sigma_\mathrm{y}}$ $\dfrac{\sigma_\theta}{\sigma_\mathrm{y}} = 1 + \ln x - \dfrac{p}{\sigma_\mathrm{y}}$ $\dfrac{\sigma_\mathrm{e}}{\sigma_\mathrm{y}} \equiv 1$	$\dfrac{\sigma_\mathrm{z}}{\sigma_\mathrm{y}} = \dfrac{k_\mathrm{j}^2}{2k^2}$ $\dfrac{\sigma_\mathrm{r}}{\sigma_\mathrm{y}} = \dfrac{k_\mathrm{j}^2}{2k^2}\left(1 - \dfrac{k^2}{x^2}\right) = \dfrac{\sigma_\mathrm{z}}{\sigma_\mathrm{y}}\left(1 - \dfrac{k^2}{x^2}\right)$ $\dfrac{\sigma_\theta}{\sigma_\mathrm{y}} = \dfrac{k_\mathrm{j}^2}{2k^2}\left(1 + \dfrac{k^2}{x^2}\right) = \dfrac{\sigma_\mathrm{z}}{\sigma_\mathrm{y}}\left(1 + \dfrac{k^2}{x^2}\right)$ $\dfrac{\sigma_\mathrm{e}}{\sigma_\mathrm{y}} = \dfrac{k_\mathrm{j}^2}{x^2}$
	Relationship between k, k_j, p, and σ_y	$\dfrac{p}{\sigma_\mathrm{y}} = \ln k_\mathrm{j} + \dfrac{1}{2} - \dfrac{k_\mathrm{j}^2}{2k^2} = \ln k_\mathrm{j} + \dfrac{k^2 - k_\mathrm{j}^2}{2k^2}$ $k_\mathrm{j} = 1$: maximum elastic load-bearing capability or initial yield pressure, p_e $\dfrac{p_\mathrm{e}}{\sigma_\mathrm{y}} = \dfrac{k^2 - 1}{2k^2}$ $k_\mathrm{j} = k$: entire yield pressure, p_y $\dfrac{p_\mathrm{y}}{\sigma_\mathrm{y}} = \dfrac{2}{\sqrt{3}}\ln k$	
Mises criterion	Elastoplastic stresses	$\dfrac{\sigma_\mathrm{z}}{\sigma_\mathrm{y}} = \dfrac{2}{\sqrt{3}}\left(\dfrac{1}{2} + \ln x\right) - \dfrac{p}{\sigma_\mathrm{y}}$ $\dfrac{\sigma_\mathrm{r}}{\sigma_\mathrm{y}} = \dfrac{2}{\sqrt{3}}\ln x - \dfrac{p}{\sigma_\mathrm{y}}$ $\dfrac{\sigma_\theta}{\sigma_\mathrm{y}} = \dfrac{2}{\sqrt{3}}(1 + \ln x) - \dfrac{p}{\sigma_\mathrm{y}}$ $\dfrac{\sigma_\mathrm{e}}{\sigma_\mathrm{y}} \equiv 1$	$\dfrac{\sigma_\mathrm{z}}{\sigma_\mathrm{y}} = \dfrac{k_\mathrm{j}^2}{\sqrt{3}k^2}$ $\dfrac{\sigma_\mathrm{r}}{\sigma_\mathrm{y}} = \dfrac{k_\mathrm{j}^2}{\sqrt{3}k^2}\left(1 - \dfrac{k^2}{x^2}\right) = \dfrac{\sigma_\mathrm{z}}{\sigma_\mathrm{y}}\left(1 - \dfrac{k^2}{x^2}\right)$ $\dfrac{\sigma_\theta}{\sigma_\mathrm{y}} = \dfrac{k_\mathrm{j}^2}{\sqrt{3}k^2}\left(1 + \dfrac{k^2}{x^2}\right) = \dfrac{\sigma_\mathrm{z}}{\sigma_\mathrm{y}}\left(1 + \dfrac{k^2}{x^2}\right)$ $\dfrac{\sigma_\mathrm{e}}{\sigma_\mathrm{y}} = \dfrac{k_\mathrm{j}^2}{x^2}$

TAB. 1.1 – (continued).

			In elastic region, when $x \le \sqrt{\sqrt{3}-1}k = x_{\mathrm{a}} = 0.8556k$, $\dfrac{\sigma_{\mathrm{e}}}{\sigma_{\mathrm{y}}} \ge \dfrac{\sigma_{\theta}}{\sigma_{\mathrm{y}}}$; when $x \geq x_{\mathrm{a}}$, $\dfrac{\sigma_{\mathrm{e}}}{\sigma_{\mathrm{y}}} \le \dfrac{\sigma_{\theta}}{\sigma_{\mathrm{y}}}$
	Relationship between k, k_{j}, p, and σ_{y}	$\dfrac{p}{\sigma_{\mathrm{y}}} = \dfrac{2}{\sqrt{3}}\ln k_{\mathrm{j}} + \dfrac{1}{\sqrt{3}} - \dfrac{k_{\mathrm{j}}^2}{\sqrt{3}k^2} = \dfrac{2}{\sqrt{3}}\ln k_{\mathrm{j}} + \dfrac{k^2-k_{\mathrm{j}}^2}{\sqrt{3}k^2}$ $\dfrac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \dfrac{k^2-1}{\sqrt{3}k^2}$; $\dfrac{p_{\mathrm{y}}}{\sigma_{\mathrm{y}}} = \dfrac{2}{\sqrt{3}}\ln k$	

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Chapter 2

Mechanical Autofrettage Technology Based on Tresca Yield Criterion

The pressure applied to the cylindrical pressure vessel, or the pressure to achieve a plastic depth k_j is called autofrettage pressure p_a. For mechanical autofrettage, the key to autofrettage technology is the residual prestresses remaining in the wall of the pressure vessels after eliminating autofrettage pressure p_a.

Based on the maximum shear stress theory (Tresca yield criterion)[1], to achieve a plastic depth k_j, the pressure exerted on a pressure vessel is evidently equation (1.30), this is just autofrettage pressure p_a, or

$$\frac{p_\mathrm{a}}{\sigma_\mathrm{y}} = \ln k_\mathrm{j} + 0.5 - \frac{k_\mathrm{j}^2}{2k^2} = \ln k_\mathrm{j} + \frac{k^2 - k_\mathrm{j}^2}{2k^2} = \frac{k^2 \ln k_\mathrm{j}^2 + k^2 - k_\mathrm{j}^2}{2k^2} \tag{2.1}$$

Obviously, the greater k_j is, the greater p_a is. It's not difficult to foresee that k_j is bound to affect residual stresses, load-bearing capability, etc., and under different strength theories, the extent and the way of its influence will be different. Therefore, it is necessary to investigate autofrettage technology based on the maximum shear stress theory (Tresca yield criterion). We proceed from the general situation.

2.1 General Study on Mechanical Autofrettage Technology

In our research work, we have obtained some research results about mechanical autofrettage technology based on the maximum shear stress theory (Tresca criterion), which are the foundation of this research and are summed up as follows.

DOI: 10.1051/978-2-7598-3111-1.c002
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2.1.1 In General Forms

1. The components of residual stresses

When a pressure vessel is subjected to an autofrettage pressure p_{a} which is greater than its initial yield pressure (p_{e}) and smaller than the entire yield pressure (p_{y}), its inner layer becomes plastic and its outer layer remains elastic. After eliminating autofrettage pressure p_{a}, the residual stresses remain in the wall of the pressure vessels. Based on the maximum shear stress theory (Tresca criterion), we obtained the residual stresses as follows[2].

Within the plastic region ($1 \leq x \leq k_{\mathrm{j}}$)

$$\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = \frac{1}{2}\left[\frac{k_{\mathrm{j}}^2}{k^2} + \ln\frac{x^2}{k_{\mathrm{j}}^2} - \left(1 - \frac{k_{\mathrm{j}}^2}{k^2} + \ln k_{\mathrm{j}}^2\right)\frac{1}{k^2 - 1}\right] \tag{2.2}$$

$$\frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = \frac{1}{2}\left[\frac{k_{\mathrm{j}}^2}{k^2} - 1 + \ln\frac{x^2}{k_{\mathrm{j}}^2} - \left(1 - \frac{k_{\mathrm{j}}^2}{k^2} + \ln k_{\mathrm{j}}^2\right)\frac{1}{k^2 - 1}\left(1 - \frac{k^2}{x^2}\right)\right] \tag{2.3}$$

$$\frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}} = \frac{1}{2}\left[\frac{k_{\mathrm{j}}^2}{k^2} + 1 + \ln\frac{x^2}{k_{\mathrm{j}}^2} - \left(1 - \frac{k_{\mathrm{j}}^2}{k^2} + \ln k_{\mathrm{j}}^2\right)\frac{1}{k^2 - 1}\left(1 + \frac{k^2}{x^2}\right)\right] \tag{2.4}$$

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2}{k^2 - 1}\frac{1}{x^2} \tag{2.5}$$

It is not difficult to learn that as x increases, σ'_{z}, σ'_{θ}, and σ'_{e} increase. When $x \leq \sqrt{\frac{k^2 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2}{k^2 - 1}}$, σ'_{r} decreases as x increases; When $x \geq \sqrt{\frac{k^2 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2}{k^2 - 1}}$, σ'_{r} increases as x increases.

From $\ln k_{\mathrm{j}}^2 < k_{\mathrm{j}}^2 - 1$, we can prove that $\sqrt{\frac{k^2 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2}{k^2 - 1}} < k_{\mathrm{j}}$[3].

In fact, $\sqrt{\frac{k^2 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2}{k^2 - 1}}$ is x_0 in the following section.

Within the elastic region ($k_{\mathrm{j}} \leq x \leq k$)

$$\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = \frac{1}{2}\left[\frac{k_{\mathrm{j}}^2}{k^2} - \left(1 - \frac{k_{\mathrm{j}}^2}{k^2} + \ln k_{\mathrm{j}}^2\right)\frac{1}{k^2 - 1}\right] = \frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{2(k^2 - 1)} = \text{constant} > 0 \tag{2.6}$$

$$\frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = \left(1 - \frac{k^2}{x^2}\right)\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = -\frac{k^2 - x^2}{x^2}\frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{2(k^2 - 1)} < 0 \tag{2.7}$$

$$\frac{\sigma'_\theta}{\sigma_y} = \left(1 + \frac{k^2}{x^2}\right)\frac{\sigma'_z}{\sigma_y} = \frac{k^2 + x^2}{x^2}\frac{k_j^2 - 1 - \ln k_j^2}{2(k^2 - 1)} > 0 \tag{2.8}$$

$$\frac{\sigma'_e}{\sigma_y} = \frac{k_j^2 - 1 - \ln k_j^2}{k^2 - 1}\frac{k^2}{x^2} > 0 \tag{2.9}$$

Obviously, as x increases, σ'_θ and σ'_e decrease, σ'_r increases, and σ'_z is kept constant.

Figure 2.1 is the typical distribution of the residual stresses where $k = 4$ and $k_j = 1.6$ are taken as an example. The meaning of each curve and main parameters have been marked in the figure.

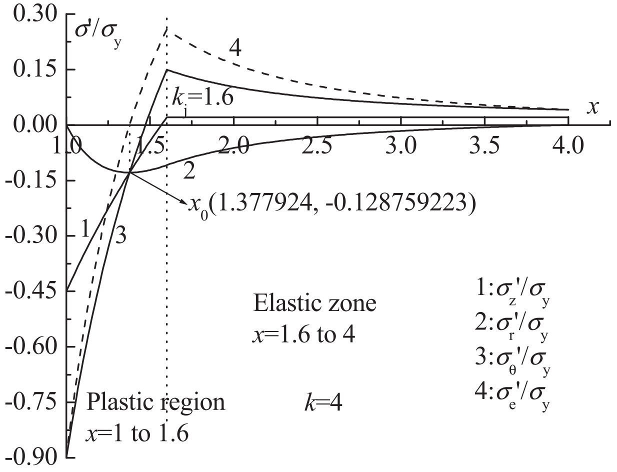

FIG. 2.1 – The typical distributions of the residual stresses.

At the inside surface, $x = r/r_i = 1$, then, from equations (2.2) to (2.5), the residual stresses on the inside surface are as follows.

$$\frac{\sigma'_{zi}}{\sigma_y} = -\frac{k^2 \ln k_j^2 - k_j^2 + 1}{2(k^2 - 1)} \leq 0 \tag{2.10}$$

$$\frac{\sigma'_{ri}}{\sigma_y} = 0 \tag{2.11}$$

$$\frac{\sigma'_{\theta i}}{\sigma_y} = -\frac{k^2 \ln k_j^2 - k_j^2 + 1}{k^2 - 1} = 2\frac{\sigma'_{zi}}{\sigma_y} \leq 0 \tag{2.12}$$

$$\sigma'_{\mathrm{ei}} = \sigma'_{\theta\mathrm{i}} - \sigma'_{\mathrm{ri}} = \sigma'_{\theta\mathrm{i}} \leq 0 \tag{2.13}$$

At the inside surface, the absolute value of σ'_{ei} or $\sigma'_{\theta\mathrm{i}}$ is the maximum.

At the elastic-plastic juncture, $x = k_{\mathrm{j}}$, the residual stresses obtained by equations (2.2)–(2.5) are equal to those obtained by equations (2.6)–(2.9). Therefore, the residual stresses in the whole wall are continuous. Substituting $x = k_{\mathrm{j}}$ into equations (2.2)–(2.5) or equations (2.6)–(2.9) leads to the residual stresses at the elastic-plastic juncture:

$$\frac{\sigma'_{\mathrm{zj}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{2(k^2-1)} \geq 0 \tag{2.14}$$

$$\frac{\sigma'_{\mathrm{ri}}}{\sigma_{\mathrm{y}}} = \left(1 - \frac{k^2}{k_{\mathrm{j}}^2}\right)\frac{\sigma'_{\mathrm{zj}}}{\sigma_{\mathrm{y}}} \leq 0 \tag{2.15}$$

$$\frac{\sigma'_{\theta\mathrm{j}}}{\sigma_{\mathrm{y}}} = \left(1 + \frac{k^2}{k_{\mathrm{j}}^2}\right)\frac{\sigma'_{\mathrm{zj}}}{\sigma_{\mathrm{y}}} > \frac{\sigma'_{\mathrm{zj}}}{\sigma_{\mathrm{y}}} \geq 0 \tag{2.16}$$

$$0 \leq \frac{\sigma'_{\theta\mathrm{j}}}{\sigma_{\mathrm{y}}} < \frac{\sigma'_{\mathrm{ej}}}{\sigma_{\mathrm{y}}} = 2\frac{k^2}{k_{\mathrm{j}}^2}\frac{\sigma'_{\mathrm{zj}}}{\sigma_{\mathrm{y}}} = \frac{k^2(k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2)}{(k^2-1)k_{\mathrm{j}}^2} \leq 1 \tag{2.17}$$

It can be proved that σ'_{θ}, σ'_{e}, and σ'_{z} monotonically increase with x increasing in the plastic region, σ'_{θ} and σ'_{e} monotonically decrease with x increasing in the elastic region, σ'_{z} remains constant with x increasing in the elastic region, σ'_{r} monotonically increases with x increasing in the elastic region[3]. At the elastic-plastic juncture, $\sigma'_{\theta\mathrm{j}}$, σ'_{ej}, and σ'_{zj} are the maximum within the whole wall (algebraic value, not absolute value).

$$\begin{aligned}\left|\frac{\sigma'_{\theta\mathrm{i}}}{\sigma_{\mathrm{y}}}\right| - \frac{\sigma'_{\theta\mathrm{j}}}{\sigma_{\mathrm{y}}} &= \frac{k^2\ln k_{\mathrm{j}}^2 - k_{\mathrm{j}}^2 + 1}{k^2-1} - \frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{2(k^2-1)}\frac{k^2+k_{\mathrm{j}}^2}{k_{\mathrm{j}}^2} \\ &= \frac{(2k^2k_{\mathrm{j}}^2 + k^2 + k_{\mathrm{j}}^2)\ln k_{\mathrm{j}}^2 - (k^2+3k_{\mathrm{j}}^2)(k_{\mathrm{j}}^2-1)}{2k_{\mathrm{j}}^2(k^2-1)}\end{aligned}$$

Since $\ln k_{\mathrm{j}}^2 > 2\frac{k_{\mathrm{j}}^2-1}{k_{\mathrm{j}}^2+1}$, $(2k^2k_{\mathrm{j}}^2 + k^2 + k_{\mathrm{j}}^2)\ln k_{\mathrm{j}}^2 > \frac{4k^2k_{\mathrm{j}}^2+2k^2+2k_{\mathrm{j}}^2}{k_{\mathrm{j}}^2+1}(k_{\mathrm{j}}^2-1)$

It is easy to prove that $\frac{4k^2k_{\mathrm{j}}^2+2k^2+2k_{\mathrm{j}}^2}{k_{\mathrm{j}}^2+1} > k^2 + 3k_{\mathrm{j}}^2$.

Therefore, $\left|\frac{\sigma'_{\theta\mathrm{i}}}{\sigma_{\mathrm{y}}}\right| > \frac{\sigma'_{\theta\mathrm{j}}}{\sigma_{\mathrm{y}}}$.

At the outside surface, $x = k$, then, from equations (2.6) to (2.9), the residual stresses on the outside surface are as follows.

$$\frac{\sigma'_{\mathrm{zo}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{2(k^2 - 1)} = \frac{\sigma'_{\mathrm{zj}}}{\sigma_{\mathrm{y}}} > 0 \tag{2.18}$$

$$\sigma'_{\mathrm{ro}} = 0 \tag{2.19}$$

$$0 < \sigma'_{\theta\mathrm{o}} = 2\sigma'_{\mathrm{zo}} < \sigma'_{\theta\mathrm{j}} > 0 \tag{2.20}$$

$$\sigma'_{\mathrm{eo}} = \sigma'_{\theta\mathrm{o}} \tag{2.21}$$

Within the whole wall, $\frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} < 0$. Because $\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}} - \frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}}$, $\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} - \frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}} = -\frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} > 0$, or within the whole wall, $\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} > \frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}}$, except $\sigma'_{\mathrm{ei}} = \sigma'_{\theta\mathrm{i}}$ and $\sigma'_{\mathrm{eo}} = \sigma'_{\theta\mathrm{o}}$.

2. Autofrettage pressure p_{a}

Autofrettage pressure is the internal pressure applied to a pressure vessel to cause a plastic depth k_{j} before it is put into production, which is

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \ln k_{\mathrm{j}} + \frac{k^2 - k_{\mathrm{j}}^2}{2k^2}$$

That is just equation (2.1).

3. The location of the intersection of the three residual stress curves or the abscissa at which $\sigma'_{\mathrm{e}} = 0$ (defined as x_0)

$$x_0 = \frac{r}{r_{\mathrm{i}}} = \sqrt{\frac{k^2 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2}{k^2 - 1}} < k_{\mathrm{j}} \tag{2.22}$$

In figure 2.1, $x_0 = \sqrt{\frac{k^2 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2}{k^2 - 1}} = 1.377924$.

The value of radial residual stresses at $x = x_0$ (σ'_{rx0}) is the minimum, which is

$$\frac{\sigma'_{\mathrm{rx0}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{rmin}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{zx0}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\theta\mathrm{x0}}}{\sigma_{\mathrm{y}}} = \ln\frac{x_0}{k_{\mathrm{j}}} + \frac{\sigma'_{\mathrm{zj}}}{\sigma_{\mathrm{y}}} = \ln\sqrt{\frac{k^2 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2}{k_{\mathrm{j}}^2(k^2 - 1)}} + \frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{2(k^2 - 1)} = y_0^{\mathrm{s}} \tag{2.23}$$

In figure 2.1, $y_0^{\mathrm{s}} = -0.128759223$

4. The location where $\sigma'_{\mathrm{z}} = 0$ (defined as x_1)

$$x_1 = \sqrt{\mathrm{e}^{\frac{1 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2}{k^2 - 1}}} \tag{2.24}$$

x_1 is obtained by letting $\sigma'_z = 0$ in equation (2.2).

From $\ln k_j^2 < k_j^2 - 1$, we can prove that $x_1 < k_j$.

On the other hand, since $\sigma'_{zi} < 0$, $\sigma'_{zj} > 0$, and $\sigma'_z > 0$ within the whole elastic region, it can be concluded that $\sigma'_z = 0$ occurs in the plastic region, or $x_1 < k_j$.

5. The total stresses

The total stresses include the residual stresses and the elastic stresses caused by internal pressure p

$$\frac{\sigma_z^T}{\sigma_y} = \frac{\sigma'_z}{\sigma_y} + \frac{\sigma_z^p}{\sigma_y} \tag{2.25}$$

$$\frac{\sigma_r^T}{\sigma_y} = \frac{\sigma'_r}{\sigma_y} + \frac{\sigma_r^p}{\sigma_y} \tag{2.26}$$

$$\frac{\sigma_\theta^T}{\sigma_y} = \frac{\sigma'_\theta}{\sigma_y} + \frac{\sigma_\theta^p}{\sigma_y} \tag{2.27}$$

$$\begin{aligned}\frac{\sigma_e^T}{\sigma_y} &= \frac{\sigma_\theta^T}{\sigma_y} - \frac{\sigma_r^T}{\sigma_y} = \left(\frac{\sigma'_\theta\sigma}{\sigma_y} + \frac{\sigma_\theta^p}{\sigma_y}\right) - \left(\frac{\sigma'_r}{\sigma_y} + \frac{\sigma_r^p}{\sigma_y}\right)\\ &= \left(\frac{\sigma'_\theta}{\sigma_y} - \frac{\sigma'_r}{\sigma_y}\right) + \left(\frac{\sigma_\theta^p}{\sigma_y} - \frac{\sigma_r^p}{\sigma_y}\right) = \frac{\sigma'_e}{\sigma_y} + \frac{\sigma_e^p}{\sigma_y}\end{aligned} \tag{2.28}$$

where $\frac{\sigma_z^p}{\sigma_y}$, $\frac{\sigma_r^p}{\sigma_y}$ and $\frac{\sigma_\theta^p}{\sigma_y}$ are expressed by equation (1.1), $\frac{\sigma_e^p}{\sigma_y}$ is equation (1.9).

Within the plastic region ($1 \leq x \leq k_j$)

$$\frac{\sigma_z^T}{\sigma_y} = \frac{1}{2}\left[\frac{k_j^2}{k^2} + \ln\frac{x^2}{k_j^2} - \left(1 - \frac{k_j^2}{k^2} + \ln k_j^2\right)\frac{1}{k^2-1}\right] + \frac{1}{k^2-1}\frac{p}{\sigma_y} \tag{2.29}$$

$$\frac{\sigma_r^T}{\sigma_y} = \frac{1}{2}\left[\frac{k_j^2}{k^2} - 1 + \ln\frac{x^2}{k_j^2} - \left(1 - \frac{k_j^2}{k^2} + \ln k_j^2\right)\frac{1}{k^2-1}\left(1 - \frac{k^2}{x^2}\right)\right] - \frac{k^2-x^2}{x^2(k^2-1)}\frac{p}{\sigma_y} \tag{2.30}$$

$$\frac{\sigma_\theta^T}{\sigma_y} = \frac{1}{2}\left[\frac{k_j^2}{k^2} + 1 + \ln\frac{x^2}{k_j^2} - \left(1 - \frac{k_j^2}{k^2} + \ln k_j^2\right)\frac{1}{k^2-1}\left(1 + \frac{k^2}{x^2}\right)\right] + \frac{k^2+x^2}{x^2(k^2-1)}\frac{p}{\sigma_y} \tag{2.31}$$

$$\frac{\sigma_e^T}{\sigma_y} = 1 - \frac{k^2 - k_j^2 + k^2\ln k_j^2}{x^2(k^2-1)} + \frac{2k^2}{x^2(k^2-1)}\frac{p}{\sigma_y} \tag{2.32}$$

Setting $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \leq 1$ obtains the load-bearing capability

$$\frac{p}{\sigma_{\mathrm{y}}} \leq \ln k_{\mathrm{j}} + \frac{k^2 - k_{\mathrm{j}}^2}{2k^2} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} \tag{2.33}$$

Setting $\frac{\mathrm{d}\left(\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\right)}{\mathrm{d}x} = \frac{k^2 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2}{k^2 - 1} \frac{2}{x^3} - \frac{4k^2}{k^2 - 1} \frac{1}{x^3} \frac{p}{\sigma_{\mathrm{y}}} \geq 0$ also obtains equation (2.33).

When $\frac{p}{\sigma_{\mathrm{y}}} = \ln k_{\mathrm{j}} + \frac{k^2 - k_{\mathrm{j}}^2}{2k^2} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \equiv 1$ = constant within the whole plastic region. Since $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} - \frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ = constant and $\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{1}{2}\left(\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\right)$, the three curves for $\sigma_{\mathrm{z}}^{\mathrm{T}}$, $\sigma_{\theta}^{\mathrm{T}}$, and $\sigma_{\mathrm{r}}^{\mathrm{T}}$ are parallel to each other.

Setting $\frac{\mathrm{d}\left(\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\right)}{\mathrm{d}x} \geq 0$ obtains

$$\frac{p}{\sigma_{\mathrm{y}}} \leq \frac{x^2(k^2 - 1)}{2k^2} + \frac{k^2 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2}{2k^2} > \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} \tag{2.34}$$

Setting $\frac{\mathrm{d}\left(\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\right)}{\mathrm{d}x} \geq 0$ obtains

$$\frac{p}{\sigma_{\mathrm{y}}} \geq \frac{k^2 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2}{2k^2} - \frac{x^2(k^2 - 1)}{2k^2} = \frac{p_{\mathrm{r}}}{\sigma_{\mathrm{y}}} < \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} \tag{2.35}$$

When $x = 1$, equation (2.35) becomes

$$\frac{p}{\sigma_{\mathrm{y}}} \geq \ln k_{\mathrm{j}} - \frac{k_{\mathrm{j}}^2 - 1}{2k^2} = \frac{p_{\mathrm{ri}}}{\sigma_{\mathrm{y}}} \tag{2.36}$$

When $x = k_{\mathrm{j}}$, equation (2.35) becomes

$$\frac{p}{\sigma_{\mathrm{y}}} \geq \ln k_{\mathrm{j}} - \frac{k_{\mathrm{j}}^2 - 1}{2} = \frac{p_{\mathrm{ro}}}{\sigma_{\mathrm{y}}} < 0 \tag{2.37}$$

Consequently, as long as the operating pressure p does not exceed autofrettage pressure p_{a}, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ increases with the increase of x. If $\frac{p}{\sigma_{\mathrm{y}}} \leq \frac{x^2(k^2-1)}{2k^2} + \frac{k^2 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2}{2k^2} > \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$, $\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ increases with the increase of x. Clearly, $\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ always increases with the increase of x. When $\frac{p}{\sigma_{\mathrm{y}}}$ satisfies equation (2.35), $\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ monotonously increases with the increase of x, otherwise, $\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ does not increase monotonously with the increase of x in the plastic region. Especially, when $p = 0$, $\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ is just $\frac{\sigma_{\mathrm{r}}'}{\sigma_{\mathrm{y}}}$, therefore, when $x \leq x_0 = \sqrt{\frac{k^2 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2}{k^2 - 1}}$, $\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ decreases as x increases; when $x \geq x_0$, $\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ increases as x increases. Besides, from equation (2.35) and equation (2.32), when $\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{r}}}{\sigma_{\mathrm{y}}}$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 0$. Since $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} - \frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$, when

$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 0$, there must be $\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$. Since $\sigma_{\mathrm{z}}^{\mathrm{T}} = \frac{\sigma_{\theta}^{\mathrm{T}} + \sigma_{\mathrm{r}}^{\mathrm{T}}}{2}$, when $\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$, there must be $\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$. Therefore, if $p \leq p_{\mathrm{r}}$, $\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ does not increase monotonously with the increase of x in the plastic region, the three curves for $\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$, $\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ and $\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ have an intersection in the plastic region, and at the intersection, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 0$. In addition, it can be proved that at the intersection, $\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ gets the minimum, or $\left|\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\right|$ gets the maximum. The abscissa (x_0^{T}) of the intersection point is solved out from equation (2.35) as follows

$$x_0^{\mathrm{T}} = \sqrt{\frac{k^2 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2}{k^2 - 1} - \frac{2k^2}{k^2 - 1} \frac{p}{\sigma_{\mathrm{y}}}} \tag{2.38}$$

Within the elastic region ($k_{\mathrm{j}} \leq x \leq k$)

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{2(k^2 - 1)} + \frac{1}{k^2 - 1} \frac{p}{\sigma_{\mathrm{y}}} = \text{constant} > 0 \tag{2.39}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = -\frac{k^2 - x^2}{x^2} \frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{2(k^2 - 1)} - \frac{k^2 - x^2}{x^2(k^2 - 1)} \frac{p}{\sigma_{\mathrm{y}}} < 0 \tag{2.40}$$

$$\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k^2 + x^2}{x^2} \frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{2(k^2 - 1)} + \frac{k^2 + x^2}{x^2(k^2 - 1)} \frac{p}{\sigma_{\mathrm{y}}} > 0 \tag{2.41}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{k^2 - 1} \frac{k^2}{x^2} + \frac{2k^2}{x^2(k^2 - 1)} \frac{p}{\sigma_{\mathrm{y}}} > 0 \tag{2.42}$$

Since within the whole wall, $\frac{\sigma_{\mathrm{r}}'}{\sigma_{\mathrm{y}}} < 0$ and $\frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} < 0$, $\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} < 0$, while $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} - \frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$, or $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} - \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = -\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} > 0$, then, within the whole wall, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} > \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$, except that $\frac{\sigma_{\mathrm{eo}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{e}\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{k^2 - 1} + \frac{2}{k^2 - 1} \frac{p}{\sigma_{\mathrm{y}}}$.

Evidently, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ and $\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ decrease as x increases, $\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ increases as x increases, and $\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ is kept constant within the elastic region.

In order to summarize and confirm the above conclusions, several figures for total stresses are given according to equations (2.29)–(2.32) and (2.39)–(2.42), as shown in figures 2.2–2.8.

(1) Add operation stresses to figure 2.1 to illustrate the total stresses as shown in figure 2.2. When $k = 4$ and $k_{\mathrm{j}} = 1.6$, $\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \ln k_{\mathrm{j}} + \frac{k^2 - k_{\mathrm{j}}^2}{2k^2} = 0.890004 > \frac{p_{\mathrm{ri}}}{\sigma_{\mathrm{y}}}$.

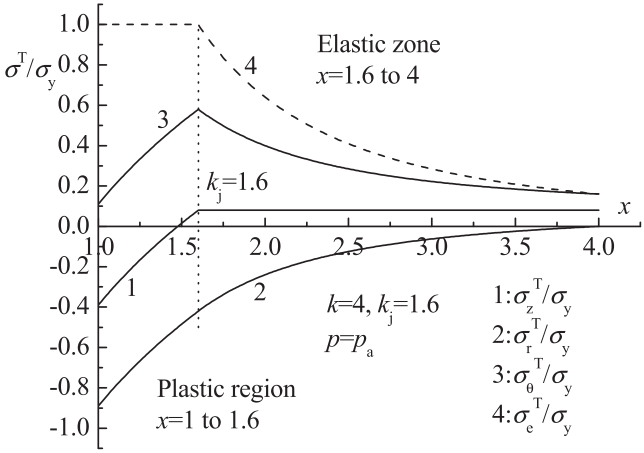

FIG. 2.2 – The total stresses when $k = 4$, $k_j = 1.6$ and $p = p_a$.

Within the whole wall, $\frac{\sigma_e^T}{\sigma_y} > \frac{\sigma_\theta^T}{\sigma_y}$ except $\frac{\sigma_{eo}^T}{\sigma_y} = \frac{\sigma_{\theta o}^T}{\sigma_y}$.

Within the plastic region, curves 1, 2, and 3 are parallel to each other because $p = p_a$.

Because $p = p_a$, $\sigma_e \equiv \sigma_y$ within the whole plastic region, and $\sigma_e < \sigma_y$ within the elastic region.

$\frac{\sigma_r^T}{\sigma_y}$ increases monotonously with the increase of x within the whole wall.

(2) When $k = 4$ and $k_j = 1.6$, $\frac{p_a}{\sigma_y} = 0.890004$, $\frac{p}{\sigma_y} = \ln k_j - \frac{k_j^2-1}{2k^2} = \frac{p_{ri}}{\sigma_y} = 0.421254 < \frac{p_e}{\sigma_y} = 0.46875 < \frac{p_a}{\sigma_y}$.

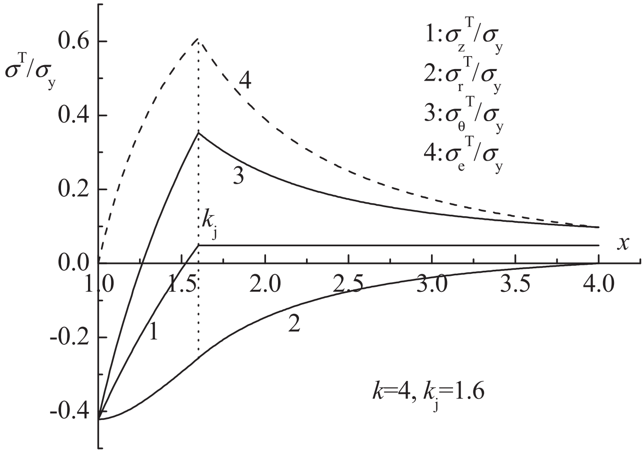

FIG. 2.3 – The total stresses when $k = 4$, $k_j = 1.6$ and $p = p_{ri}$.

Within the whole wall, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} > \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ except $\frac{\sigma_{\mathrm{eo}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\theta\mathrm{o}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$.

Within the plastic region, curves 1, 2, and 3 are not parallel to each other because $p \neq p_{\mathrm{a}}$.

When $p < p_{\mathrm{a}}$, all stresses are lower than σ_{y} within the whole wall.

$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ increases monotonously with the increase of x in the plastic region, but it is in a critical state because $p = p_{\mathrm{ri}}$. Therefore, $x_0^{\mathrm{T}} = 0$.

(3) When $k = 4$ and $k_{\mathrm{j}} = 1.6$, $\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = 0.270082 < \frac{p_{\mathrm{ri}}}{\sigma_{\mathrm{y}}} < \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = 0.46875 < \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = 0.890004$.

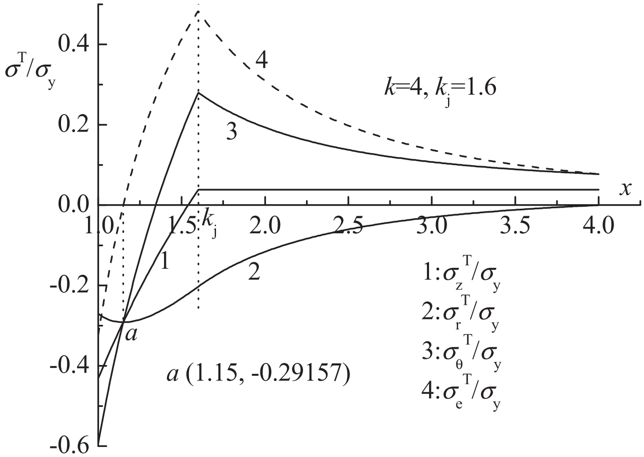

FIG. 2.4 – The total stresses when $k = 4$, $k_{\mathrm{j}} = 1.6$ and $p < p_{\mathrm{ri}}$.

Within the whole wall, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} > \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ except $\frac{\sigma_{\mathrm{eo}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\theta\mathrm{o}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$.

Within the plastic region, curves 1, 2, and 3 are not parallel to each other because $p \neq p_{\mathrm{a}}$.

When $p < p_{\mathrm{a}}$, all stresses are lower than σ_{y} within the whole wall.

$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ does not increase monotonously with the increase of x in the plastic region, the three curves for total stresses have an intersection in the plastic region because $p < p_{\mathrm{ri}}$, at the intersection, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 0$ and $\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ gets the minimum. $x_0^{\mathrm{T}} = 1.15 > 0$. The coordinates of the intersection point a are $a\ (1.15, -0.29157)$.

(4) When $k = 4$ and $k_{\mathrm{j}} = 1.6$, $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = 0.890004 > \frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = 0.46875 > \frac{p_{\mathrm{ri}}}{\sigma_{\mathrm{y}}} = 0.421254$.

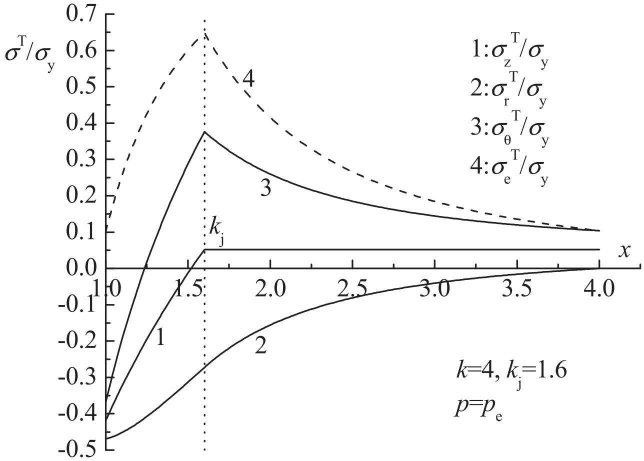

FIG. 2.5 – The total stresses when $k = 4$, $k_j = 1.6$ and $p = p_e > p_{ri}$.

Within the whole wall, $\frac{\sigma_e^T}{\sigma_y} > \frac{\sigma_\theta^T}{\sigma_y}$ except $\frac{\sigma_{eo}^T}{\sigma_y} = \frac{\sigma_{\theta o}^T}{\sigma_y}$.

When $p < p_a$, all stresses are lower than σ_y within the whole wall.

Within the whole wall, curves 1, 2, and 3 are not parallel to each other because σ_e is not constant.

$\frac{\sigma_r^T}{\sigma_y}$ increases monotonously with the increase of x within the whole wall because $p > p_{ri}$.

(5) When $k = 4$ and $k_j = 1.6$, $\frac{p}{\sigma_y} = 0.95 > \frac{p_a}{\sigma_y} > \frac{p_{ri}}{\sigma_y}$.

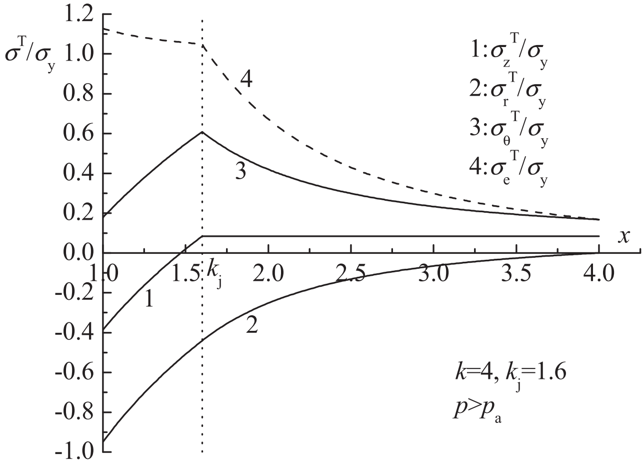

FIG. 2.6 – The total stresses when $k = 4$, $k_j = 1.6$ and $p > p_a$.

Within the whole wall, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} > \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ except $\frac{\sigma_{\mathrm{eo}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\theta\mathrm{o}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$.

Within the whole wall, curves 1, 2 and 3 are not parallel to each other because σ_{e} is not constant.

$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ increases monotonously with the increase of x within the whole wall because $p > p_{\mathrm{ri}}$.

Since $p > p_{\mathrm{a}}$, $\sigma_{\mathrm{e}} > \sigma_{\mathrm{y}}$ within the whole plastic region.

(6) When $k = 4$ and $k_{\mathrm{j}} = 2$, $\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = 0.46875 < \frac{p_{\mathrm{ri}}}{\sigma_{\mathrm{y}}} = 0.599397 < \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = 1.068147$.

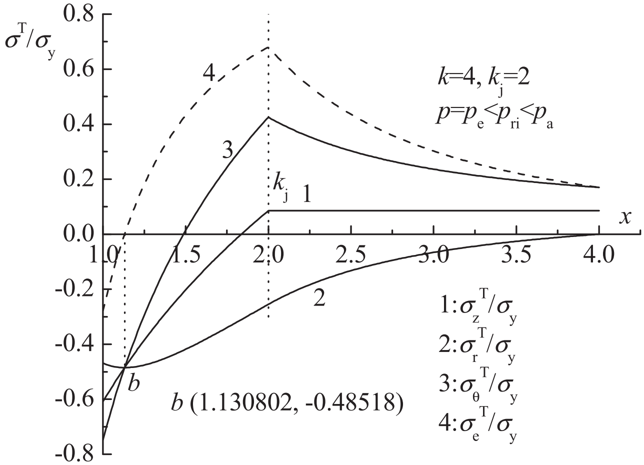

FIG. 2.7 – The total stresses when $k = 4$, $k_{\mathrm{j}} = 2$ and $p = p_{\mathrm{e}} < p_{\mathrm{ri}}$.

Within the whole wall, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} > \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ except $\frac{\sigma_{\mathrm{eo}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\theta\mathrm{o}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$.

$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ does not increase monotonously with the increase of x in the plastic region, the three curves for total stresses have an intersection in the plastic region, at the intersection, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 0$ and $\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ gets the minimum. $x_0^{\mathrm{T}} = 1.130802 > 0$. The coordinates of the intersection point b are b $(1.130802, -0.48518)$.

Within the whole wall, curves 1, 2, and 3 are not parallel to each other because $p \neq p_{\mathrm{a}}$ thus σ_{e} is not constant.

Since $p < p_{\mathrm{a}}$, $\sigma_{\mathrm{e}} < \sigma_{\mathrm{y}}$ within the whole plastic region.

(7) When $k = 4$ and $k_{\mathrm{j}} = 1.694172$, $\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = 0.46875 = \frac{p_{\mathrm{ri}}}{\sigma_{\mathrm{y}}} = 0.46875 < \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = 0.9375$.

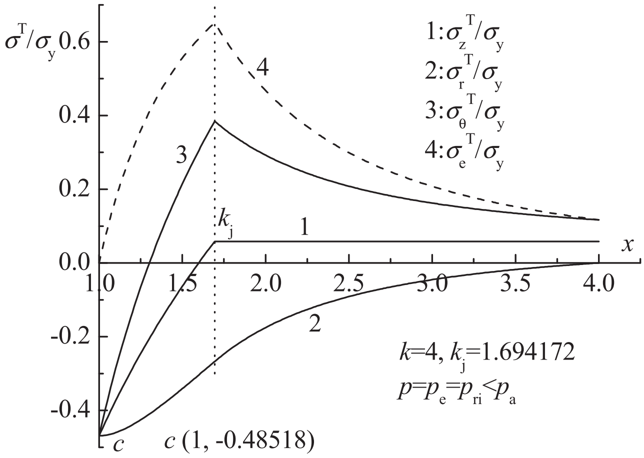

FIG. 2.8 – The total stresses when $k = 4$, $k_{\mathrm{j}} = 1.694172$ and $p = p_{\mathrm{e}} = p_{\mathrm{ri}} < p_{\mathrm{a}}$.

Within the whole wall, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} > \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ except $\frac{\sigma_{\mathrm{eo}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\theta\mathrm{o}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$.

$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ increase monotonously with the increase of x within the whole wall, but it is in a critical state, the intersection point of the three curves for total stresses is on the vertical axis, where $x_0^{\mathrm{T}} = 1$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 0$ and $\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ gets the minimum. The coordinates of the intersection point c are $c\ (1, -\frac{p}{\sigma_{\mathrm{y}}}) = c\ (1, -\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}) = c\ (1, -0.46875)$.

Within the whole wall, curves 1, 2, and 3 are not parallel to each other because $p \neq p_{\mathrm{a}}$ thus σ_{e} is not constant.

2.1.2 The Critical Radius Ratio

For equation (2.10), setting $k_{\mathrm{j}} = k$ leads to

$$\frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2 \ln k^2}{k^2 - 1} \tag{2.43}$$

For equation (2.43), letting $\frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}} \geq -1$

$$\frac{k^2 \ln k}{k^2 - 1} \leq 1 \tag{2.44}$$

The solution of equation (2.44) is

$$k \leq k_{\mathrm{c}} = 2.218\,457\,489\,916\,7\ldots \tag{2.45}$$

Therefore, when $k \leq k_{\mathrm{c}}$, irrespective of k_{j}, even if $k_{\mathrm{j}} = k$, $-\sigma_{\mathrm{y}} \leq \sigma'_{\mathrm{ei}} \leq 0$. In fact, $\frac{k^2 \ln k}{k^2 - 1} = 1$ is equivalent to $2p_{\mathrm{e}} = p_{\mathrm{y}}$, *i.e.*

$$2\frac{k^2-1}{2k^2}=\ln k \quad \text{or} \quad 2\frac{k^2-1}{\sqrt{3}k^2}=\frac{2\ln k}{\sqrt{3}}$$

where p_{e} is a maximum elastic load-bearing capability or initial yield pressure, p_{y} is entire yield pressure.

2.1.3 The Optimum Plastic Depth k_{j*} (k_j is Written as k_{j*})

For equation (2.10), setting $\frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}} \geq -1$ and writing k_{j} as $k_{\mathrm{j}*}$ obtain the optimum plastic depth $k_{\mathrm{j}*}$

$$k^2\ln k_{j*}^2-k^2-k_{j*}^2+2\leq 0 \quad \text{or} \quad k^2\leq\frac{k_{\mathrm{j}}^2-2}{\ln k_{\mathrm{j}}^2-1} \quad (\mathrm{e}^{0.5}\leq k_{\mathrm{j}*}\leq k_{\mathrm{c}}\text{ and }k\geq k_{\mathrm{c}}) \tag{2.46}$$

From equation (2.46), when $k\rightarrow\infty$, $k_{\mathrm{j}*}=\sqrt{\mathrm{e}}=k_{\mathrm{j}\infty}$; letting $k_{\mathrm{j}*}=k$ just results in equation (2.44).

When $k \geq k_{\mathrm{c}}$, if $k_{\mathrm{j}} \leq k_{\mathrm{j}*}$, which is the plastic depth calculated by equation (2.46), $-\sigma_{\mathrm{y}} \leq \sigma'_{\mathrm{ei}} \leq 0$. When $k \leq k_{\mathrm{c}}$, even if $k_{\mathrm{j}} = k$, $-\sigma_{\mathrm{y}} \leq \sigma'_{\mathrm{ei}} \leq 0$. The optimum plastic depth $k_{\mathrm{j}*}$ is illustrated in figure 2.9. From figure 2.9 and equation (2.46), we know that when $k \geq k_{\mathrm{c}}$, as k increases, $k_{\mathrm{j}*}$ decreases, which means that for $\sigma'_{\mathrm{ei}} \geq -\sigma_{\mathrm{y}}$, thicker pressure vessels need shallower plastic depth.

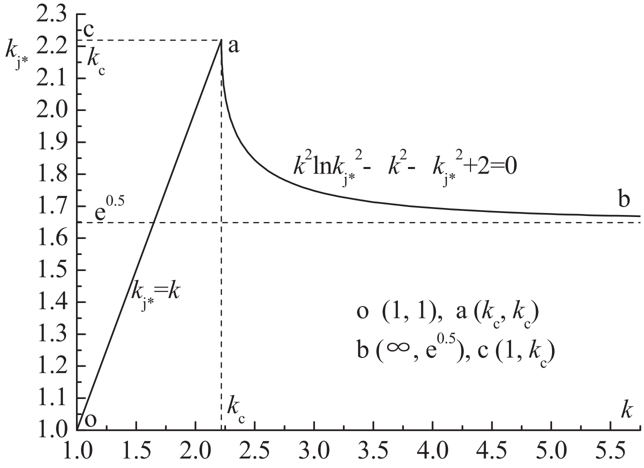

FIG. 2.9 – The optimum plastic depth $k_{\mathrm{j}*}$.

Interestingly enough, letting $\frac{p_{\mathrm{ri}}}{\sigma_{\mathrm{y}}}=\ln k_{\mathrm{j}}-\frac{k_{\mathrm{j}}^2-1}{2k^2}=\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}=\frac{k^2-1}{2k^2}$ just results in equation (2.46). When $k_{\mathrm{j}} < k_{\mathrm{j}*}$, $\frac{p_{\mathrm{ri}}}{\sigma_{\mathrm{y}}} < \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$; when $k_{\mathrm{j}} > k_{\mathrm{j}*}$, $\frac{p_{\mathrm{ri}}}{\sigma_{\mathrm{y}}} > \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$. Therefore, when $k_{\mathrm{j}} \leq k_{\mathrm{j}*}$, if $\frac{p}{\sigma_{\mathrm{y}}} > \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$, there must be $\frac{p}{\sigma_{\mathrm{y}}} > \frac{p_{\mathrm{ri}}}{\sigma_{\mathrm{y}}}$, thus, $\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ increases monotonously with the increase of x in the plastic region.

2.1.4 The Results When $k_j = k_{j*}$

1. The components of residual stresses

Within the plastic region ($1 \le x \le k_{\mathrm{j}*}$), when $k_{\mathrm{j}} = k_{\mathrm{j}*}$,

$$\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = \ln x - 0.5 \tag{2.47}$$

$$\frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = \ln x + x^{-2} - 1 \tag{2.48}$$

$$\frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}} = \ln x - x^{-2} \tag{2.49}$$

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = 1 - \frac{2}{x^2} \tag{2.50}$$

Within the elastic region ($k_{\mathrm{j}*} \le x \le k$), when $k_{\mathrm{j}} = k_{\mathrm{j}*}$,

$$\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}*}^2 - 2}{2k^2} \tag{2.51}$$

$$\frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = \left(1 - \frac{k^2}{x^2}\right)\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} \tag{2.52}$$

$$\frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}} = \left(1 + \frac{k^2}{x^2}\right)\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} \tag{2.53}$$

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}*}^2 - 2}{x^2} \tag{2.54}$$

On the inner surface, equations (2.47)–(2.50) become:

$$\frac{\sigma'_{\mathrm{zi}}}{\sigma_{\mathrm{y}}} = -\frac{1}{2} \tag{2.55}$$

$$\sigma'_{\mathrm{ri}} = 0 \tag{2.56}$$

$$\frac{\sigma'_{\theta\mathrm{i}}}{\sigma_{\mathrm{y}}} = -1 \tag{2.57}$$

$$\frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\theta\mathrm{i}}}{\sigma_{\mathrm{y}}} = -1 \tag{2.58}$$

Therefore, if $k_{\mathrm{j}} = k_{\mathrm{j}*}$, the residual stresses at the inner surface are always compressive and constant irrespective of k and k_{j}, or irrespective of the thickness and overstrain of a pressure vessel.

2. The optimum load-bearing capability

When the load exerted on the pressure vessel is just autofrettage pressure p_{a}, $\sigma'_{\mathrm{ei}} \equiv$ in the whole plastic region irrespective of k and k_{j}. When $k \geq k_{\mathrm{c}}$ and $k_{\mathrm{j}} = k_{\mathrm{j}*}$, autofrettage pressure p_{a} (see equation (2.1)) becomes $\frac{k^2-1}{k^2}$. Then, when $k \geq k_{\mathrm{c}}$ and $k_{\mathrm{j}} = k_{\mathrm{j}*}$, the optimum load-bearing capability is

$$\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = 2\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{k^2} \quad (k \geq k_{\mathrm{c}}) \tag{2.59}$$

When $k \rightarrow \infty$, $\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = 2\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{k^2} \rightarrow 1$, and $\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{2k^2} \rightarrow \frac{1}{2}$.

When $k \leq k_{\mathrm{c}}$, $k_{\mathrm{j}} = k_{\mathrm{j}*} = k$, autofrettage pressure p_{a} or equation (2.1) becomes $\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \ln k$. Then, the optimum load-bearing capability is

$$\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \ln k \quad (k \leq k_{\mathrm{c}}) \tag{2.60}$$

The optimum load-bearing capability is illustrated in figure 2.10, where the coordinates of some key points are marked.

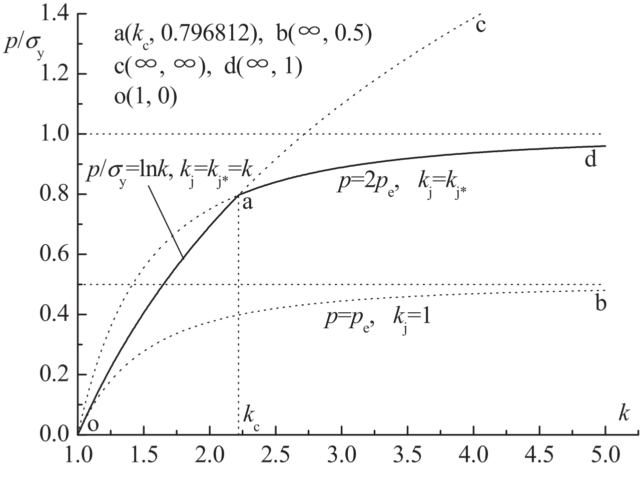

FIG. 2.10 – The optimum load-bearing capacity of an autofrettaged pressure vessel.

In figure 2.10, point a is the intersection of curve oac illustrating $\frac{p}{\sigma_{\mathrm{y}}} = \ln k$ and curve oad illustrating $\frac{p}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{k^2}$. Setting $\frac{p}{\sigma_{\mathrm{y}}} = \ln k = \frac{p}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{k^2}$ just obtains $\frac{k^2 \ln k}{k^2-1} = 1$.

3. The location of the intersection of the three residual stress curves or the abscissa at which $\sigma'_{\mathrm{e}} = 0$, defined as x_0

For equation (2.50), letting $\sigma'_{\mathrm{e}} = 0$ just results in $x = x_0 \equiv \sqrt{2}$. x_0 is not related to k and k_{j}. On the other hand, when $k_{\mathrm{j}} = k_{\mathrm{j}*}$, or when the plastic depth is expressed by equation (2.46), equation (2.22) becomes

$$x_0 \equiv \sqrt{2} \tag{2.61}$$

The value of residual stresses when $x = x_0$

$$y_0^{\mathrm{s}} = \frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{rmin}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{rx0}}}{\sigma_{\mathrm{y}}} \equiv \ln\sqrt{2} - 0.5 = -\frac{1-\ln 2}{2} = \text{constant} \tag{2.62}$$

y_0^{s} is not related to k and k_{j}.

This means that when $k_{\mathrm{j}} = k_{\mathrm{j}*}$, for any k, the three curves for the residual stress collect at the same point within the plastic region and the intersection is:

$$\left(\sqrt{2}, -\frac{1-\ln 2}{2}\right) = (1.414214\ldots, -0.1534264\ldots)$$

4. The location where $\sigma'_{\mathrm{z}} = 0$, defined as x_1

For equation (2.42), letting $\sigma'_{\mathrm{z}} = 0$ just results in $x = x_1 \equiv \sqrt{\mathrm{e}}$. x_1 is not related to k and k_{j}. On the other hand, when $k_{\mathrm{j}} = k_{\mathrm{j}*}$, or when the plastic depth is expressed by equation (2.46), equation (2.24) becomes

$$x_1 \equiv \sqrt{\mathrm{e}} \tag{2.63}$$

When $k = 4$, from equation (2.46), $k_{\mathrm{j}*} = 1.694172$, the residual stresses are illustrated in figure 2.11. The meaning of each curve and main parameters have been marked in the figure.

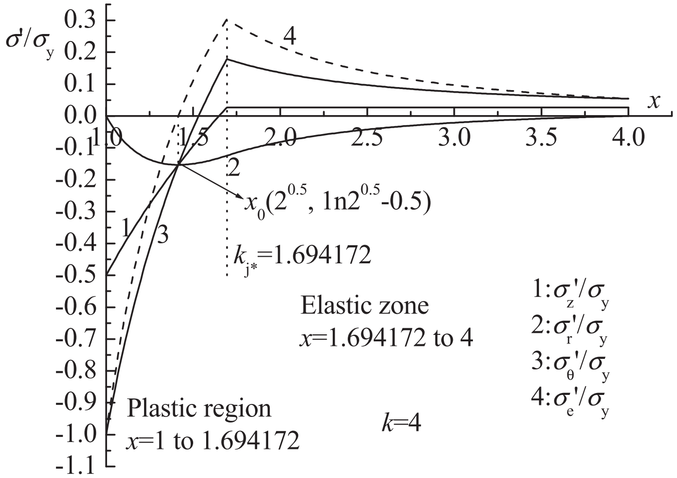

FIG. 2.11 – The residual stresses when $k_{\mathrm{j}} = k_{\mathrm{j}*}$.

5. The components of total stresses when $k \geq k_{\mathrm{c}}$, $k_{\mathrm{j}} = k_{\mathrm{j}*}$ and $p = p_{\mathrm{a}} = 2p_{\mathrm{e}}$

Within the plastic region $(1 \leq x \leq k_{\mathrm{j}*})$

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \ln x - \frac{1}{2} + \frac{1}{k^2} \tag{2.64}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \ln x - 1 + \frac{1}{k^2} \tag{2.65}$$

$$\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \ln x + \frac{1}{k^2} \tag{2.66}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \equiv 1 \tag{2.67}$$

On the inner surface $(x = 1)$,

$$\frac{\sigma_{\mathrm{zi}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = -\frac{1}{2} + \frac{1}{k^2} \tag{2.68}$$

$$\frac{\sigma_{\mathrm{ri}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = -1 + \frac{1}{k^2} \tag{2.69}$$

$$\frac{\sigma_{\theta\mathrm{i}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{1}{k^2} \tag{2.70}$$

$$\frac{\sigma_{\mathrm{ei}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1 \tag{2.71}$$

Clearly, $-\sigma_{\mathrm{y}} < \sigma_{\mathrm{zi}}^{\mathrm{T}} < \sigma_{\mathrm{y}}$, $-\sigma_{\mathrm{y}} < \sigma_{\mathrm{ri}}^{\mathrm{T}} < \sigma_{\mathrm{y}}$, and $0 < \sigma_{\theta\mathrm{i}}^{\mathrm{T}} < \sigma_{\mathrm{y}}$ in the range of $1 \leq k \leq \infty$.

On the elastic-plastic juncture $(x = k_{\mathrm{j}*})$,

$$\frac{\sigma_{\mathrm{zj}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \ln k_{\mathrm{j}*} - \frac{1}{2} + \frac{1}{k^2} \tag{2.72}$$

$$\frac{\sigma_{\mathrm{rj}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \ln k_{\mathrm{j}*} - 1 + \frac{1}{k^2} \tag{2.73}$$

$$\frac{\sigma_{\theta\mathrm{j}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \ln k_{\mathrm{j}*} + \frac{1}{k^2} > 0 \tag{2.74}$$

$$\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1 \tag{2.75}$$

The above expressions are quite concise.

$\frac{\sigma_{\mathrm{rj}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = -\frac{k^2-k_{\mathrm{j}}^2}{2k^2} > -1$. $\frac{\sigma_{\theta\mathrm{j}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \ln k_{\mathrm{j}} + \frac{1}{k^2} = \frac{k^2 \ln k_{\mathrm{j}} + 1}{k^2}$. When $k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 = 0$, $k^2 \ln k_{\mathrm{j}} + 1 = \frac{k^2 + k_{\mathrm{j}}^2}{2}$, therefore, $\frac{\sigma_{\theta\mathrm{j}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k^2 + k_{\mathrm{j}}^2}{2k^2} < 1$.

Within the elastic region ($k_{\mathrm{j}*} \leq x \leq k$)

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}*}^2}{2k^2} = \text{constant} > 0 \tag{2.76}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = -\frac{k_{\mathrm{j}*}^2}{2}\left(\frac{1}{x^2} - \frac{1}{k^2}\right) = \left(1 - \frac{k^2}{x^2}\right)\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \tag{2.77}$$

$$\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}*}^2}{2}\left(\frac{1}{x^2} + \frac{1}{k^2}\right) = \left(1 + \frac{k^2}{x^2}\right)\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \leq 1 \tag{2.78}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}*}^2}{x^2} \leq 1 \tag{2.79}$$

On the outer surface ($x = k$),

$$\frac{\sigma_{\mathrm{zo}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}*}^2}{2k^2} = \text{constant} \tag{2.80}$$

$$\frac{\sigma_{\mathrm{ro}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 0 \tag{2.81}$$

$$\frac{\sigma_{\theta\mathrm{o}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 2\frac{\sigma_{\mathrm{zo}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}*}^2}{k^2} = \frac{\sigma_{\mathrm{eo}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \leq 1 \tag{2.82}$$

$$\frac{\sigma_{\mathrm{eo}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\theta\mathrm{o}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}*}^2}{k^2} \leq 1 \tag{2.83}$$

All in all, if $k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 \leq 0$, all stresses are in a safe range.

The total stresses with $k = 4$, $k_{\mathrm{j}} = k_{\mathrm{j}*} = 1.694172$, and $p = p_{\mathrm{a}} = 2p_{\mathrm{e}}$ are illustrated in figure 2.12. The meaning of each curve and main parameters have been marked in the figure.

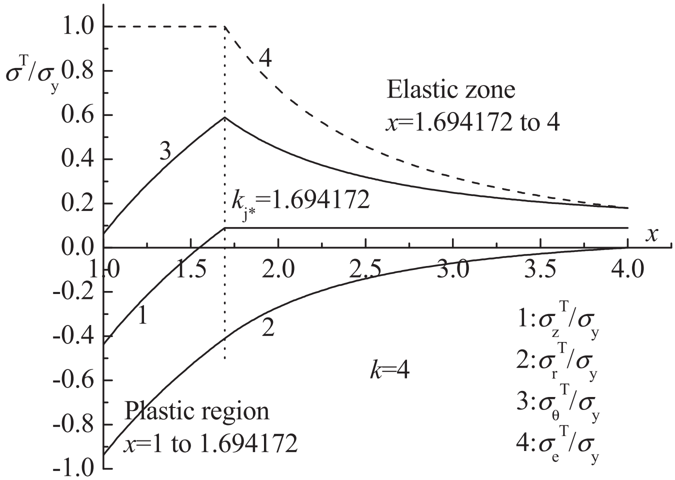

FIG. 2.12 – The total stresses with $k = 4$, $k_{\mathrm{j}} = k_{\mathrm{j}*}$ and $p = p_{\mathrm{a}} = 2p_{\mathrm{e}}$.

The following equations will be applied hereinafter.

Since $\ln x^2 > 2\frac{x^2-1}{x^2+1}$, or $\frac{x^2+1}{2(x^2-1)}\ln x^2 > 1$, thus,

$$\frac{k^2 \ln k^2}{k^2 - 1} > 1 \tag{2.84}$$

In addition,

$$k^2 \ln k_{\mathrm{j}}^2 > \frac{2k^2}{k_{\mathrm{j}}^2 + 1}(k_{\mathrm{j}}^2 - 1) > k_{\mathrm{j}}^2 - 1 \tag{2.85}$$

Therefore,

$$k^2 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2 > k^2 - 1 \tag{2.86}$$

Since $\mathrm{e}^{x^2-1} > 1 + x^2 - 1 = x^2$, then

$$\mathrm{e}^{k_{\mathrm{j}}^2-1} > 1 + k_{\mathrm{j}}^2 - 1 = k_{\mathrm{j}}^2 \tag{2.87}$$

and

$$\ln k_{\mathrm{j}}^2 < k_{\mathrm{j}}^2 - 1 \quad \text{or} \quad k^2 \ln k_{\mathrm{j}}^2 < k^2 k_{\mathrm{j}}^2 - k^2 \quad \text{or} \quad k^2 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2 < k^2 k_{\mathrm{j}}^2 - k_{\mathrm{j}}^2 \tag{2.88}$$

thus, $\frac{k^2 \ln k^2}{k^2-1} > 1$.

2.2 Mechanical Autofrettage Technology Under Entire Yield State

Overstrain is defined as

$$\varepsilon = \frac{r_{\mathrm{j}} - r_{\mathrm{i}}}{r_{\mathrm{o}} - r} = \frac{k_{\mathrm{j}} - 1}{k - 1} \tag{2.89}$$

When $k \leq k_{\mathrm{c}}$ (=2.2184574899167), k_{j} can be k. When $k_{\mathrm{j}} = k$, or overstrain $\varepsilon = 100\%$, there is only a plastic zone and no elastic zone within the whole wall of a pressure vessel, then, the components of residual stresses or equations (2.2)–(2.5) become

Within the whole wall ($1 \leq x \leq k$)

$$\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = \frac{1}{2}\left(1 + \ln\frac{x^2}{k^2} - \frac{\ln k^2}{k^2 - 1}\right) \tag{2.90}$$

$$\frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = \frac{1}{2}\left(\ln\frac{x^2}{k^2} - \frac{\ln k^2}{k^2 - 1} + \frac{k^2 \ln k^2}{(k^2 - 1)x^2}\right) \tag{2.91}$$

$$\frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}} = \frac{1}{2}\left(2 + \ln\frac{x^2}{k^2} - \frac{\ln k^2}{k^2 - 1} - \frac{k^2 \ln k^2}{(k^2 - 1)x^2}\right) \tag{2.92}$$

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2 \ln k^2}{(k^2 - 1)x^2} \tag{2.93}$$

At the inner surface, or $x = 1$, the absolute value of σ'_{ei} $(= \sigma'_{\theta\mathrm{i}} = 2\sigma'_{\mathrm{zi}})$ is the maximum, and

$$\frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2 \ln k^2}{k^2 - 1} \tag{2.94}$$

Clearly, as x increases, $\frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}}$ increases. $\frac{\mathrm{d}\left(\frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}}\right)}{\mathrm{d}k} = -\frac{2k(k^2-1-\ln k^2)}{(k^2-1)^2} < 0$, and when $k = 1$, $\frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}} = 0$; in addition, $\frac{k^2 \ln k^2}{k^2-1} > 1$. For any k, $\frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\theta\mathrm{i}}}{\sigma_{\mathrm{y}}} - \frac{\sigma'_{\mathrm{ri}}}{\sigma_{\mathrm{y}}} \leq 0$.

At the outer surface,

$$\frac{\sigma'_{\mathrm{eo}}}{\sigma_{\mathrm{y}}} = 1 - \frac{\ln k^2}{k^2 - 1} \tag{2.95}$$

Since $k^2 - 1 > \ln k^2$, $\frac{\sigma'_{\mathrm{eo}}}{\sigma_{\mathrm{y}}} > 0$.

Setting $\frac{\sigma'_{\text{ei}}}{\sigma_{\text{y}}} = 1 - \frac{k^2 \ln k^2}{k^2-1} \geq -1$ just obtains $\frac{k^2 \ln k}{k^2-1} \leq 1$, *i.e.* equation (2.44). Thus, under entire yield, or $k_{\text{j}} = k$, to avoid reverse yielding or $\sigma'_{\text{ei}} \geq -\sigma_{\text{y}}$, there must be $k \leq k_{\text{c}}$.

Figures 2.13 and 2.14 show the distributions of the residual stresses for $k = k_{\text{c}}$ and $k = 2$, respectively. To show the reverse yielding, figure 2.15 shows the distributions of the residual stresses for $k = 2.5$ ($>k_{\text{c}}$). The meaning of each curve and coordinates of key points as well as the main parameters are marked in the figures.

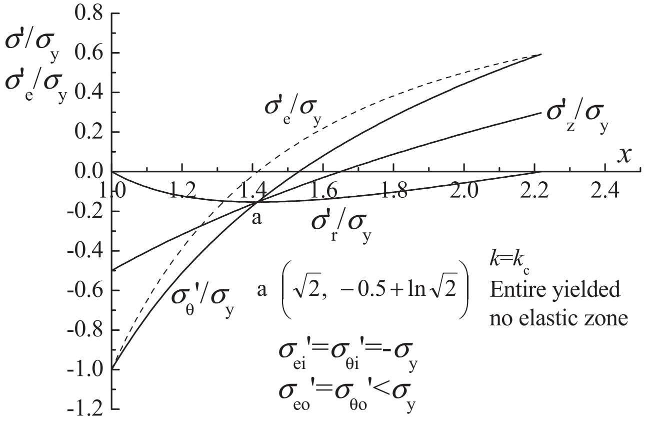

FIG. 2.13 – The distributions of the residual stresses for $k = k_{\text{c}}$ and $k_{\text{j}} = k$.

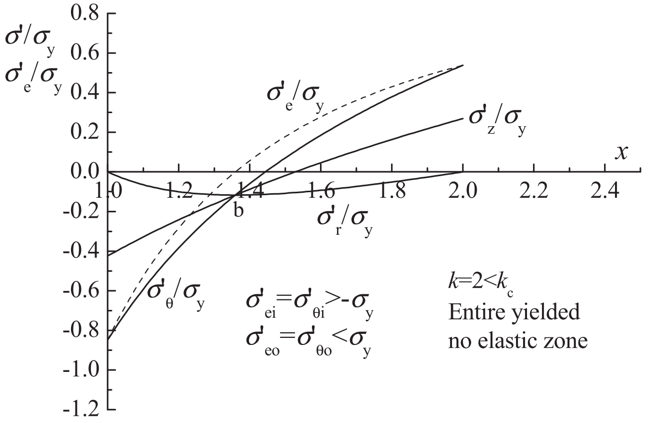

FIG. 2.14 – The distributions of the residual stresses for $k = 2$ and $k_{\text{j}} = k$.

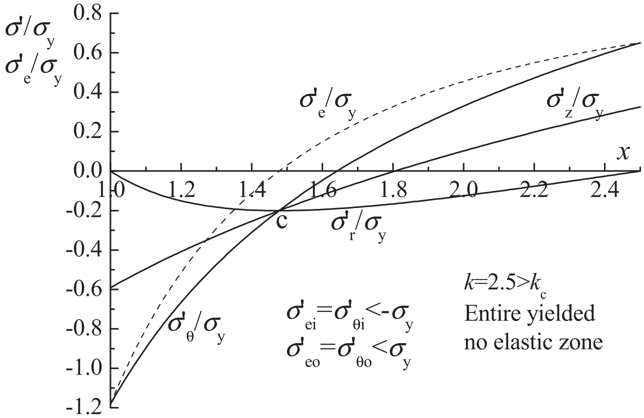

FIG. 2.15 – The distributions of the residual stresses for $k = 2.5$ and $k_\mathrm{j} = k$.

The total stresses within the whole wall ($1 \leq x \leq k$)

$$\frac{\sigma_\mathrm{z}^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{1}{2}\left(1 + \ln\frac{x^2}{k^2} - \frac{\ln k^2}{k^2-1}\right) + \frac{1}{k^2-1}\frac{p}{\sigma_\mathrm{y}} \tag{2.96}$$

$$\frac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{1}{2}\left(\ln\frac{x^2}{k^2} - \frac{\ln k^2}{k^2-1} + \frac{k^2\ln k^2}{(k^2-1)x^2}\right) - \frac{k^2-x^2}{x^2(k^2-1)}\frac{p}{\sigma_\mathrm{y}} \tag{2.97}$$

$$\frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{1}{2}\left(2 + \ln\frac{x^2}{k^2} - \frac{\ln k^2}{k^2-1} - \frac{k^2\ln k^2}{(k^2-1)x^2}\right) + \frac{k^2+x^2}{x^2(k^2-1)}\frac{p}{\sigma_\mathrm{y}} \tag{2.98}$$

$$\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} = 1 - \frac{k^2\ln k^2}{(k^2-1)x^2} + \frac{2k^2}{x^2(k^2-1)}\frac{p}{\sigma_\mathrm{y}} = 1 + \left(\frac{2k^2}{k^2-1}\frac{p}{\sigma_\mathrm{y}} - \frac{k^2\ln k^2}{k^2-1}\right)\frac{1}{x^2} \tag{2.99}$$

From equation (2.99), when $\frac{p}{\sigma_\mathrm{y}} \leq \ln k = \frac{p_\mathrm{y}}{\sigma_\mathrm{y}}$, the larger x is, the larger $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}}$ is. Setting $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} \leq 1$ leads to $\frac{p}{\sigma_\mathrm{y}} \leq \ln k = \frac{p_\mathrm{y}}{\sigma_\mathrm{y}}$. That is to say, when $k_\mathrm{j} = k$ and $k \leq k_\mathrm{c}$, the maximum load is the entire yield pressure. When a pressure vessel is subjected to entire yield pressure $p_\mathrm{y} = \sigma_\mathrm{y}\ln k$, Lamé equations become

$$\frac{\sigma_\mathrm{z}^\mathrm{p}}{\sigma_\mathrm{y}} = \frac{\ln k}{k^2-1} \tag{2.100}$$

$$\frac{\sigma_\mathrm{r}^\mathrm{p}}{\sigma_\mathrm{y}} = -\frac{k^2-x^2}{x^2(k^2-1)}\ln k \tag{2.101}$$

$$\frac{\sigma_\theta^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{k^2 + x^2}{x^2(k^2 - 1)} \ln k \tag{2.102}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{k^2 \ln k^2}{(k^2 - 1)x^2} \tag{2.103}$$

At the inner surface, $\frac{\sigma_{\mathrm{ei}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{k^2 \ln \mathrm{k}^2}{k^2 - 1} \geq 1$. It is seen that when $k_{\mathrm{j}} = k$ and $p = \sigma_{\mathrm{y}} \ln k$, even if $k \leq k_{\mathrm{c}}$, $\sigma_{\mathrm{ei}}^{\mathrm{p}} \geq \sigma_{\mathrm{y}}$ if autofrettage process is not carried out.

When $k_{\mathrm{j}} = k$ and $p = p_{\mathrm{y}} = \sigma_{\mathrm{y}} \ln k$, the total stresses are as follows

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{1}{2} + \ln x - \ln k \tag{2.104}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \ln x - \ln k \tag{2.105}$$

$$\frac{\sigma_\theta^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1 + \ln x - \ln k \tag{2.106}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \equiv 1 \tag{2.107}$$

as x increases, $\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$, $\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ and $\frac{\sigma_\theta^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ increase. At the inner surface,

$$\frac{\sigma_{\mathrm{zi}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{1}{2} - \ln k \tag{2.108}$$

$$\frac{\sigma_{\mathrm{ri}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = -\ln k \tag{2.109}$$

$$\frac{\sigma_{\theta\mathrm{i}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1 - \ln k \tag{2.110}$$

$$\frac{\sigma_{\mathrm{ei}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1 \tag{2.111}$$

For $\frac{\sigma_{\mathrm{zi}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \geq -1,\ k \leq \mathrm{e}^{1.5}\ (k_{\mathrm{c}} < \mathrm{e}^{1.5})$; for $\frac{\sigma_{\theta\mathrm{i}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \geq -1,\ k \leq \mathrm{e}^2\ (k_{\mathrm{c}} < \mathrm{e}^2)$; for $\frac{\sigma_{\mathrm{ri}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \geq -1$, $k \leq \mathrm{e}\ (k_{\mathrm{c}} < \mathrm{e})$. At the outer surface,

$$\sigma_{\mathrm{zo}}^{\mathrm{T}} = 0.5\sigma_{\mathrm{y}} \tag{2.112}$$

$$\sigma_{\mathrm{ro}}^{\mathrm{T}} = 0 \tag{2.113}$$

$$\sigma_{\theta\mathrm{o}}^{\mathrm{T}} = \sigma_{\mathrm{y}} \tag{2.114}$$

$$\sigma_{\mathrm{eo}}^{\mathrm{T}} = \sigma_{\mathrm{y}} \tag{2.115}$$

Seemingly, as long as $k \le \mathrm{e}$ $(k_{\mathrm{c}} < \mathrm{e})$, there certainly are $|\sigma_{\mathrm{z}}^{\mathrm{T}}| \le \sigma_{\mathrm{y}}$, $|\sigma_{\mathrm{r}}^{\mathrm{T}}| \le \sigma_{\mathrm{y}}$, $|\sigma_{\theta}^{\mathrm{T}}| \le \sigma_{\mathrm{y}}$, and $|\sigma_{\mathrm{e}}^{\mathrm{T}}| \le \sigma_{\mathrm{y}}$, but if $k \ge k_{\mathrm{c}}$ and $k_{\mathrm{j}} = k$, reverse yield will occur. When $k_{\mathrm{j}} = k$, $p_{\mathrm{y}} = \sigma_{\mathrm{y}} \ln k$ and $k \le k_{\mathrm{c}}$, the distributions of the total stresses are plotted in figures 2.16 and 2.17 for $k = k_{\mathrm{c}}$ and $k = 2$, respectively. The meaning of each curve and the main parameters are marked in the figures.

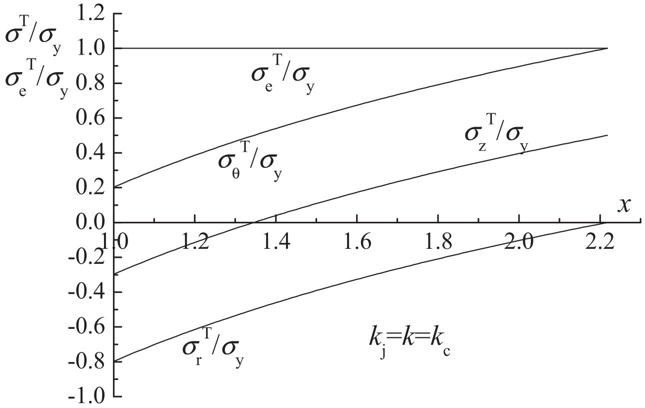

FIG. 2.16 – The distributions of the total stresses when $k = k_{\mathrm{c}}$ and $k_{\mathrm{j}} = k$.

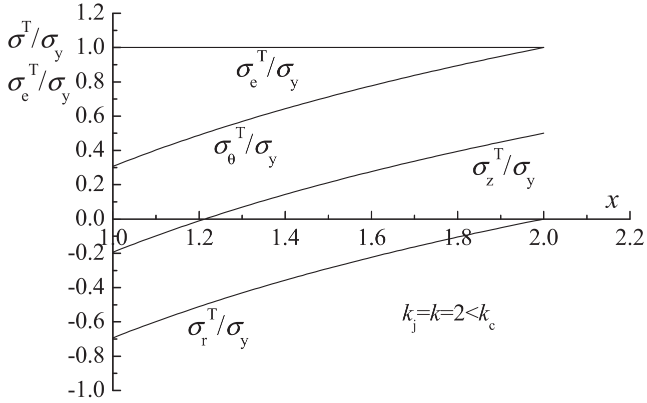

FIG. 2.17 – The distributions of the total stresses when $k = 2$ and $k_{\mathrm{j}} = k$.

All in all, the premise of $k_{\mathrm{j}} = k$ is $k \leq k_{\mathrm{c}}$.

In addition, when $k_{\mathrm{j}} = k$,

(1) Autofrettage pressure or equation (2.1) becomes $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \ln k$, which is the maximum load when $k_{\mathrm{j}} = k$;

(2) $x_0 = \sqrt{\frac{k^2 \ln k^2}{k^2-1}} < k$;

(3) $\frac{\sigma'_{\mathrm{zx0}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{\theta x0}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{rx0}}}{\sigma_{\mathrm{y}}} = \ln \sqrt{\frac{\ln k^2}{k^2-1}} + \frac{k^2-1-\ln k^2}{2(k^2-1)}$;

(4) $x_1 = \sqrt{\mathrm{e}^{\frac{1-k^2+k^2 \ln k^2}{k^2-1}}}$.

2.3 Mechanical Autofrettage Technology with Radius of Elastic-Plastic Juncture Being Arithmetic Mean Radius of Inside Radius and Outside Radius

Sometimes, some designers will be inclined to use the average radius as the radius of the plastic region. When the radius of the plastic region is an arithmetic mean value of the inner and outer radius of the pressure vessel, there is

$$r_{\mathrm{j}} = \frac{r_{\mathrm{i}} + r_{\mathrm{o}}}{2} \quad \text{or} \quad k_{\mathrm{j}} = \frac{r_{\mathrm{j}}}{r_{\mathrm{i}}} = \frac{k+1}{2} \quad \text{or} \quad \varepsilon = 0.5 \tag{2.116}$$

Substituting equation (2.116) into equations (2.2)–(2.5) obtains the residual stresses within the plastic region

$$\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = \frac{1}{2}\left[\frac{(k+1)^2}{4k^2} + \ln\frac{4x^2}{(k+1)^2} - \left(1 - \frac{(k+1)^2}{4k^2} + \ln\frac{(k+1)^2}{4}\right)\frac{1}{k^2-1}\right] \tag{2.117}$$

$$\frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = \frac{1}{2}\left[\frac{(k+1)^2}{4k^2} - 1 + \ln\frac{4x^2}{(k+1)^2} - \left(1 - \frac{(k+1)^2}{4k^2} + \ln\frac{(k+1)^2}{4}\right)\frac{1}{k^2-1}\left(1 - \frac{k^2}{x^2}\right)\right] \tag{2.118}$$

$$\frac{\sigma'_{\mathrm{\theta}}}{\sigma_{\mathrm{y}}} = \frac{1}{2}\left[\frac{(k+1)^2}{4k^2} + 1 + \ln\frac{4x^2}{(k+1)^2} - \left(1 - \frac{(k+1)^2}{4k^2} + \ln\frac{(k+1)^2}{4}\right)\frac{1}{k^2-1}\left(1 + \frac{k^2}{x^2}\right)\right] \tag{2.119}$$

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2 - \frac{(k+1)^2}{4} + k^2 \ln\frac{(k+1)^2}{4}}{k^2-1}\frac{1}{x^2} \tag{2.120}$$

Substituting equation (2.116) into equation (2.6) into equation (2.9) obtains the residual stresses within the elastic region

$$\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}}=\frac{\frac{(k+1)^2}{4}-1-\ln\frac{(k+1)^2}{4}}{2(k^2-1)}>0 \tag{2.121}$$

$$\frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}}=\left(1-\frac{k^2}{x^2}\right)\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}}<0 \tag{2.122}$$

$$\frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}}=\left(1+\frac{k^2}{x^2}\right)\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}}>0 \tag{2.123}$$

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}}=\frac{\frac{(k+1)^2}{4}-1-\ln\frac{(k+1)^2}{4}}{k^2-1}\frac{k^2}{x^2}>0 \tag{2.124}$$

When $k=2,\ k_{\mathrm{j}}=\frac{k+1}{2}=1.5$. The residual stresses within the whole wall are plotted in figure 2.18. The meaning of each curve and the main parameters are marked in the figure.

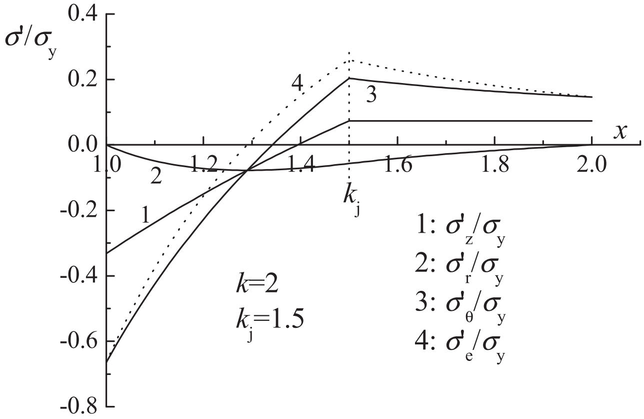

FIG. 2.18 – The residual stresses within the whole wall when $k=2$ and $k_{\mathrm{j}}=\frac{k+1}{2}$.

When $k=3,\ k_{\mathrm{j}}=\frac{k+1}{2}=2$. The residual stresses within the whole wall are plotted in figure 2.19. The meaning of each curve and the main parameters are marked in the figure.

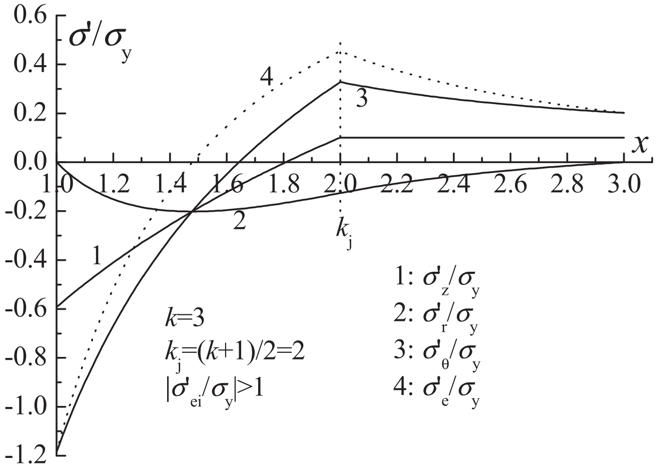

FIG. 2.19 – The residual stresses within the whole wall when $k = 3$ and $k_\mathrm{j} = \frac{k+1}{2}$.

At the inner surface ($x = 1$),

$$\frac{\sigma'_\mathrm{ei}}{\sigma_\mathrm{y}} = \frac{\sigma'_{\theta\mathrm{i}}}{\sigma_\mathrm{y}} = \frac{\frac{(k-1)(k+3)}{4} - k^2 \ln \frac{(k+1)^2}{4}}{k^2 - 1} \tag{2.125}$$

Letting $\sigma'_\mathrm{ei} \geq -\sigma_\mathrm{y}$ obtains

$$5k^2 + 2k - 7 - 4k^2 \ln \frac{(k+1)^2}{4} = c \geq 0 \tag{2.126}$$

The image of $c = 0$ is plotted in figure 2.20.

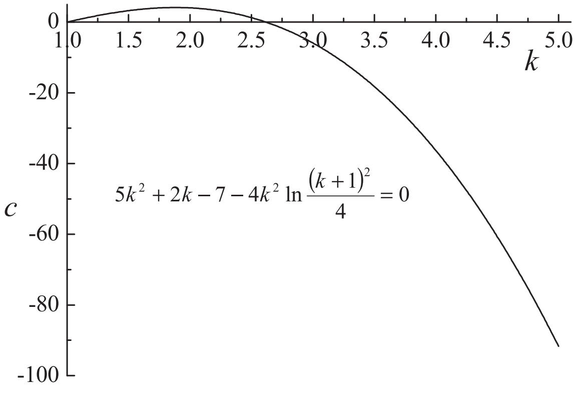

FIG. 2.20 – The image of $5k^2 + 2k - 7 - 4k^2 \ln \frac{(k+1)^2}{4} = 0$.

From figure 2.20, it is seen that when k is greater than a certain value, $c < 0$, or $\sigma'_{\mathrm{ei}}/\sigma_{\mathrm{y}} < -1$. Solving equation (2.126) obtains

$$k \leq 2.618253\ldots = k_{\mathrm{a}} \tag{2.127}$$

When $k \leq k_{\mathrm{a}}$, $|\sigma'_{\mathrm{ei}}| \leq \sigma_{\mathrm{y}}$. When $k = k_{\mathrm{a}}$, $k_{\mathrm{j}} = \frac{k+1}{2} = 1.809126\ldots$. The residual stresses within the whole wall are plotted in figure 2.21. The meaning of each curve and the main parameters are marked in the figure.

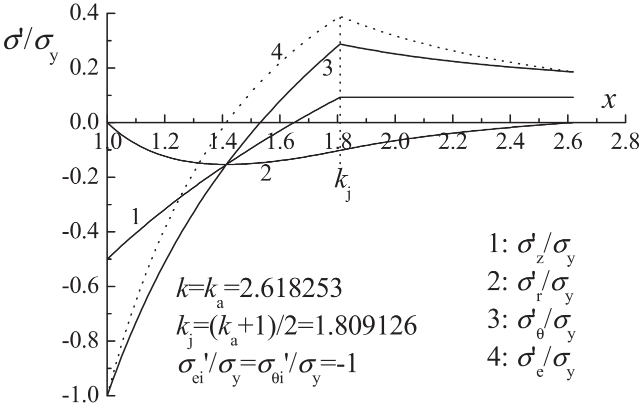

Fig. 2.21 – The residual stresses within the whole wall when $k = k_{\mathrm{a}}$ and $k_{\mathrm{j}} = \frac{k+1}{2}$.

When $k \geq k_{\mathrm{c}}$, equation (2.46) or $k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 = 0$ can guarantee a pressure vessel against reverse yield when processed with autofrettage. The graph of $k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 = 0$ is shown as a curve ad in figure 2.22. As mentioned above, when $k \leq k_{\mathrm{c}}$, k_{j} can be k. In figure 2.22, the straight line oa is for $k_{\mathrm{j}} = k$.

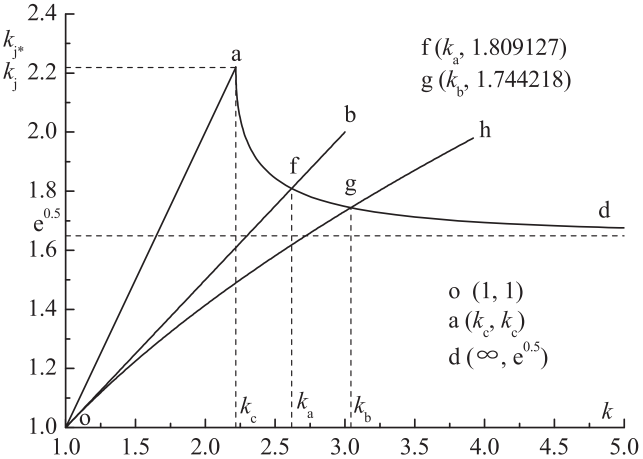

Fig. 2.22 – The relationship between k_{j} and k.

Figure 2.22 is applicable for both the maximum shear stress theory (Tresca yield criterion) and the maximum distortion strain energy theory (Mises yield criterion).

Substituting $k_{\mathrm{j}} = \frac{k+1}{2}$ into $k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 = 0$ just results in $5k^2 + 2k - 7 - 4k^2 \ln \frac{(k+1)^2}{4} = 0$, or equation (2.126). $k_{\mathrm{j}} = \frac{k+1}{2}$ is also plotted in figure 2.22 as straight-line ob. The coordinates of the intersection point of the straight line ob and the curve ad are $f(k_{\mathrm{a}}, 1.809127)$. In figure 2.22, if $k \geq k_{\mathrm{a}}$, such as in section fb, the pressure vessel will undergo reverse yield when processed with autofrettage.

From figure 2.22, it is known that when $k < k_{\mathrm{a}}$, the plastic depth $k_{\mathrm{j}} = \frac{k+1}{2}$ is smaller than $k_{\mathrm{j}*}$. However, this may be at the expense of reducing load-bearing capacity.

The equivalent stress of the total stresses

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2 - \frac{(k+1)^2}{4} + k^2 \ln \frac{(k+1)^2}{4}}{k^2 - 1} \frac{1}{x^2} + \frac{2k^2}{x^2(k^2 - 1)} \frac{p}{\sigma_{\mathrm{y}}} \tag{2.128}$$

Setting $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \leq 1$ gives

$$\frac{p}{\sigma_{\mathrm{y}}} \leq \frac{1}{2} - \frac{(k+1)^2}{8k^2} + \ln \frac{k+1}{2} \tag{2.129}$$

Accordingly, when $k_{\mathrm{j}} = \frac{k+1}{2}$, the maximum load-bearing capacity of a pressure vessel is

$$\frac{p}{\sigma_{\mathrm{y}}} = \frac{1}{2} - \frac{(k+1)^2}{8k^2} + \ln \frac{k+1}{2} \tag{2.130}$$

Effectively, substituting $k_{\mathrm{j}} = \frac{k+1}{2}$ into equation (2.1) also results in equation (2.130).

Letting $\frac{p}{\sigma_{\mathrm{y}}} = \frac{1}{2} - \frac{(k+1)^2}{8k^2} + \ln \frac{k+1}{2} \leq 1$ obtains

$$k \leq 3.102921\ldots = k_{\mathrm{z}} \tag{2.131}$$

When $k \leq 3.102921\ldots = k_{\mathrm{z}}$, $p \leq \sigma_{\mathrm{y}}$ and $\sigma_{\mathrm{ri}}^{\mathrm{T}} \geq -\sigma_{\mathrm{y}}$; When $k \geq k_{\mathrm{z}}$, $p \geq \sigma_{\mathrm{y}}$ and $\sigma_{\mathrm{ri}}^{\mathrm{T}} \leq -\sigma_{\mathrm{y}}$. When $k \leq k_{\mathrm{a}}$, there must be $k \leq k_{\mathrm{z}}$.

As mentioned above, when $k \leq k_{\mathrm{c}}$, $k_{\mathrm{j}} = k$, and the maximum load-bearing capacity of a pressure vessel is the entire yield load $\frac{p}{\sigma_{\mathrm{y}}} = \ln k$, that is plotted in figure 2.23 as curve oa; and when $k \geq k_{\mathrm{c}}$, k_{j} is determined by $k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 = 0$, and the maximum load-bearing capacity of a pressure vessel is $\frac{p}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{k^2}$, that is plotted in figure 2.23 as curve ad.

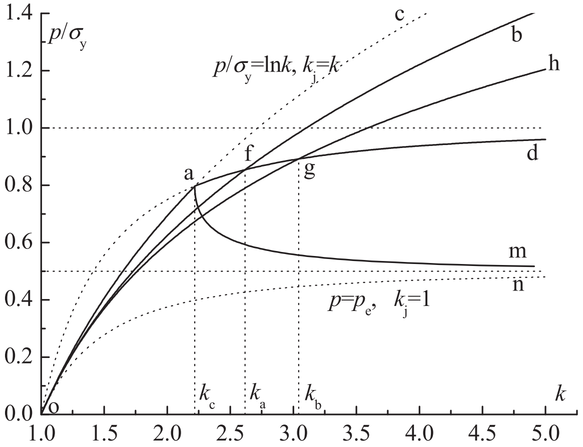

FIG. 2.23 – The load-bearing capacity under various conditions.

Letting $\frac{p}{\sigma_y} = \frac{1}{2} - \frac{(k+1)^2}{8k^2} + \ln\frac{k+1}{2} = \frac{k^2-1}{k^2}$ leads to $5k^2 + 2k - 7 - 4k^2 \ln\frac{(k+1)^2}{4} = 0$, or equation (2.126). $\frac{p}{\sigma_y} = \frac{1}{2} - \frac{(k+1)^2}{8k^2} + \ln\frac{k+1}{2}$ is also plotted in figure 2.23 as curve ob. The coordinates of the intersection point of the curve ob and the curve ad are f(k_a, 0.854126). In figure 2.23, if $k \geq k_a$, such as in the section fb, $\sigma_e^T > \sigma_y$.

For comparison, the maximum elastic load or initial yield load $\frac{p_e}{\sigma_y} = \frac{k^2-1}{2k^2}$ is also plotted in figure 2.23 as the dash curve. As k increases, point n and point m gradually coincide with one point.

$$\ln k - \left(\frac{1}{2} - \frac{(k+1)^2}{8k^2} + \ln\frac{k+1}{2}\right) = \ln\frac{2k}{k+1} - \frac{1}{2} + \frac{(k+1)^2}{8k^2} > 2\frac{\frac{2k}{k+1} - 1}{\frac{2k}{k+1} + 1}$$

$$- \frac{1}{2} + \frac{(k+1)^2}{8k^2} = \frac{2(k-1)^2(7k-1)}{8k^2(6k+2)} \geq 0$$

This means that the load-bearing capacity $\frac{p}{\sigma_y}$ when $k_j = \frac{k+1}{2}$ is smaller than ln$k$, as is seen in figure 2.23. This is because $k_{j*} > \frac{k+1}{2}$, or the plastic depth when $k_j = \frac{k+1}{2}$ is smaller than that when $k_j = k_{j*}$.

From figure 2.23, we can also see that when $k \leq k_a$, $\frac{1}{2} - \frac{(k+1)^2}{8k^2} + \ln\frac{k+1}{2} < \frac{k^2-1}{k^2}$; when $k \geq k_a$, $\frac{1}{2} - \frac{(k+1)^2}{8k^2} + \ln\frac{k+1}{2} > \frac{k^2-1}{k^2}$. However, when $k \geq k_a$, $|\sigma'_{ei}| > \sigma_y$ or reverse yield occurs due to $k_j = \frac{k+1}{2}$.

In addition, when $k_j = \frac{k+1}{2}$, autofrettage pressure $\frac{p_a}{\sigma_y} = \frac{1}{2} - \frac{(k+1)^2}{8k^2} + \ln\frac{k+1}{2} = \frac{p}{\sigma_y}$, which is just equation (2.130). This is because the general equation of autofrettage pressure $\frac{p_a}{\sigma_y} = \ln k_j + \frac{k^2 - k_j^2}{2k^2}$ is obtained just by setting $\frac{\sigma_e^T}{\sigma_y} = 1$ in a plastic region.

2.4 Mechanical Autofrettage Technology with Radius of Elastic-Plastic Juncture Being Geometrical Mean Radius of Inside Radius and Outside Radius

Taking the geometric mean value of the inner and outer radius as the plastic radius is a measure worthy of consideration for the design of autofrettaged pressure vessels. When the radius of the plastic region is a geometric mean value of the inner and outer radius of the pressure vessel, then

$$r_{\mathrm{j}} = \sqrt{r_{\mathrm{o}} r_{\mathrm{i}}} \quad \text{or} \quad k_{\mathrm{j}} = \sqrt{k} \quad \text{or} \quad \varepsilon = \frac{\sqrt{k}-1}{k-1} = \frac{1}{\sqrt{k}+1} \tag{2.132}$$

Substituting equation (2.132) into equations (2.2)–(2.5) obtains the residual stresses within the plastic region

$$\frac{\sigma_{\mathrm{z}}'}{\sigma_{\mathrm{y}}} = \frac{1}{2}\left(\ln\frac{x^2}{k} + \frac{k-1-\ln k}{k^2-1}\right) \tag{2.133}$$

$$\frac{\sigma_{\mathrm{r}}'}{\sigma_{\mathrm{y}}} = \frac{1}{2}\left[\ln\frac{x^2}{k} + \frac{1}{k} - 1 + \left(\frac{1}{k} - 1 - \ln k\right)\frac{1}{k^2-1}\left(1 - \frac{k^2}{x^2}\right)\right] \tag{2.134}$$

$$\frac{\sigma_{\theta}'}{\sigma_{\mathrm{y}}} = \frac{1}{2}\left[\ln\frac{x^2}{k} + \frac{1}{k} + 1 + \left(\frac{1}{k} - 1 - \ln k\right)\frac{1}{k^2-1}\left(1 + \frac{k^2}{x^2}\right)\right] \tag{2.135}$$

$$\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2 - k + k^2\ln k}{k^2-1}\frac{1}{x^2} \tag{2.136}$$

Substituting equation (2.132) into equations (2.6)–(2.9) obtains the residual stresses within the elastic region

$$\frac{\sigma_{\mathrm{z}}'}{\sigma_{\mathrm{y}}} = \frac{k-1-\ln k}{2(k^2-1)} = \text{constant} > 0 \tag{2.137}$$

$$\frac{\sigma_{\mathrm{r}}'}{\sigma_{\mathrm{y}}} = \left(1 - \frac{k^2}{x^2}\right)\frac{\sigma_{\mathrm{z}}'}{\sigma_{\mathrm{y}}} < 0 \tag{2.138}$$

$$\frac{\sigma_{\theta}'}{\sigma_{\mathrm{y}}} = \left(1 + \frac{k^2}{x^2}\right)\frac{\sigma_{\mathrm{z}}'}{\sigma_{\mathrm{y}}} > 0 \tag{2.139}$$

$$\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} = \frac{k-1-\ln k}{k^2-1}\frac{k^2}{x^2} > 0 \tag{2.140}$$

When $k = 3$, $k_j = \sqrt{k} = 1.732051\ldots$. The residual stresses within the whole wall are plotted in figure 2.24, where the meaning of each curve and the main parameters are marked.

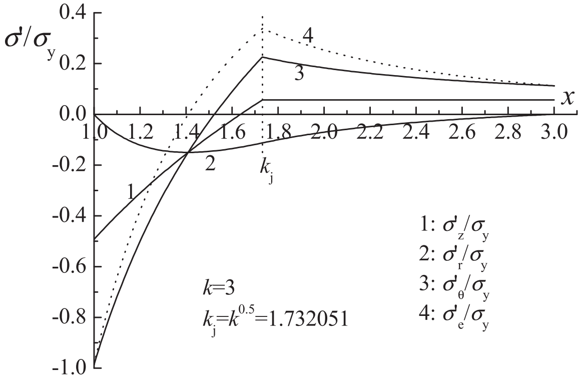

FIG. 2.24 – The residual stresses within the whole wall when $k = 3$ and $k_j = \sqrt{k}$.

When $k = 3.5$, $k_j = \sqrt{k} = 1.870829\ldots$. The residual stresses within the whole wall are plotted in figure 2.25, in which it is shown that $|\sigma'_{ei}| > \sigma_y$ when $k = 3.5$. The meaning of each curve and the main parameters are marked in the figure.

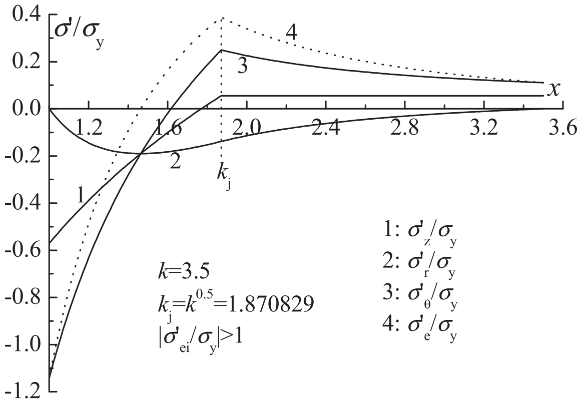

FIG. 2.25 – The residual stresses within the whole wall when $k = 3.5$ and $k_j = \sqrt{k}$.

At the inner surface ($x = 1$), from equation (2.136),

$$\frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\theta\mathrm{i}}}{\sigma_{\mathrm{y}}} = \frac{k - 1 - k^2 \ln k}{k^2 - 1} \tag{2.141}$$

Letting $\sigma'_{\mathrm{ei}} \geq -\sigma_{\mathrm{y}}$ obtains

$$k^2 + k - 2 - k^2 \ln k = d \geq 0 \tag{2.142}$$

The image of $d = 0$ is plotted in figure 2.26.

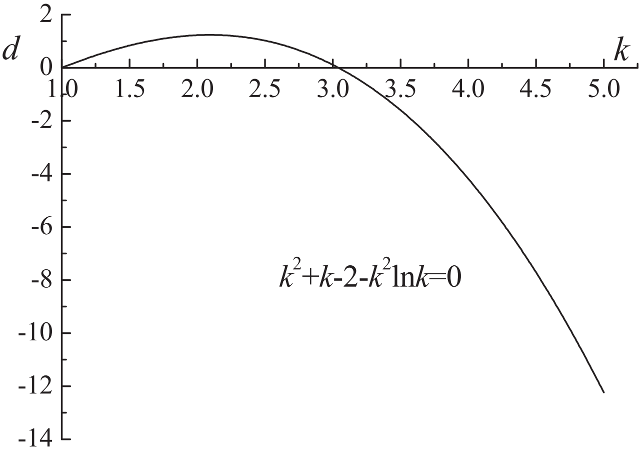

FIG. 2.26 – The image of $k^2 + k - 2 - k^2 \ln k = 0$.

From figure 2.26, it is seen that when k is greater than a certain value, $d < 0$. This indicates $\sigma'_{\mathrm{ei}}/\sigma_{\mathrm{y}} < -1$. Solving equation (2.142) obtains

$$k \leq 3.042297\ldots = k_{\mathrm{b}} \tag{2.143}$$

When $k \leq k_{\mathrm{b}}$, $|\sigma'_{\mathrm{ei}}/\sigma_{\mathrm{y}}| \leq 1$. When $k = k_{\mathrm{b}}$, $k_{\mathrm{j}} = \sqrt{k} = 1.744218\ldots$. The residual stresses within the whole wall are plotted in figure 2.27, where the meaning of each curve and the main parameters are marked.

Substituting $k_{\mathrm{j}} = \sqrt{k}$ into $k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 = 0$ just results in $k^2 + k - 2 - k^2 \ln k = 0$, or equation (2.142). $k_{\mathrm{j}} = \sqrt{k}$ has been plotted in figure 2.22 as curve oh. The coordinates of the intersection point of the curve oh and the curve ad are g (k_{b}, 1.744218). In figure 2.22, if $k \geq k_{\mathrm{b}}$, such as in section gh, the pressure vessel will undergo reverse yield when processed with autofrettage.

From figure 2.22, it is known that when $k < k_{\mathrm{b}}$, the plastic depth $k_{\mathrm{j}} = \sqrt{k}$ is smaller than $k_{\mathrm{j}*}$. Nevertheless, this may be at the expense of reducing load-bearing capacity.

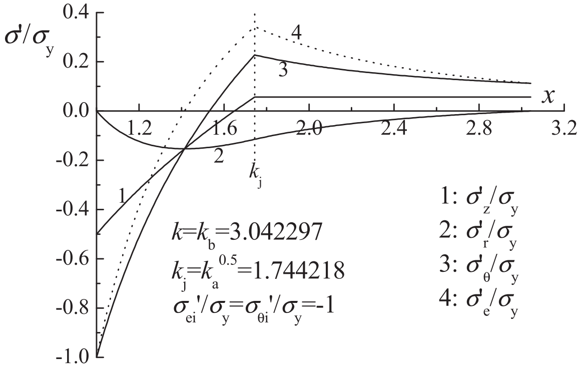

FIG. 2.27 – The residual stresses within the whole wall when $k = k_\mathrm{b}$ and $k_\mathrm{j} = \sqrt{k}$.

The equivalent stress of the total stresses

$$\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{\sigma_\mathrm{e}'}{\sigma_\mathrm{y}} + \frac{\sigma_\mathrm{e}^\mathrm{p}}{\sigma_\mathrm{y}} = 1 - \frac{k^2 - k + k^2 \ln k}{k^2 - 1}\frac{1}{x^2} + \frac{2k^2}{x^2(k^2-1)}\frac{p}{\sigma_\mathrm{y}} \tag{2.144}$$

Setting $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} \leq 1$ gives

$$\frac{p}{\sigma_\mathrm{y}} \leq \frac{k-1}{2k} + \ln\sqrt{k} \tag{2.145}$$

Accordingly, the maximum load-bearing capacity of a pressure vessel is $\frac{p}{\sigma_\mathrm{y}} = \frac{k-1}{2k} + \ln\sqrt{k}$ when $k_\mathrm{j} = \sqrt{k}$.

Effectively, substituting $k_\mathrm{j} = \sqrt{k}$ into equation (2.1) also results in equation (2.145).

Letting $\frac{p}{\sigma_\mathrm{y}} \leq \frac{k-1}{2k} + \ln\sqrt{k} \leq 1$ obtains

$$k \leq 3.591121\ldots = k_\mathrm{x} \tag{2.146}$$

$k_\mathrm{x} > k_\mathrm{b}$, therefore, when $k \leq k_\mathrm{b}$, there must be $k < k_\mathrm{x}$, thus $p \leq \sigma_\mathrm{y}$, accordingly $\frac{\sigma_\mathrm{ri}^\mathrm{T}}{\sigma_\mathrm{y}} \geq -1$.

Letting $\frac{k-1}{2k} + \ln\sqrt{k} = \frac{k^2-1}{k^2}$ leads to $k^2 + k - 2 - k^2 \ln k = 0$, or equation (2.142). $\frac{p}{\sigma_\mathrm{y}} = \frac{k-1}{2k} + \ln\sqrt{k}$ is also plotted in figure 2.23 as curve oh. The coordinates of the intersection point of the curve oh and the curve ad are g (k_b, 0.891957). In figure 2.23, if $k \geq k_\mathrm{b}$, such as in the section gh, $\sigma_\mathrm{e}^\mathrm{T} > \sigma_\mathrm{y}$.

$$
\begin{aligned}
\frac{1}{2}-\frac{(k+1)^2}{8k^2}+\ln\frac{k+1}{2}-\left(\frac{k-1}{2k}+\ln\sqrt{k}\right) &= \frac{1}{2}-\frac{(k+1)^2}{8k^2}-\frac{k-1}{2k}+\ln\frac{k+1}{2\sqrt{k}} \\
&> \frac{1}{2}-\frac{(k+1)^2}{8k^2}-\frac{k-1}{2k}+2\frac{\frac{k+1}{2\sqrt{k}}-1}{\frac{k+1}{2\sqrt{k}}+1} \\
&= \frac{(\sqrt{k}-1)^3(15k\sqrt{k}+11k+5\sqrt{k}+1)}{8k^2(k+1+2\sqrt{k})} \geq 0
\end{aligned}
$$

This means that the load-bearing capacity when $k_{\mathrm{j}}=\sqrt{k}$ is smaller than that when $k_{\mathrm{j}}=\frac{k+1}{2}$, as is seen in figure 2.23. This is because $\frac{k+1}{2}>\sqrt{k}$, or the plastic depth when $k_{\mathrm{j}}=\frac{k+1}{2}$ is larger than that when $k_{\mathrm{j}}=\sqrt{k}$.

From figure 2.23, we can also see that when $k \leq k_{\mathrm{b}}$, $\frac{k-1}{2k}+\ln\sqrt{k}<\frac{k^2-1}{k^2}$; when $k \geq k_{\mathrm{b}}$, $\frac{k-1}{2k}+\ln\sqrt{k}>\frac{k^2-1}{k^2}$. However, when $k \geq k_{\mathrm{b}}$, $|\sigma'_{\mathrm{ei}}|>\sigma_{\mathrm{y}}$ or reverse yield occur if $k_{\mathrm{j}}=\sqrt{k}$.

In addition, when $k_{\mathrm{j}}=\sqrt{k}$, there are conclusions that

(1) Autofrettage pressure $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{k-1}{2k}+\ln\sqrt{k}=\frac{p}{\sigma_{\mathrm{y}}}$, which is just equation (2.145). As mentioned above, this is because the general equation of autofrettage pressure $\frac{p_a}{\sigma_{\mathrm{y}}}=\ln k_{\mathrm{j}}+\frac{k^2-k_{\mathrm{j}}^2}{2k^2}$ is obtained just by setting $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=1$ in plastic region, and $\frac{p}{\sigma_{\mathrm{y}}}=\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$ also results in $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=1$;

(2) $x_0=\sqrt{\frac{k^2-k+k^2\ln k}{k^2-1}}<k$;

(3) $\frac{\sigma'_{\mathrm{zx0}}}{\sigma_{\mathrm{y}}}=\frac{\sigma'_{\theta\mathrm{x0}}}{\sigma_{\mathrm{y}}}=\frac{\sigma'_{\mathrm{rx0}}}{\sigma_{\mathrm{y}}}=\ln\sqrt{\frac{k-1+k\ln k}{k^2-1}}+\frac{k-1-\ln k}{2(k^2-1)}$;

(4) $x_1=\sqrt{\mathrm{e}^{\frac{1-k+k^2\ln k}{k^2-1}}}$.

2.5 Mechanical Autofrettage Technology with Minimum Equivalent Total Stress on Elastic-Plastic Juncture

On the elastic-plastic juncture, $x=r/r_{\mathrm{i}}=k_{\mathrm{j}}$, from equation (2.32) or equation (2.42), we obtain

$$
\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=1-\frac{1}{k^2-1}\left(\frac{k^2}{k_{\mathrm{j}}^2}-1+\frac{2k^2}{k_{\mathrm{j}}^2}\ln k_{\mathrm{j}}\right)+\frac{2k^2}{(k^2-1)k_{\mathrm{j}}^2}\frac{p}{\sigma_{\mathrm{y}}} \tag{2.147}
$$

Setting $\frac{\mathrm{d}\left(\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\right)}{\mathrm{d}k_{\mathrm{j}}}=0$ results in

$$k_{\mathrm{j}}=\frac{r_{\mathrm{j}}}{r_{\mathrm{i}}}=\exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right) \quad \text{or} \quad \varepsilon=\frac{\exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)-1}{k-1} \tag{2.148}$$

When $k_{\mathrm{j}}=\exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)$ and $1 \leq k_{\mathrm{j}} \leq k$, the equivalent total stress at the elastic-plastic juncture $\sigma_{\mathrm{ej}}^{\mathrm{T}}$ is the minimum, the minimum value is

$$\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{k^2}{k^2-1}\frac{\exp\left(2\frac{p}{\sigma_{\mathrm{y}}}\right)-1}{\exp\left(2\frac{p}{\sigma_{\mathrm{y}}}\right)}=\frac{k^2}{k^2-1}\frac{k_{\mathrm{j}}^2-1}{k_{\mathrm{j}}^2} \tag{2.149}$$

When $\frac{p}{\sigma_{\mathrm{y}}}=\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}=\frac{k^2-1}{2k^2}$,

$$k_{\mathrm{j}}=\frac{r_{\mathrm{j}}}{r_{\mathrm{i}}}=\exp\left(\frac{k^2-1}{2k^2}\right) \tag{2.150}$$

Letting $k_{\mathrm{j}}=\exp\left(\frac{k^2-1}{2k^2}\right) \leq k$ obtains

$$k^2 \ln k^2 \geq k^2-1 \tag{2.151}$$

Equation (2.151) is natural, because $\ln k^2 > 2\frac{k^2-1}{k^2+1}$. Therefore,

$$k^2 \ln k^2 > 2k^2\frac{k^2-1}{k^2+1}=\frac{2k^2}{k^2+1}\left(k^2-1\right) > k^2-1$$

Thus, when the load is initial yield pressure p_{e}, the plastic depth determined by equation (2.150) cannot exceed the radius ratio k.

When $\frac{p}{\sigma_{\mathrm{y}}}=\frac{k^2-1}{k^2}$,

$$k_{\mathrm{j}}=\frac{r_{\mathrm{j}}}{r_{\mathrm{i}}}=\exp\left(\frac{k^2-1}{k^2}\right) \tag{2.152}$$

Letting $k_{\mathrm{j}}=\exp\left(\frac{k^2-1}{k^2}\right) \leq k$ obtains

$$\frac{k^2 \ln k}{k^2-1} \geq 1 \tag{2.153}$$

$\frac{\mathrm{d}\left(\frac{k^2 \ln k}{k^2-1}\right)}{\mathrm{d}k}=\frac{k}{\left(k^2-1\right)^2}\left(k^2-1-\ln k^2\right) > 0$, when $k \rightarrow 1$, $\frac{k^2 \ln k}{k^2-1} \rightarrow \frac{1}{2}$, and solving $\frac{k^2 \ln k}{k^2-1}=1$ obtains

$$k=2.2184574899167\ldots=k_{\mathrm{c}} \tag{2.154}$$

Therefore, when $\frac{p}{\sigma_{\mathrm{y}}}=\frac{k^2-1}{k^2}$, the plastic depth determined by equation (2.152) cannot exceed radius ratio k if $k > k_{\mathrm{c}}$. Equation (2.152) is plotted in figure 2.28.

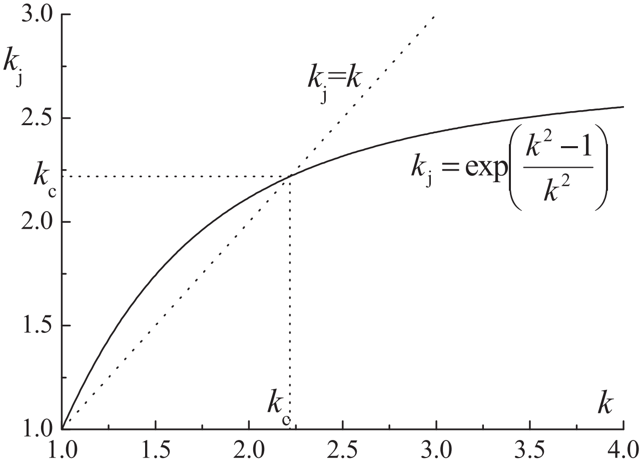

FIG. 2.28 – Comparison between k_j and k.

When $k < k_c$, for equation (2.148), setting $k_j = \exp\left(\frac{p}{\sigma_y}\right) \leq k$ leads to

$$\frac{p}{\sigma_y} \leq \ln k = \frac{p_y}{\sigma_y} \tag{2.155}$$

Setting $\frac{p}{\sigma_y} = \frac{k^2-1}{k^2} = \ln k$ just obtains equation (2.44) or equation (2.153), or $\frac{k^2 \ln k}{k^2-1} = 1$. This indicates that when $k > k_c$, $\ln k > \frac{k^2-1}{k^2}$, if $\frac{p}{\sigma_y} = \frac{k^2-1}{k^2}$, $k_j < k$, but the maximum load is unlikely $\frac{p}{\sigma_y} = \frac{k^2-1}{k^2}$; when $k < k_c$, $\ln k < \frac{k^2-1}{k^2}$, the maximum load is $\frac{p}{\sigma_y} = \ln k$, when $\frac{p}{\sigma_y} = \ln k$, $k_j = k$; when $k = k_c$, $\ln k = \frac{k^2-1}{k^2}$, the maximum load is $\frac{p}{\sigma_y} = \frac{k_c^2-1}{k_c^2} = \ln k_c$.

Substituting equation (2.148) into equations (2.2)–(2.5) obtains the residual stresses within the plastic region

$$\frac{\sigma_z'}{\sigma_y} = \frac{1}{2}\left[\ln x^2 + \frac{\exp\left(2\frac{p}{\sigma_y}\right) - 1}{k^2 - 1} - \frac{2k^2}{k^2 - 1}\frac{p}{\sigma_y}\right] \tag{2.156}$$

$$\frac{\sigma_r'}{\sigma_y} = \frac{1}{2}\left[\frac{\exp\left(2\frac{p}{\sigma_y}\right)}{k^2} - 1 + \ln x^2 - 2\frac{p}{\sigma_y} - \left(1 - \frac{\exp\left(2\frac{p}{\sigma_y}\right)}{k^2} + 2\frac{p}{\sigma_y}\right)\frac{1}{k^2 - 1}\left(1 - \frac{k^2}{x^2}\right)\right] \tag{2.157}$$

$$\frac{\sigma_\theta'}{\sigma_y} = \frac{1}{2}\left[\frac{\exp\left(2\frac{p}{\sigma_y}\right)}{k^2} + 1 + \ln x^2 - 2\frac{p}{\sigma_y} - \left(1 - \frac{\exp\left(2\frac{p}{\sigma_y}\right)}{k^2} + 2\frac{p}{\sigma_y}\right)\frac{1}{k^2 - 1}\left(1 + \frac{k^2}{x^2}\right)\right] \tag{2.158}$$

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2 - \exp\left(2\frac{p}{\sigma_{\mathrm{y}}}\right) + 2k^2\frac{p}{\sigma_{\mathrm{y}}}}{k^2 - 1}\frac{1}{x^2} \tag{2.159}$$

Substituting equation (2.148) into equations (2.6)–(2.9) obtains the residual stresses within the elastic region

$$\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = \frac{\exp\left(2\frac{p}{\sigma_{\mathrm{y}}}\right) - 1 - 2\frac{p}{\sigma_{\mathrm{y}}}}{2(k^2 - 1)} = \text{constant} > 0 \tag{2.160}$$

$$\frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = \left(1 - \frac{k^2}{x^2}\right)\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} < 0 \tag{2.161}$$

$$\frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}} = \left(1 + \frac{k^2}{x^2}\right)\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} > 0 \tag{2.162}$$

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{\exp\left(2\frac{p}{\sigma_{\mathrm{y}}}\right) - 1 - 2\frac{p}{\sigma_{\mathrm{y}}}}{k^2 - 1}\frac{k^2}{x^2} > 0 \tag{2.163}$$

When $k = 2$, $\frac{p}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{2k^2} = 0.375$, $k_{\mathrm{j}} = \exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right) = 1.454991\ldots$. The residual stresses within the whole wall are plotted in figure 2.29, where the meaning of each curve and the main parameters are marked.

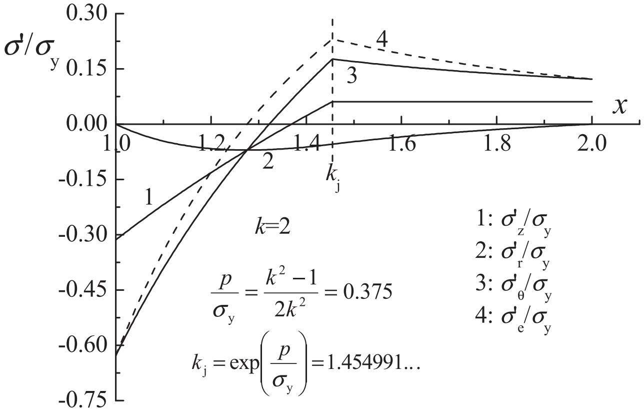

FIG. 2.29 – The residual stresses within the whole wall when $\frac{p}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{2k^2} = 0.375$.

When $k = 2$, $\frac{p}{\sigma_y} = \ln k = 0.693147...$, $k_j = k = 2$. The whole wall is plastic, and the residual stresses within the whole wall are plotted in figure 2.30, where the meaning of each curve and the main parameters are marked.

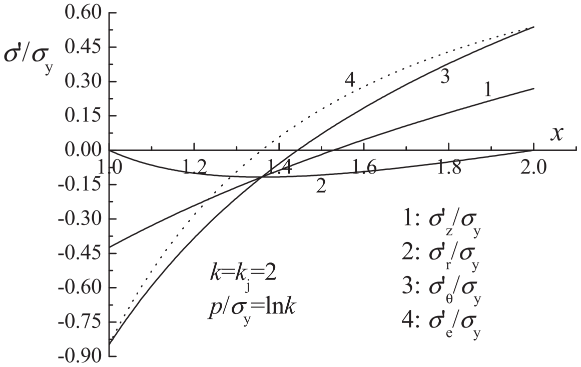

FIG. 2.30 – The residual stresses within the whole wall when $k_j = k = 2$.

When $k = 3$, $\frac{p}{\sigma_y} = \frac{k^2-1}{k^2} = \frac{8}{9}$, $k_j = \exp\left(\frac{p}{\sigma_y}\right) = 2.432425\ldots$. The residual stresses within the whole wall are plotted in figure 2.31, where the meaning of each curve and the main parameters are marked.

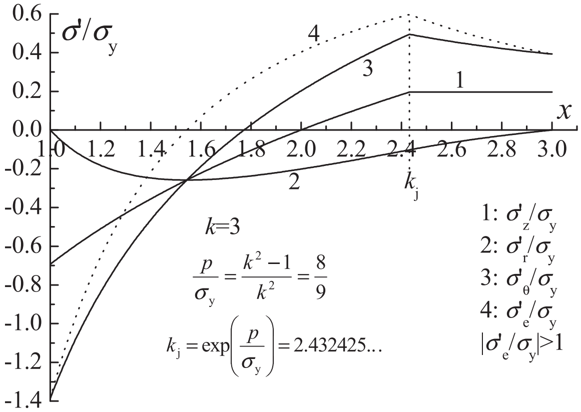

FIG. 2.31 – The residual stresses within the whole wall when $\frac{p}{\sigma_y} = \frac{k^2-1}{k^2} = \frac{8}{9}$.

When $k = 3$, $\frac{p}{\sigma_\mathrm{y}} = \frac{k^2-1}{k^2} = \frac{8}{9}$, $\frac{\sigma'_\mathrm{ei}}{\sigma_\mathrm{y}} < -1$. In light of $k_\mathrm{j} = \exp\left(\frac{p}{\sigma_\mathrm{y}}\right)$, we cannot directly determine whether the equivalent residual stress on the inner surface σ'_ei exceeds strength limit σ_y. When $k \geq k_\mathrm{c}$, in order to avoid reverse yield, for equation (2.159), setting $\frac{\sigma'_\mathrm{ei}}{\sigma_\mathrm{y}} \geq -1$ obtains

$$\exp\left(2\frac{p}{\sigma_\mathrm{y}}\right) + k^2 - 2k^2\frac{p}{\sigma_\mathrm{y}} - 2 \geq 0 \quad \text{or} \quad k^2 \leq \frac{\exp\left(2\frac{p}{\sigma_\mathrm{y}}\right) - 2}{2\frac{p}{\sigma_\mathrm{y}} - 1} \tag{2.164}$$

For a certain k, the load-bearing capacity $\frac{p}{\sigma_\mathrm{y}}$ when $k \geq k_\mathrm{c}$ should be determined by equation (2.164), as is shown by curve am in figure 2.23. For equation (2.164), the domain of k^2 is $\frac{p}{\sigma_\mathrm{y}} \geq \frac{1}{2}$ and $\frac{p}{\sigma_\mathrm{y}} \leq \ln\sqrt{2}$. The function $k^2 = \frac{\exp\left(2\frac{p}{\sigma_\mathrm{y}}\right)-2}{2\frac{p}{\sigma_\mathrm{y}}-1}$ is plotted in figure 2.32. There are two branches for the graphs of the function: curve og and curve bah. In addition, the function $k_\mathrm{j} = \exp\left(\frac{p}{\sigma_\mathrm{y}}\right)$ is plotted in figure 2.32 as the solid curve odai and the function $\frac{p}{\sigma_\mathrm{y}} = \frac{k^2-1}{k^2}$ is also plotted in figure 2.32 as the dash curve ofac. In figure 2.32, curve og corresponds to the domain of $\frac{p}{\sigma_\mathrm{y}} \leq \ln\sqrt{2}$, and $k < 1$ on curve og, thus it is of no practical significance; curve bah corresponds to the domain of $\frac{p}{\sigma_\mathrm{y}} \geq \frac{1}{2}$. For the function $k^2 = \frac{\exp\left(2\frac{p}{\sigma_\mathrm{y}}\right)-2}{2\frac{p}{\sigma_\mathrm{y}}-1}$, setting $\frac{\mathrm{d}k^2}{\mathrm{d}\left(2\frac{p}{\sigma_\mathrm{y}}\right)} = 0$ gives

$$\frac{p}{\sigma_\mathrm{y}}\exp\left(2\frac{p}{\sigma_\mathrm{y}}\right) - \exp\left(2\frac{p}{\sigma_\mathrm{y}}\right) + 1 = 0 \tag{2.165}$$

Solving equation (2.165) obtains the horizontal ordinate of the minimum value point a

$$\frac{p}{\sigma_\mathrm{y}} = 0.796812\ldots \tag{2.166}$$

Substituting $\frac{p}{\sigma_\mathrm{y}} = 0.796812\ldots$ into $k^2 = \frac{\exp\left(2\frac{p}{\sigma_\mathrm{y}}\right)-2}{2\frac{p}{\sigma_\mathrm{y}}-1}$ leads to $k = k_\mathrm{c}$.

Letting $k^2 = \frac{\exp\left(2\frac{p}{\sigma_\mathrm{y}}\right)-2}{2\frac{p}{\sigma_\mathrm{y}}-1} = k_\mathrm{j}^2 = \exp\left(2\frac{p}{\sigma_\mathrm{y}}\right)$ also obtains equation (2.165). When $k = k_\mathrm{j}$, substituting $k^2 = k_\mathrm{j}^2 = \exp\left(2\frac{p}{\sigma_\mathrm{y}}\right)$ into $\frac{p}{\sigma_\mathrm{y}} = \frac{k^2-1}{k^2}$ obtains equation (2.165) again. This shows that the three curves bah, ofac, and odai intersect at one point. This point is just the minimum value point (point a) of curve bah. From figure 2.32, when $\frac{p}{\sigma_\mathrm{y}} < 0.796812...$, $k > k_\mathrm{c}$, and k is larger than k_j determined by $k_\mathrm{j} = \exp\left(\frac{p}{\sigma_\mathrm{y}}\right)$, and $\frac{p}{\sigma_\mathrm{y}}$ is larger than the value determined by $\frac{p}{\sigma_\mathrm{y}} = \frac{k^2-1}{k^2}$. This is reasonable and makes sense. When $\frac{p}{\sigma_\mathrm{y}} > 0.796812...$, $k > k_\mathrm{c}$, and k is smaller than k_j determined by

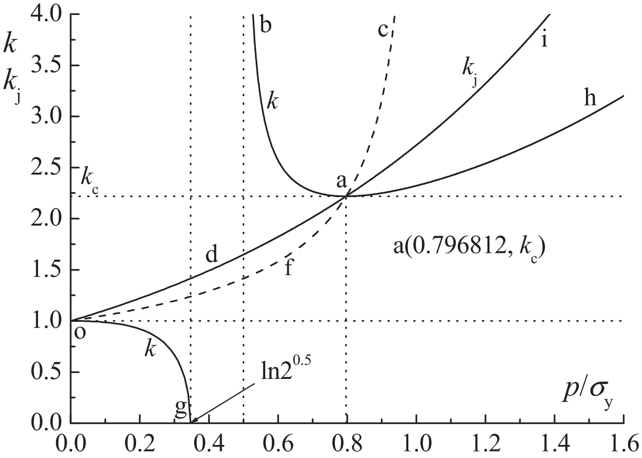

FIG. 2.32 – The relationship between k (k_j) and $\frac{p}{\sigma_\mathrm{y}}$.

$k_\mathrm{j} = \exp\left(\frac{p}{\sigma_\mathrm{y}}\right)$. This is unreasonable and makes no sense. Curve odac can ensure $k > k_\mathrm{j}$, but cannot ensure $\sigma'_\mathrm{ei} \geq -\sigma_\mathrm{y}$. Therefore, when $k > k_\mathrm{c}$, the meaningful control curve is curve ba, which is also curve am in figure 2.23. If the design conditions are controlled by curve ba when $k > k_\mathrm{c}$, there are both $k > k_\mathrm{j}$ and $\sigma'_\mathrm{ei} \geq -\sigma_\mathrm{y}$. For example, when $k = 3$, from equation (2.164), $\frac{p}{\sigma_\mathrm{y}} = 0.558725....$ When $k = 3$ and $\frac{p}{\sigma_\mathrm{y}} = 0.558725...$, $k_\mathrm{j} = \exp\left(\frac{p}{\sigma_\mathrm{y}}\right) = 1.748442\ldots = k_{\mathrm{j}*}$. The residual stresses within the whole wall are plotted in figure 2.33, where the meaning of each curve and the main parameters are marked.

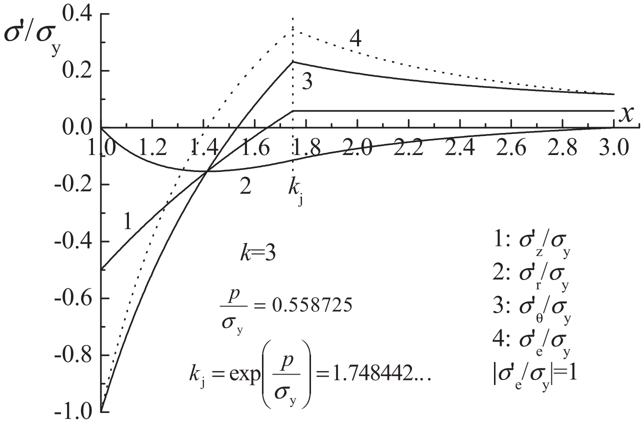

FIG. 2.33 – The residual stresses within the whole wall when $k = 3$ and $\frac{p}{\sigma_\mathrm{y}} = 0.558725$.

Compared with figure 2.31, the residual stresses are enormously improved.

In reality, when $k = k_{\mathrm{j}} = \exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)$, equation (2.165) and equation (2.164) are equivalent. When $k_{\mathrm{j}} = \exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)$ (but $k > k_{\mathrm{j}}$) equation (2.164) becomes equation (2.46), *i.e.*, $k^2 \ln k_{\mathrm{j}}^2 - k^2 - k_{\mathrm{j}}^2 + 2 \leq 0$ *i.e.* $k^2 \leq \frac{k_{\mathrm{j}}^2 - 2}{\ln k_{\mathrm{j}}^2 - 1}$.

Therefore, curve ad in figure 2.22 is also the relation between k and k_{j} when $k_{\mathrm{j}} = \exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)$ and $\sigma'_{\mathrm{ei}} = -\sigma_{\mathrm{y}}$. However, when the load-bearing capacity $\frac{p}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{k^2}$, there is $\sigma_{\mathrm{e}}^{\mathrm{T}} \equiv \sigma_{\mathrm{y}}$ within the whole plastic region; when the load-bearing capacity $\frac{p}{\sigma_{\mathrm{y}}} = \ln k_{\mathrm{j}}$ $\left(\text{equivalent to } k_{\mathrm{j}} = \exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)\right)$, where k_{j} is restrained by $k^2 \ln k_{\mathrm{j}}^2 - k^2 - k_{\mathrm{j}}^2 + 2 \leq 0$, $\sigma_{\mathrm{ej}}^{\mathrm{T}} < \sigma_{\mathrm{y}}$ at the elastic-plastic juncture.

Equation (2.164) can ensure $k_{\mathrm{j}} < k$ and $\sigma'_{\mathrm{ei}} \geq -\sigma_{\mathrm{y}}$, but under the premise of ensuring $k_{\mathrm{j}} < k$ and $\sigma'_{\mathrm{ei}} \geq -\sigma_{\mathrm{y}}$ as well as $\sigma_{\mathrm{ej}}^{\mathrm{T}}$ being the minimum, the load-bearing capacity $\frac{p}{\sigma_{\mathrm{y}}}$ calculated by $k_{\mathrm{j}} = \exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)$ may not be the most satisfactory. From equation (2.164), the greater k is, the smaller k_{j} is, thus the smaller $\frac{p}{\sigma_{\mathrm{y}}}$ is, because $k_{\mathrm{j}} = \exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)$. When $k = \infty$, $k_{\mathrm{j}} = \mathrm{e}^{0.5}$, hence $\frac{p}{\sigma_{\mathrm{y}}} = 0.5$, that is merely the maximum elastic load with $k = \infty$. Besides, when $k_{\mathrm{j}} = \exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)$, from figure 2.23, the load-bearing capacity under the case $k > k_{\mathrm{c}}$ is unexpectedly smaller than that under the case $k < k_{\mathrm{c}}$. So, if it is not necessary, $k_{\mathrm{j}} = \exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)$ is not a satisfactory solution.

When $\exp\left(2\frac{p}{\sigma_{\mathrm{y}}}\right) + k^2 - 2k^2\frac{p}{\sigma_{\mathrm{y}}} - 2 = 0$, which is equation (2.164), equation (2.159) become

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2 - \exp\left(2\frac{p}{\sigma_{\mathrm{y}}}\right) + 2k^2\frac{p}{\sigma_{\mathrm{y}}}}{k^2-1}\frac{1}{x^2} = 1 - \frac{2}{x^2} \tag{2.167}$$

The equivalent residual stress expressed by equation (2.167) is exactly that under the condition of $k^2 \ln k_{\mathrm{j}}^2 - k^2 - k_{\mathrm{j}}^2 + 2 \leq 0$.

When $k_{\mathrm{j}} = \exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)$, autofrettage pressure or equation (2.1) becomes

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{p}{\sigma_{\mathrm{y}}} + \frac{1}{2} - \frac{1}{2k^2}\exp\left(2\frac{p}{\sigma_{\mathrm{y}}}\right) \tag{2.168}$$

$\frac{\mathrm{d}(p_{\mathrm{a}}/\sigma_{\mathrm{y}})}{\mathrm{d}(p/\sigma_{\mathrm{y}})} = 1 - \frac{1}{k^2}\exp\left(2\frac{p}{\sigma_{\mathrm{y}}}\right)$. Setting $\frac{\mathrm{d}(p_{\mathrm{a}}/\sigma_{\mathrm{y}})}{\mathrm{d}(p/\sigma_{\mathrm{y}})} = 1 - \frac{1}{k^2}\exp\left(2\frac{p}{\sigma_{\mathrm{y}}}\right) = 0$ gives $\frac{p}{\sigma_{\mathrm{y}}} = \ln k$. $\frac{\mathrm{d}^2(p_{\mathrm{a}}/\sigma_{\mathrm{y}})}{\mathrm{d}(p/\sigma_{\mathrm{y}})^2} = -\frac{2}{k^2}\exp\left(2\frac{p}{\sigma_{\mathrm{y}}}\right) < 0$. Consequently, when $\frac{p}{\sigma_{\mathrm{y}}} = \ln k$, $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$ reaches its maximum value, which is $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{p}{\sigma_{\mathrm{y}}} = \ln k$. It should be noted that only if $k \leq k_{\mathrm{c}}$ can we have $\frac{p}{\sigma_{\mathrm{y}}} = \ln k$. For $k \geq k_{\mathrm{c}}$, from equation (2.164), we obtain $\frac{p}{\sigma_{\mathrm{y}}} - \frac{1}{2k^2}\exp\left(2\frac{p}{\sigma_{\mathrm{y}}}\right) = \frac{k^2-2}{2k^2}$. Substituting this equation into equation (2.168) gives

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{k^2-1}{k^2} \tag{2.169}$$

Equation (2.33) indicates that in general when the load-bearing capacity $\frac{p}{\sigma_{\mathrm{y}}}$ equals autofrettage pressure, or $\frac{p}{\sigma_{\mathrm{y}}}=\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$, it is the optimum, in this case $\sigma_{\mathrm{e}}^{\mathrm{T}}\equiv\sigma_{\mathrm{y}}$ in the whole plastic region. So, when $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{k^2-1}{k^2}$, the load-bearing capacity is also $\frac{p}{\sigma_{\mathrm{y}}}=\frac{k^2-1}{k^2}$. In this case *i.e.* $k_{\mathrm{j}}=\exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)$ and $k\geq k_{\mathrm{c}}$, when $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{k^2-1}{k^2}$, if $\frac{p}{\sigma_{\mathrm{y}}}$ is controlled by equation (2.164), $\frac{p}{\sigma_{\mathrm{y}}}<\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$, and $\sigma_{\mathrm{ej}}^{\mathrm{T}}<\sigma_{\mathrm{y}}$. For example, when $k=3$, from equation (2.164), $\frac{p}{\sigma_{\mathrm{y}}}=0.558724$, and from equation (2.169) $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{8}{9}$, from equation (2.149) $\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=0.756998$.

When $k\leq k_{\mathrm{c}}$, the optimum load-bearing capacity is also $\frac{p}{\sigma_{\mathrm{y}}}=\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\ln k$, and in this case $\sigma_{\mathrm{e}}^{\mathrm{T}}\equiv\sigma_{\mathrm{y}}$ in the whole plastic region.

When $k=\infty$, from equation (2.169), $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=1$; from equation (2.164), $\frac{p}{\sigma_{\mathrm{y}}}=0.5$; when $\frac{p}{\sigma_{\mathrm{y}}}=0.5$, from equation (2.168), we also have $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=1$. When $k=\infty$, equation (2.168) become

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{p}{\sigma_{\mathrm{y}}}+\frac{1}{2} \tag{2.170}$$

For equation (2.168), when $k\leq k_{\mathrm{c}}$, $\frac{p}{\sigma_{\mathrm{y}}}=0$ to $\ln k_{\mathrm{c}}$; when $k\geq k_{\mathrm{c}}$, $\frac{p}{\sigma_{\mathrm{y}}}=0$ to the value determined by equation (2.164). When $\frac{p}{\sigma_{\mathrm{y}}}=0$, $k_{\mathrm{j}}=\exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)=1$. The pressure vessel is at the initial yield stage, in which the pressure is the initial yield load $\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}=\frac{k^2-1}{2k^2}$. For a pressure vessel with a certain radius ratio $k\geq k_{\mathrm{c}}$, the maximum load for $\sigma_{\mathrm{ej}}^{\mathrm{T}}$ to reach the minimum is the value determined by equation (2.164). When $\frac{p}{\sigma_{\mathrm{y}}}$ equals the value determined by equation (2.164), $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=2\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}=\frac{k^2-1}{k^2}$. For the convenience of engineering applications, some data calculated by equations (2.164) and (2.168) are listed in table 2.1.

Equation (2.168) is plotted in figure 2.34. In figure 2.34, curve ab is the connection of the ends of the curves described by equation (2.168), which shows the maximum load-bearing capacity $\frac{p}{\sigma_{\mathrm{y}}}$ and the maximum autofrettage pressure $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$ for $k\geq k_{\mathrm{c}}$. From equation (2.164) we can find k^2 expressed by $\frac{p}{\sigma_{\mathrm{y}}}$, substituting k^2 into equation (2.169) we obtain $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$, as a result, the equation of curve ab in figure 2.34 is generated, that is

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{\exp\left(2\frac{p}{\sigma_{\mathrm{y}}}\right)-2\frac{p}{\sigma_{\mathrm{y}}}-1}{\exp\left(2\frac{p}{\sigma_{\mathrm{y}}}\right)-2} \tag{2.171}$$

Tab. 2.1 – Numerical value for $\frac{p}{\sigma_y}$ and $\frac{p_a}{\sigma_y}$.

k	$\frac{p}{\sigma_y}$	$\frac{p_a}{\sigma_y}$	k	$\frac{p}{\sigma_y}$	$\frac{p_a}{\sigma_y}$
2.3	0.67708	0.810964	3.3	0.544632	0.908173
2.4	0.636288	0.826389	3.4	0.541154	0.913495
2.5	0.612134	0.84	3.5	0.538103	0.918367
2.6	0.595387	0.852071	3.6	0.535407	0.92284
2.7	0.582876	0.862826	3.8	0.530865	0.930748
2.8	0.573101	0.872449	4	0.527194	$15/16 = 0.9375$
2.9	0.565222	0.881094	4.5	0.520552	0.950617
3	0.558724	$8/9 = 0.888889$	5	0.51615	$24/25 = 0.96$
3.1	0.553272	0.895942	10	0.503692	0.99
3.2	0.54863	0.902344	∞	0.5	1

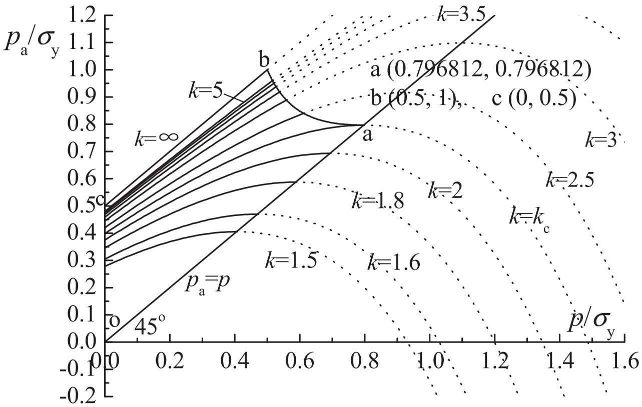

Fig. 2.34 – The graph of $\frac{p_a}{\sigma_y} = \frac{p}{\sigma_y} + \frac{1}{2} - \frac{1}{2k^2}\exp\left(2\frac{p}{\sigma_y}\right)$.

The coordinates of point a is a(ln k_c, ln k_c). As mentioned above, the meaningful domain of k^2 is $\frac{p}{\sigma_y} \geq \frac{1}{2}$, and when $\frac{p}{\sigma_y} = 0.5$, $k^2 = \infty$, consequently from equation (2.168) or equation (2.169), $\frac{p_a}{\sigma_y} = 1$, or the coordinates of point b is b(0.5, 1). The coordinates of point c is c(0, 0.5). The practically significant curves for relation between $\frac{p}{\sigma_y}$ and $\frac{p_a}{\sigma_y}$ are located in a quadrilateral abco, where the straight line bc is parallel to the straight line ao.

For $k \leq k_c$, the ends of the curves described by equation (2.168) are on the straight line ao where $\frac{p_a}{\sigma_y} = \frac{p}{\sigma_y}$. Thereupon the included angle between the straight line ao and the abscissa axis or vertical axis is 45°.

In figure 2.34, the dash curves have no practical significance, they are an extension of solid curves or only the elongation of the mathematical relation expressed by equation (2.168).

The total stresses are expressed as follows

$$\begin{cases} \dfrac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \dfrac{\sigma_{\mathrm{z}}'}{\sigma_{\mathrm{y}}} + \dfrac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} \\ \dfrac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \dfrac{\sigma_{\mathrm{r}}'}{\sigma_{\mathrm{y}}} + \dfrac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} \\ \dfrac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \dfrac{\sigma_{\theta}'}{\sigma_{\mathrm{y}}} + \dfrac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} \\ \dfrac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \dfrac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} + \dfrac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} \end{cases} \tag{2.172}$$

If the pressure contained in a pressure vessel does not exceed the initial yield pressure, or $p \leq p_{\mathrm{e}}$, all stresses will not exceed the yield strength σ_{y}, then autofrettage is not necessary. However, if autofrettage is performed, the stresses will be evened and reduced. Between 1 and k, if $k_{\mathrm{j}} = \exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)$ (when $p = p_{\mathrm{e}}$, $k_{\mathrm{j}} = \exp\left(\frac{k^2-1}{2k^2}\right)$), the equivalent total stress at the elastic-plastic juncture $\sigma_{\mathrm{ej}}^{\mathrm{T}}$ is the minimum, the minimum value is equation (2.149).

When $p = p_{\mathrm{e}}$, $k_{\mathrm{j}} = \exp\left(\frac{k^2-1}{2k^2}\right)$, then, equation (2.149) becomes

$$\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k^2}{k^2-1}\,\frac{\exp\left(2\frac{p}{\sigma_{\mathrm{y}}}\right)-1}{\exp\left(2\frac{p}{\sigma_{\mathrm{y}}}\right)} = \frac{k^2}{k^2-1}\,\frac{\exp\left(\frac{k^2-1}{k^2}\right)-1}{\exp\left(\frac{k^2-1}{k^2}\right)} \tag{2.173}$$

Several figures are presented to show the above-mentioned conclusions.

For $k = 3$, $\frac{p}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{2k^2} = 0.444444$, $k_{\mathrm{j}} = \exp\left(\frac{k^2-1}{2k^2}\right) = 1.559623$, the operation stresses, residual stresses, and total stresses are shown in figures 2.35–2.37, respectively.

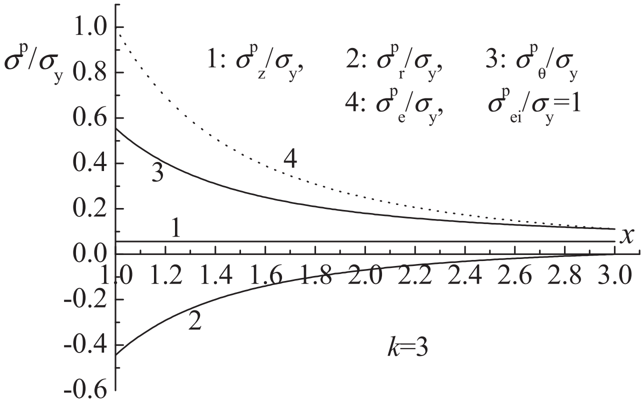

FIG. 2.35 – The distribution of the operation stresses for $k = 3$, $p = p_{\mathrm{e}}$ and $k_{\mathrm{j}} = \exp\left(\frac{k^2-1}{2k^2}\right)$.

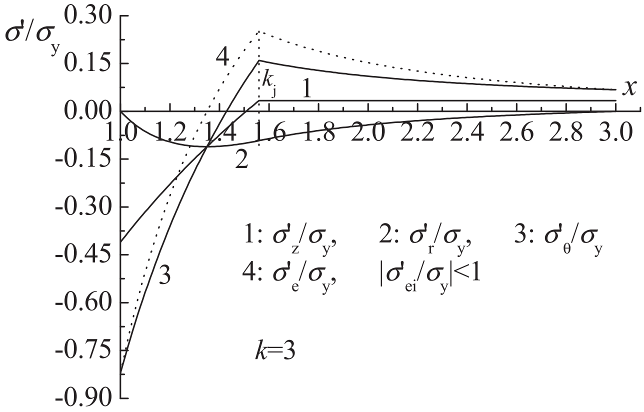

FIG. 2.36 – The distribution of the residual stresses for $k = 3$, $p = p_{\mathrm{e}}$, and $k_{\mathrm{j}} = \exp\left(\frac{k^2-1}{2k^2}\right)$.

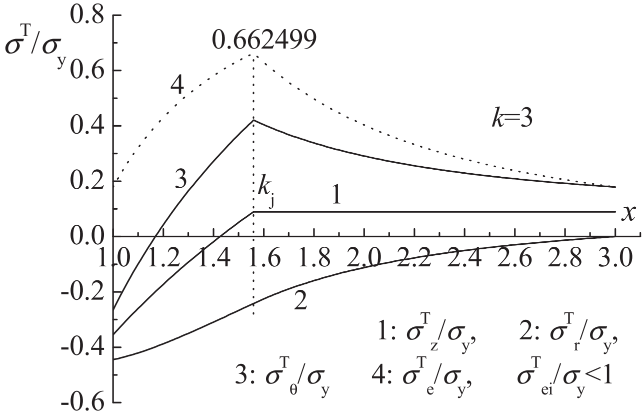

FIG. 2.37 – The distribution of the total stresses for $k = 3$, $p = p_{\mathrm{e}}$, and $k_{\mathrm{j}} = \exp\left(\frac{k^2-1}{2k^2}\right)$.

In figure 2.37, $\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 0.662499$. This must be the minimum.

For $k = 3$, $\frac{p}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{2k^2} = \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = 0.444444$, $k_{\mathrm{j}} = 1.4 < \exp\left(\frac{k^2-1}{2k^2}\right)$, the total stresses are shown in figure 2.38.

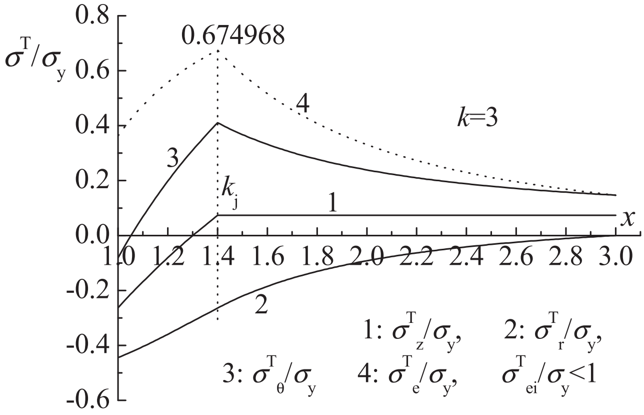

FIG. 2.38 – The distribution of the total stresses for $k = 3$, $p = p_{\mathrm{e}}$, and $k_{\mathrm{j}} = 1.4$.

In figure 2.38, $\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 0.674968 > 0.662499$.

For $k = 3$, $\frac{p}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{2k^2} = \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = 0.444444$, $\exp\left(\frac{k^2-1}{2k^2}\right) < k_{\mathrm{j}} = 1.7 < k_{\mathrm{j}*} = 1.748442$, the total stresses are shown in figure 2.39.

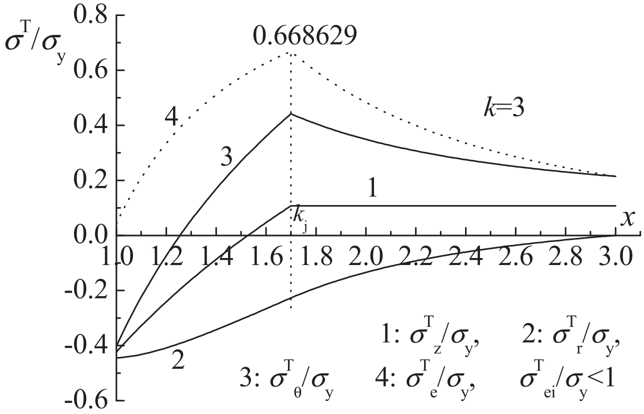

FIG. 2.39 – The distribution of the total stresses for $k = 3$, $p = p_{\mathrm{e}}$, and $k_{\mathrm{j}} = 1.7$.

In figure 2.39, $\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 0.668629 > 0.662499$.

For $k = 3$, from equation (2.164) or table 2.1, $\frac{p}{\sigma_{\mathrm{y}}} = 0.558724 > \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$, $k_{\mathrm{j}} = \exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right) = k_{\mathrm{j}*} = 1.748442$, the total stresses are shown in figure 2.40.

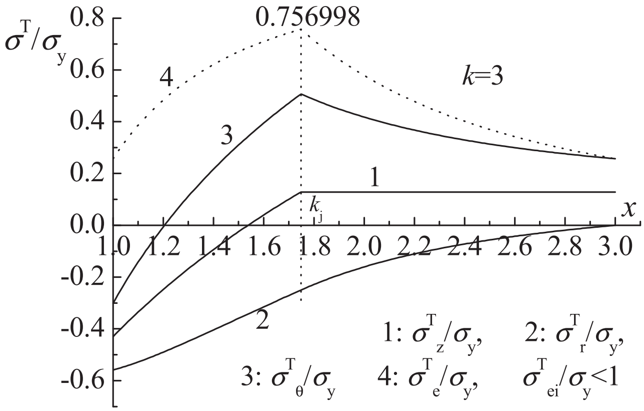

FIG. 2.40 – The distribution of the total stresses for $k = 3$, $p = 0.558724$ and $k_j = k_{j^*}$.

In figure 2.40, $\frac{\sigma^T_{ej}}{\sigma_y} = 0.756998$.

For $k = 3$, from equation (2.164) or table 2.1, $\frac{p}{\sigma_y} = 0.558724 > \frac{p_e}{\sigma_y}$, $k_j = 1.6 < k_{j^*}$, the total stresses are shown in figure 2.41.

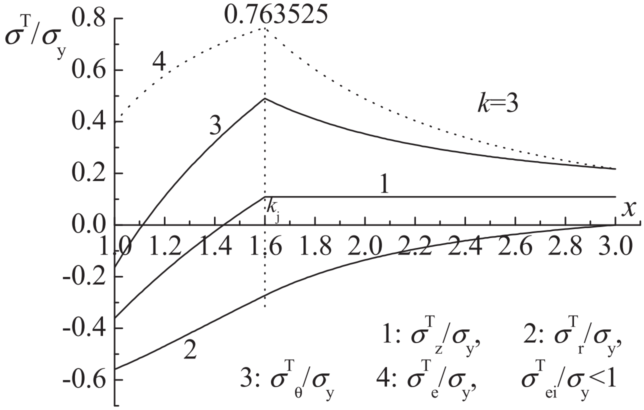

FIG. 2.41 – The distribution of the total stresses for $k = 3$, $p = 0.558724$, and $k_j = 1.6$.

In figure 2.41, $\frac{\sigma^T_{ej}}{\sigma_y} = 0.763525 > 0.756998$.

For $k = 3$, from equation (2.164) or table 2.1, $\frac{p}{\sigma_y} = 0.558724 > \frac{p_e}{\sigma_y}$, $k_j = 1.8 > k_{j^*}$, the total stresses are shown in figure 2.42.

In figure 2.42, $\frac{\sigma^T_{ej}}{\sigma_y} = 0.757596 > 0.756998$.

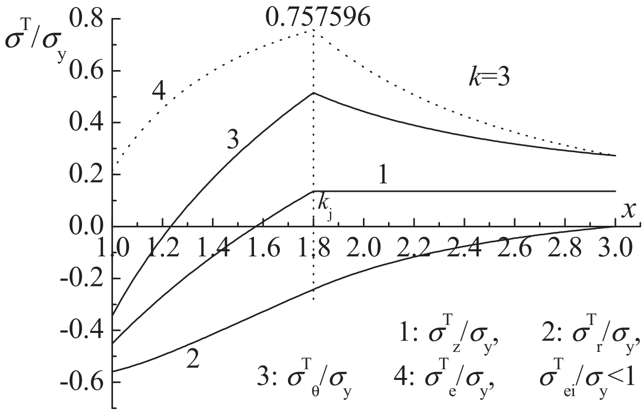

FIG. 2.42 – The distribution of the total stresses for $k = 3$, $p = 0.558724$ and $k_j = 1.8$.

2.6 Comparison Between Three Cases

So far, we have investigated autofrettage under four cases: $k_j = k$, $k_j = \frac{k+1}{2}$, $k_j = \sqrt{k}$ and $k_j = \exp\left(\frac{p}{\sigma_y}\right)$. Some of the results obtained here are similar to those in ref.[4–6]. $k_j = k$ is for $k < k_c$, and when $k < k_c$, $\frac{p}{\sigma_y} = \ln k$, that is equivalent to $k = \exp\left(\frac{p}{\sigma_y}\right)$. Therefore, when $k < k_c$, $k_j = k$ is equivalent to $k_j = \exp\left(\frac{p}{\sigma_y}\right)$. Then, we do a comparison between the three cases: $k_j = \frac{k+1}{2}$, $k_j = \sqrt{k}$ and $k_j = \exp\left(\frac{p}{\sigma_y}\right)$.

I. For $k = 2.5 > k_c$

1. $k_j = \frac{k+1}{2}$

$$k_j = \frac{k+1}{2} \triangleq k_{j1} = 1.75$$

The equivalent residual stress

From equation (2.120), in the plastic region,

$$\frac{\sigma'_e}{\sigma_y} = 1 - \frac{k^2 - \frac{(k+1)^2}{4} + k^2 \ln \frac{(k+1)^2}{4}}{k^2 - 1} \frac{1}{x^2} = 1 - \frac{1.939561}{x^2}$$

In the elastic region, from equation (2.88),

$$\frac{\sigma'_e}{\sigma_y} = \frac{\frac{(k+1)^2}{4} - 1 - \ln \frac{(k+1)^2}{4}}{k^2 - 1} \frac{k^2}{x^2} = \frac{1.122939}{x^2}$$

The load-bearing capacity, from equation (2.130),

$$\frac{p}{\sigma_{\mathrm{y}}}=\frac{1}{2}-\frac{(k+1)^2}{8k^2}+\ln\frac{k+1}{2}=0.814616$$

The equivalent operation stress, from equation (1.9),

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}=\frac{2k^2}{x^2(k^2-1)}\frac{p}{\sigma_{\mathrm{y}}}=\frac{1.939561}{x^2}$$

The equivalent total stress, from equation (2.15),

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}}+\frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}$$

Then, in the plastic region,

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\equiv 1$$

In the elastic region,

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{3.0625}{x^2}$$

The autofrettage pressure, from equation (2.130),

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{p}{\sigma_{\mathrm{y}}}$$

2. $k_{\mathrm{j}}=\sqrt{k}$

$$k_{\mathrm{j}}=\sqrt{k}\triangleq k_{\mathrm{j}2}=1.581139$$

The equivalent residual stress
From equation (2.136), in the plastic region,

$$\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}}=1-\frac{k^2-k+k^2\ln k}{k^2-1}\frac{1}{x^2}=1-\frac{1.805108}{x^2}$$

From equation (2.140), in the elastic region,

$$\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}}=\frac{k-1-\ln k}{k^2-1}\frac{k^2}{x^2}=\frac{0.694892}{x^2}$$

The load-bearing capacity, from equation (2.145),

$$\frac{p}{\sigma_{\mathrm{y}}} = \frac{k-1}{2k} + \ln\sqrt{k} = 0.758145$$

The equivalent operation stress, from equation (1.9),

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{2k^2}{x^2(k^2-1)}\frac{p}{\sigma_{\mathrm{y}}} = \frac{1.805108}{x^2}$$

The equivalent total stress, from equation (2.28),

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}$$

Thus, in the plastic region,

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \equiv 1$$

In the elastic region,

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{2.5}{x^2}$$

The autofrettage pressure

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{p}{\sigma_{\mathrm{y}}}$$

3. $k_{\mathrm{j}} = \exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)$

When $k = 2.5$, from equation (2.164) or table 2.1, the load-bearing capacity $\frac{p}{\sigma_{\mathrm{y}}} = 0.612134$. When $\frac{p}{\sigma_{\mathrm{y}}} = 0.612134$, from equation (2.148), $k_{\mathrm{j}} = \exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right) \triangleq k_{\mathrm{j}3} = 1.844363$.

The equivalent residual stress

From equation (2.159), in the plastic region,

$$\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} = 1 - \frac{2}{x^2}$$

From equation (2.163), in the elastic region,

$$\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} = \frac{k-1-\ln k}{k^2-1}\frac{k^2}{x^2} = \frac{1.401675}{x^2}$$

The equivalent operation stress, from equation (1.9),

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}=\frac{2k^2}{x^2(k^2-1)}\frac{p}{\sigma_{\mathrm{y}}}=\frac{1.805108}{x^2}$$

The equivalent total stress, from equation (2.15),

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}}+\frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}$$

Then, in the plastic region,

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=1-\frac{0.194892}{x^2}$$

In the elastic region,

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{3.206783}{x^2}$$

The autofrettage pressure, from equation (2.169),

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{k^2-1}{k^2}=0.84$$

When $k=2.5$, the equivalent residual stress and the equivalent total stress are plotted in figures 2.43 and 2.44, respectively, where curve beh is the equivalent residual stress for $k_{\mathrm{j}}=\frac{k+1}{2}$, curve cdl is the equivalent residual stress for $k_{\mathrm{j}}=\sqrt{k}$, curve afg is the equivalent residual stress for $k_{\mathrm{j}}=\exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)$; curve nqv is the equivalent total stress for $k_{\mathrm{j}}=\frac{k+1}{2}$, curve nzw is the equivalent total stress for $k_{\mathrm{j}}=\sqrt{k}$, curve mst is the equivalent total stress for $k_{\mathrm{j}}=\exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)$. The main parameters are marked in these two figures. Using these figures, we can compare the value and shape of the stress images.

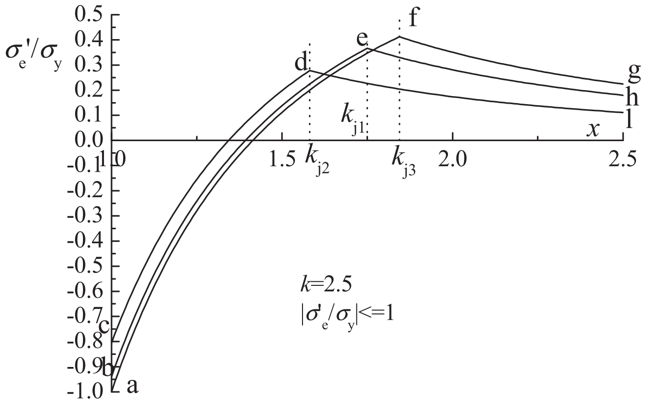

FIG. 2.43 – Comparison of the equivalent residual stress when $k=2.5$.

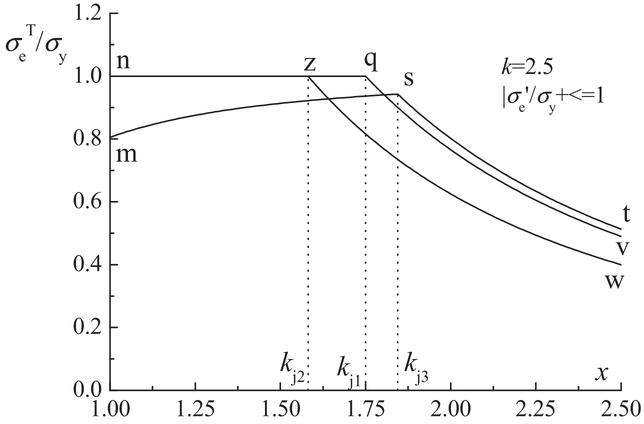

FIG. 2.44 – Comparison of the equivalent total stress when $k = 2.5$.

II. For $k = 2 < k_c$

1. $k_j = \frac{k+1}{2}$

$$k_j = \frac{k+1}{2} \triangleq k_{j1} = 1.5$$

The equivalent residual stress

From equation (2.120), in the plastic region,

$$\frac{\sigma_e'}{\sigma_y} = 1 - \frac{k^2 - \frac{(k+1)^2}{4} + k^2 \ln \frac{(k+1)^2}{4}}{k^2 - 1} \frac{1}{x^2} = 1 - 1.664574 \frac{1}{x^2}$$

From equation (2.88), in the elastic region

$$\frac{\sigma_e'}{\sigma_y} = \frac{\frac{(k+1)^2}{4} - 1 - \ln \frac{(k+1)^2}{4}}{k^2 - 1} \frac{k^2}{x^2} = \frac{0.585426}{x^2}$$

The load-bearing capacity, from equation (2.130),

$$\frac{p}{\sigma_y} = \frac{1}{2} - \frac{(k+1)^2}{8k^2} + \ln \frac{k+1}{2} = 0.624215$$

The equivalent operation stress, from equation (1.9),

$$\frac{\sigma_e^p}{\sigma_y} = \frac{2k^2}{x^2(k^2 - 1)} \frac{p}{\sigma_y} = \frac{1.664574}{x^2}$$

The equivalent total stress, from equation (2.15),

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}$$

Thus, in the plastic region,

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \equiv 1$$

In the elastic region,

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{2.25}{x^2}$$

The autofrettage pressure

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{p}{\sigma_{\mathrm{y}}}$$

2. $k_{\mathrm{j}} = \sqrt{k}$

$$k_{\mathrm{j}} = \sqrt{k} \triangleq k_{\mathrm{j}2} = 1.414214$$

The equivalent residual stress
From equation (2.136), in the plastic region,

$$\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2 - k + k^2 \ln k}{k^2 - 1}\frac{1}{x^2} = 1 - \frac{1.590863}{x^2}$$

From equation (2.140), in the elastic region,

$$\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} = \frac{k - 1 - \ln k}{k^2 - 1}\frac{k^2}{x^2} = \frac{0.409137}{x^2}$$

The load-bearing capacity, from equation (2.145),

$$\frac{p}{\sigma_{\mathrm{y}}} = \frac{k-1}{2k} + \ln\sqrt{k} = 0.596574$$

The equivalent operation stress, from equation (1.9),

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{2k^2}{x^2(k^2-1)}\frac{p}{\sigma_{\mathrm{y}}} = \frac{1.590863}{x^2}$$

The equivalent total stress, from equation (2.15),

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}$$

Then, in the plastic region,

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \equiv 1$$

In the elastic region,

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{2}{x^2}$$

The autofrettage pressure

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{p}{\sigma_{\mathrm{y}}}$$

3. $k_{\mathrm{j}} = \exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)$ and $k_{\mathrm{j}} = k$

When $k = 2 < k_{\mathrm{c}}$, $k_{\mathrm{j}} = k$, the load-bearing capacity $\frac{p}{\sigma_{\mathrm{y}}} = \ln k = 0.693147$, which is greater than the load-bearing capacity when $k = 2.5$ and $k_{\mathrm{j}} = \exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)$. When $\frac{p}{\sigma_{\mathrm{y}}} = \ln k$, $k_{\mathrm{j}3} = \exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right) = k$. There is no elastic region when $k_{\mathrm{j}} = k$.

The equivalent residual stress

From equation (2.93),

$$\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2 \ln k^2}{(k^2 - 1)x^2} = 1 - \frac{1.848392}{x^2}$$

The equivalent operation stress, from equation (1.9),

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{1.848392}{x^2}$$

The equivalent total stress

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \equiv 1$$

The autofrettage pressure

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{p}{\sigma_{\mathrm{y}}} = 0.693147$$

When $k = 2$, the equivalent residual stress and the equivalent total stress are plotted in figures 2.45 and 2.46, respectively, where curve beh is the equivalent residual stress for $k_{\mathrm{j}} = \frac{k+1}{2}$, curve cdg is the equivalent residual stress for $k_{\mathrm{j}} = \sqrt{k}$, curve af is the equivalent residual stress for $k_{\mathrm{j}} = \exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)$ or $k_{\mathrm{j}} = k$; curve nqv is the equivalent total stress for $k_j = \frac{k+1}{2}$, curve nzw is the equivalent total stress for $k_j = \sqrt{k}$, curve nm is the equivalent total stress for $k_j = \exp\left(\frac{p}{\sigma_y}\right)$ or $k_j = k$. The main parameters are marked in these two figures. Using these figures, we can compare the value and shape of the stress images.

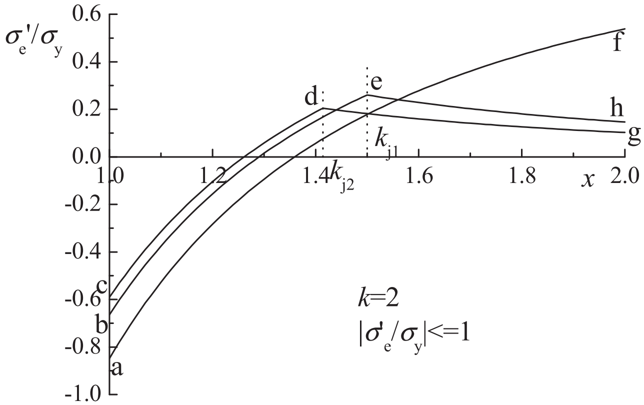

FIG. 2.45 – Comparison of the equivalent residual stress when $k = 2$.

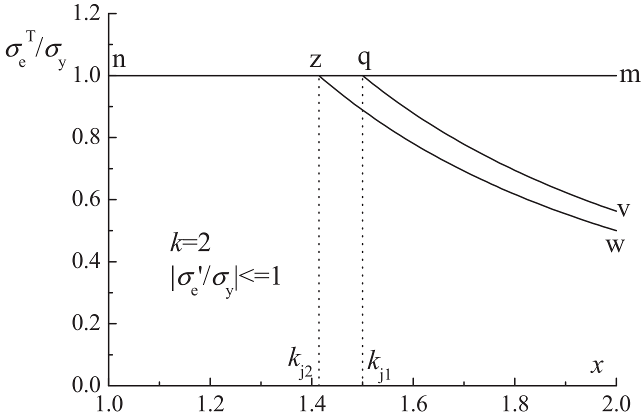

FIG. 2.46 – Comparison of the equivalent total stress when $k = 2$.

2.7 Chapter Summary

The main equations and conclusion are listed in table 2.2.

TAB. 2.2 – The main equations and conclusion of this chapter.

The components of residual stresses in general forms

Within the plastic region ($1 \le x \le k_j$)

$$\frac{\sigma'_z}{\sigma_y} = \frac{1}{2}\left[\frac{k_j^2}{k^2} + \ln\frac{x^2}{k_j^2} - \left(1 - \frac{k_j^2}{k^2} + \ln k_j^2\right)\frac{1}{k^2-1}\right]$$

$$\frac{\sigma'_r}{\sigma_y} = \frac{1}{2}\left[\frac{k_j^2}{k^2} - 1 + \ln\frac{x^2}{k_j^2} - \left(1 - \frac{k_j^2}{k^2} + \ln k_j^2\right)\frac{1}{k^2-1}\left(1 - \frac{k^2}{x^2}\right)\right]$$

$$\frac{\sigma'_\theta}{\sigma_y} = \frac{1}{2}\left[\frac{k_j^2}{k^2} + 1 + \ln\frac{x^2}{k_j^2} - \left(1 - \frac{k_j^2}{k^2} + \ln k_j^2\right)\frac{1}{k^2-1}\left(1 + \frac{k^2}{x^2}\right)\right]$$

Within the elastic region ($k_j \le x \le k$)

$$\frac{\sigma'_z}{\sigma_y} = \frac{k_j^2 - 1 - \ln k_j^2}{2(k^2-1)} = \text{constant} > 0$$

$$\frac{\sigma'_r}{\sigma_y} = \left(1 - \frac{k^2}{x^2}\right)\frac{\sigma'_z}{\sigma_y} < 0$$

$$\frac{\sigma'_\theta}{\sigma_y} = \left(1 + \frac{k^2}{x^2}\right)\frac{\sigma'_z}{\sigma_y} > 0$$

General equivalent residual stress

$$\frac{\sigma'_e}{\sigma_y} = 1 - \frac{k^2 - k_j^2 + k^2\ln k_j^2}{k^2-1}\frac{1}{x^2} \quad (1 \le x \le k_j)$$

$$\frac{\sigma'_e}{\sigma_y} = \frac{k_j^2 - 1 - \ln k_j^2}{k^2-1}\frac{k^2}{x^2} > 0 \quad (k_j \le x \le k)$$

General autofrettage pressure p_a

$$\frac{p_a}{\sigma_y} = \ln k_j + \frac{k^2 - k_j^2}{2k^2}$$

The abscissa of the intersection of curves of the residual stress or the abscissa at which $\sigma'_e = 0$

$$x_0 = \frac{r}{r_i} = \sqrt{\frac{k^2 - k_j^2 + k^2\ln k_j^2}{k^2-1}} < k_j$$

The value of residual stresses when $x = x_0$

$$\frac{\sigma'_{zx0}}{\sigma_y} = \frac{\sigma'_{\theta x0}}{\sigma_y} = \frac{\sigma'_{rx0}}{\sigma_y} = \frac{\sigma'_{r\min}}{\sigma_y} = \ln\frac{x_0}{k_j} + \frac{\sigma'_{zj}}{\sigma_y} = \ln\sqrt{\frac{k^2 - k_j^2 + k^2\ln k_j^2}{k_j^2(k^2-1)}} + \frac{k_j^2 - 1 - \ln k_j^2}{2(k^2-1)}$$

The abscissa at which $\sigma'_z = 0$

$$x_1 = \sqrt{e^{\frac{1 - k_j^2 + k^2\ln k_j^2}{k^2-1}}}$$

Equivalent operation stress

$$\frac{\sigma_e^p}{\sigma_y} = \frac{\sigma_\theta^p}{\sigma_y} - \frac{\sigma_r^p}{\sigma_y} = \frac{2k^2}{x^2}\frac{\sigma_z^p}{\sigma_y} = \frac{2k^2}{x^2(k^2-1)}\frac{p}{\sigma_y}$$

The critical radius ratio

$$k_c = 2.218\ 457\ 489\ 916\ 7\ldots$$

k_c is the solution of $\dfrac{k^2\ln k}{k^2-1} = 1$, and $\dfrac{k^2\ln k}{k^2-1} = 1$ is equivalent to $2p_e = p_y$, *i.e.*

$$2\frac{k^2-1}{2k^2} = \ln k$$

TAB. 2.2 – (continued).

The optimum plastic depth k_{j*}

$$k^2 \ln k_{j*}^2 - k^2 - k_{j*}^2 + 2 = 0 \ (e^{0.5} \le k_{j*} \le k_c \text{ and } k \ge k_c)$$

When $k \le k_c$, $k_j = k$

The results when $k_j = k_{j*}$

1. Residual stresses in axial, radial, and circumferential direction

$$\begin{cases} \dfrac{\sigma'_z}{\sigma_y} = \ln x - 0.5 \\ \dfrac{\sigma'_r}{\sigma_y} = \ln x + x^{-2} - 1 \quad (1 \le x \le k_{j*}) \\ \dfrac{\sigma'_\theta}{\sigma_y} = \ln x - x^{-2} \end{cases} \begin{cases} \dfrac{\sigma'_z}{\sigma_y} = \dfrac{k_{j*}^2 - 2}{2k^2} \\ \dfrac{\sigma'_r}{\sigma_y} = \left(1 - \dfrac{k^2}{x^2}\right)\dfrac{\sigma'_z}{\sigma_y} \quad (k_{j*} \le x \le k) \\ \dfrac{\sigma'_\theta}{\sigma_y} = \left(1 + \dfrac{k^2}{x^2}\right)\dfrac{\sigma'_z}{\sigma_y} \end{cases}$$

2. Equivalent residual stress

$$\frac{\sigma'_e}{\sigma_y} = 1 - \frac{2}{x^2} \quad (1 \le x \le k_{j*}) \qquad \frac{\sigma'_e}{\sigma_y} = \frac{k_{j*}^2 - 2}{x^2} \quad (k_{j*} \le x \le k)$$

3. The optimum load-bearing capability and autofrettage pressure

$$\frac{p}{\sigma_y} = \frac{p_a}{\sigma_y} = 2\frac{p_e}{\sigma_y} = \frac{k^2 - 1}{k^2}$$

4. $x_0 \equiv \sqrt{2} = \text{constant}$ $\qquad x_1 \equiv \sqrt{e} = \text{constant}$

5. $y_0^s = \dfrac{\sigma'_z}{\sigma_y} = \dfrac{\sigma'_r}{\sigma_y} = \dfrac{\sigma'_\theta}{\sigma_y} = \dfrac{\sigma'_{rmin}}{\sigma_y} = \dfrac{\sigma'_{rx0}}{\sigma_y} \equiv \ln\sqrt{2} - 0.5 = -\dfrac{1 - \ln 2}{2} = \text{constant}$

y_0^s is not related to k and k_j

The results when $k_j = k_{j*}$, $p = p_a = 2p_e$

1. The components of total stresses

$$\begin{cases} \dfrac{\sigma_z^T}{\sigma_y} = \ln x - \dfrac{1}{2} + \dfrac{1}{k^2} \\ \dfrac{\sigma_r^T}{\sigma_y} = \ln x - 1 + \dfrac{1}{k^2} \quad (1 \le x \le k_{j*}) \\ \dfrac{\sigma_\theta^T}{\sigma_y} = \ln x + \dfrac{1}{k^2} \end{cases} \begin{cases} \dfrac{\sigma_z^T}{\sigma_y} = \dfrac{k_j^2}{2k^2} = \text{constant} \\ \dfrac{\sigma_r^T}{\sigma_y} = -\dfrac{k_j^2}{2}\left(\dfrac{1}{x^2} - \dfrac{1}{k^2}\right) \quad (k_{j*} \le x \le k) \\ \dfrac{\sigma_\theta^T}{\sigma_y} = \dfrac{k_j^2}{2}\left(\dfrac{1}{x^2} + \dfrac{1}{k^2}\right) \end{cases}$$

2. Equivalent residual stress

$$\frac{\sigma_e^T}{\sigma_y} \equiv 1 \quad (1 \le x \le k_{j*}) \qquad \frac{\sigma_e^T}{\sigma_y} = \frac{k_j^2}{x^2} \quad (k_{j*} \le x \le k)$$

The results when $k_j = k$

1. The optimum load-bearing capability and autofrettage pressure

$$\frac{p}{\sigma_y} = \frac{p_a}{\sigma_y} = \ln k = \frac{p_y}{\sigma_y}$$

2. The total stresses when $k_j = k$ and $p = p_y = \sigma_y \ln k$

$$\frac{\sigma_z^T}{\sigma_y} = \frac{1}{2} + \ln x - \ln k$$

$$\frac{\sigma_r^T}{\sigma_y} = \ln x - \ln k$$

$$\frac{\sigma_\theta^T}{\sigma_y} = 1 + \ln x - \ln k$$

Tab. 2.2 – (continued).

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \equiv 1$$

3. The total stresses on an inner surface when $k_{\mathrm{j}} = k$ and $p = p_{\mathrm{y}} = \sigma_{\mathrm{y}} \ln k$

$$\frac{\sigma_{\mathrm{zi}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{1}{2} - \ln k$$

$$\frac{\sigma_{\mathrm{ri}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = -\ln k = -\frac{p}{\sigma_{\mathrm{y}}}$$

$$\frac{\sigma_{\theta\mathrm{i}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1 - \ln k$$

$$\frac{\sigma_{\mathrm{ei}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1$$

The results when $k_{\mathrm{j}} = \dfrac{k+1}{2}$

1. When $k \leq k_{\mathrm{a}} = 2.618253\ldots$, $|\sigma'_{\mathrm{ei}}| \leq \sigma_{\mathrm{y}}$; when $k \geq k_{\mathrm{a}}$, $|\sigma'_{\mathrm{ei}}| \geq \sigma_{\mathrm{y}}$
2. When $k = k_{\mathrm{a}}$, $k_{\mathrm{j}} = \dfrac{k+1}{2} = 1.809126\ldots$
3. The maximum load-bearing capacity and autofrettage pressure

$$\frac{p}{\sigma_{\mathrm{y}}} = \frac{1}{2} - \frac{(k+1)^2}{8k^2} + \ln\frac{k+1}{2}$$

4. When $k \leq k_{\mathrm{a}}$, $\frac{1}{2} - \dfrac{(k+1)^2}{8k^2} + \ln\dfrac{k+1}{2} < \dfrac{k^2-1}{k^2}$; when $k \geq k_{\mathrm{a}}$, $\dfrac{1}{2} - \dfrac{(k+1)^2}{8k^2} + \ln\frac{k+1}{2} > \frac{k^2-1}{k^2}$, but when $k \geq k_{\mathrm{a}}$, $|\sigma'_{\mathrm{ei}}| > \sigma_{\mathrm{y}}$
5. When $k \leq 3.102921\ldots = k_{\mathrm{z}}$, $p \leq \sigma_{\mathrm{y}}$ and $\sigma_{\mathrm{ri}}^{\mathrm{T}} \geq -\sigma_{\mathrm{y}}$; when $k \geq k_{\mathrm{z}}$, $p \geq \sigma_{\mathrm{y}}$ and $\sigma_{\mathrm{ri}}^{\mathrm{T}} \leq -\sigma_{\mathrm{y}}$. When $k \leq k_{\mathrm{a}}$, there must be $k \leq k_{\mathrm{z}}$
6. When $k \geq k_{\mathrm{a}}$, $\sigma_{\mathrm{e}}^{\mathrm{T}} > \sigma_{\mathrm{y}}$
7. Autofrettage pressure $\dfrac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \dfrac{1}{2} - \dfrac{(k+1)^2}{8k^2} + \ln\dfrac{k+1}{2} = \dfrac{p}{\sigma_{\mathrm{y}}}$
8. $\dfrac{p}{\sigma_{\mathrm{y}}} = \dfrac{1}{2} - \dfrac{(k+1)^2}{8k^2} + \ln\dfrac{k+1}{2}\left(k_{\mathrm{j}} = \dfrac{k+1}{2}\right) > \dfrac{p}{\sigma_{\mathrm{y}}} = \dfrac{k-1}{2k} + \ln\sqrt{k}\ (k_{\mathrm{j}} = \sqrt{k})$
9. When $k \leq k_{\mathrm{a}}$, $\dfrac{1}{2} - \dfrac{(k+1)^2}{8k^2} + \ln\dfrac{k+1}{2} < \dfrac{k^2-1}{k^2}$; when $k \geq k_{\mathrm{a}}$, $\dfrac{1}{2} - \dfrac{(k+1)^2}{8k^2} + \ln\frac{k+1}{2} > \frac{k^2-1}{k^2}$, but when $k \geq k_{\mathrm{a}}$, $|\sigma'_{\mathrm{ei}}| > \sigma_{\mathrm{y}}$

The results when $k_{\mathrm{j}} = \sqrt{k}$

1. When $k \leq k_{\mathrm{b}} = 3.042297\ldots$, $|\sigma'_{\mathrm{ei}}| \leq \sigma_{\mathrm{y}}$; when $k \geq k_{\mathrm{b}}$, $|\sigma'_{\mathrm{ei}}| \geq \sigma_{\mathrm{y}}$
2. When $k = k_{\mathrm{b}}$, $k_{\mathrm{j}} = \sqrt{k} = 1.744218\ldots$
3. The maximum load-bearing capacity and autofrettage pressure

$$\frac{p}{\sigma_{\mathrm{y}}} = \frac{k-1}{2k} + \ln\sqrt{k}$$

4. $\dfrac{p}{\sigma_{\mathrm{y}}} = \dfrac{k-1}{2k} + \ln\sqrt{k}\ (k_{\mathrm{j}} = \sqrt{k}) < \dfrac{p}{\sigma_{\mathrm{y}}} = \dfrac{1}{2} - \dfrac{(k+1)^2}{8k^2} + \ln\dfrac{k+1}{2}\left(k_{\mathrm{j}} = \dfrac{k+1}{2}\right)$
5. When $k \leq k_{\mathrm{b}}$, $\dfrac{p}{\sigma_{\mathrm{y}}} = \dfrac{k-1}{2k} + \ln\sqrt{k} < \dfrac{k^2-1}{k^2}$; when $k \geq k_{\mathrm{b}}$, $\dfrac{p}{\sigma_{\mathrm{y}}} = \dfrac{k-1}{2k} + \ln\sqrt{k} > \dfrac{k^2-1}{k^2}$, but when $k \geq k_{\mathrm{b}}$, $|\sigma'_{\mathrm{ei}}| > \sigma_{\mathrm{y}}$

TAB. 2.2 – (continued).

The results when $k_{\mathrm{j}}=\dfrac{r_{\mathrm{j}}}{r_{\mathrm{i}}}=\exp\left(\dfrac{p}{\sigma_{\mathrm{y}}}\right)$
1. Overstrain $\varepsilon=\frac{\exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)-1}{k-1}$
2. When $k_{\mathrm{j}}=\exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)$, the equivalent stress σ_{ej} is the minimum, that is $\dfrac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\dfrac{k^2}{k^2-1}\dfrac{\exp\left(2\frac{p}{\sigma_{\mathrm{y}}}\right)-1}{\exp\left(2\frac{p}{\sigma_{\mathrm{y}}}\right)}=\dfrac{k^2}{k^2-1}\dfrac{k_{\mathrm{j}}^2-1}{k_{\mathrm{j}}^2}$
3. The allowable load in order for $\sigma'_{\mathrm{ei}}/\sigma_{\mathrm{y}} \geq -1$ when $k \geq k_{\mathrm{c}}$ $\exp\left(2\frac{p}{\sigma_{\mathrm{y}}}\right)+k^2-2k^2\frac{p}{\sigma_{\mathrm{y}}}-2\geq 0$ or $k^2\leq\frac{\exp\left(2\frac{p}{\sigma_{\mathrm{y}}}\right)-2}{2\frac{p}{\sigma_{\mathrm{y}}}-1}$
4. The relationship between autofrettage pressure and the allowable load $\dfrac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\dfrac{p}{\sigma_{\mathrm{y}}}+\dfrac{1}{2}-\dfrac{1}{2k^2}\exp\left(2\dfrac{p}{\sigma_{\mathrm{y}}}\right)$

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Chapter 3

Mechanical Autofrettage Technology Based on Mises Yield Criterion

Based on the maximum distortion strain energy theory (Mises yield criterion), to achieve a plastic depth k_j, the pressure exerted on a pressure vessel is evidently equation (1.24), this is just autofrettage pressure p_a[1, 2], or

$$\frac{p_\mathrm{a}}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}} \ln k_\mathrm{j} + \frac{1}{\sqrt{3}} - \frac{k_\mathrm{j}^2}{\sqrt{3}k^2} = \frac{2}{\sqrt{3}} \ln k_\mathrm{j} + \frac{k^2 - k_\mathrm{j}^2}{\sqrt{3}k^2} = \frac{k^2 \ln k_\mathrm{j}^2 + k^2 - k_\mathrm{j}^2}{\sqrt{3}k^2} \tag{3.1}$$

Obviously, as in the case of the maximum shear stress theory (Tresca yield criterion), the greater k_j is, the greater p_a is. It's not hard to foresee that k_j is bound to affect residual stresses, load-bearing capability, etc., and under different strength theories, the extent and the way of its influence will be different. Therefore, it is necessary to investigate autofrettage technology based on the maximum distortion strain energy theory. We also proceed from the general situation.

3.1 General Study on Mechanical Autofrettage Technology

In our research, we obtained some results about mechanical autofrettage technology based on the maximum distortion strain energy theory[3–7], which is the foundation of this research.

3.1.1 In General Forms

1. The components of residual stresses

When a pressure vessel is subjected to an autofrettage pressure p_a which is greater than its initial yield pressure (p_e) and smaller than the entire yield pressure (p_y), its inner layer becomes plastic and its outer layer remains elastic. After

DOI: 10.1051/978-2-7598-3111-1.c003
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eliminating autofrettage pressure p_{a}, the residual stresses remain in the wall of the pressure vessels. Based on the maximum distortion strain energy theory (Mises yield criterion), the residual stresses can be expressed as follows[2].

Within the plastic region ($1 \leq x \leq k_{\mathrm{j}}$)

$$\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}\left[\frac{k_{\mathrm{j}}^2}{k^2} + \ln\frac{x^2}{k_{\mathrm{j}}^2} - \left(1 - \frac{k_{\mathrm{j}}^2}{k^2} + \ln k_{\mathrm{j}}^2\right)\frac{1}{k^2-1}\right] \tag{3.2}$$

$$\frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}\left[\frac{k_{\mathrm{j}}^2}{k^2} - 1 + \ln\frac{x^2}{k_{\mathrm{j}}^2} - \left(1 - \frac{k_{\mathrm{j}}^2}{k^2} + \ln k_{\mathrm{j}}^2\right)\frac{1}{k^2-1}\left(1 - \frac{k^2}{x^2}\right)\right] \tag{3.3}$$

$$\frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}\left[\frac{k_{\mathrm{j}}^2}{k^2} + 1 + \ln\frac{x^2}{k_{\mathrm{j}}^2} - \left(1 - \frac{k_{\mathrm{j}}^2}{k^2} + \ln k_{\mathrm{j}}^2\right)\frac{1}{k^2-1}\left(1 + \frac{k^2}{x^2}\right)\right] \tag{3.4}$$

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2 - k_{\mathrm{j}}^2 + k^2\ln k_{\mathrm{j}}^2}{k^2-1}\frac{1}{x^2} \tag{3.5}$$

Based on the maximum shear stress theory (Tresca yield criterion), it is easy to know[8] that as x increases, σ'_{z}, σ'_{θ}, and σ'_{e} increase within the plastic region ($1 \leq x \leq k_{\mathrm{j}}$). When $x \leq \sqrt{\frac{k^2-k_{\mathrm{j}}^2+k^2\ln k_{\mathrm{j}}^2}{k^2-1}}$, σ'_{r} decreases as x increases; When $x \geq \sqrt{\frac{k^2-k_{\mathrm{j}}^2+k^2\ln k_{\mathrm{j}}^2}{k^2-1}}$, σ'_{r} increases as x increases. From $\ln k_{\mathrm{j}}^2 < k_{\mathrm{j}}^2 - 1$, we can prove that $\sqrt{\frac{k^2-k_{\mathrm{j}}^2+k^2\ln k_{\mathrm{j}}^2}{k^2-1}} < k_{\mathrm{j}}$.

In fact, $\sqrt{\frac{k^2-k_{\mathrm{j}}^2+k^2\ln k_{\mathrm{j}}^2}{k^2-1}}$ is just the relative location of the intersection of the three residual stress curves, *i.e.*, the abscissa at which $\sigma'_{\mathrm{e}} = 0$, or x_0 in the following section.

Within the elastic region ($k_{\mathrm{j}} \leq x \leq k$)

$$\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}\left[\frac{k_{\mathrm{j}}^2}{k^2} - \left(1 - \frac{k_{\mathrm{j}}^2}{k^2} + \ln k_{\mathrm{j}}^2\right)\frac{1}{k^2-1}\right] = \frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{\sqrt{3}(k^2-1)} = \text{constant} > 0 \tag{3.6}$$

$$\frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = \left(1 - \frac{k^2}{x^2}\right)\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = -\frac{k^2 - x^2}{x^2}\frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{\sqrt{3}(k^2-1)} < 0 \tag{3.7}$$

$$\frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}} = \left(1 + \frac{k^2}{x^2}\right)\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = \frac{k^2 + x^2}{x^2}\frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{\sqrt{3}(k^2-1)} > 0 \tag{3.8}$$

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{k^2-1}\frac{k^2}{x^2} > 0 \tag{3.9}$$

Clearly, as x increases, σ'_θ and σ'_e decrease, σ'_r increases, and σ'_z is kept as a constant.

Figure 3.1 is the typical distribution of the residual stresses, where $k = 4$ and $k_j = 1.6$. The meaning of each curve and main parameters are marked in the figure.

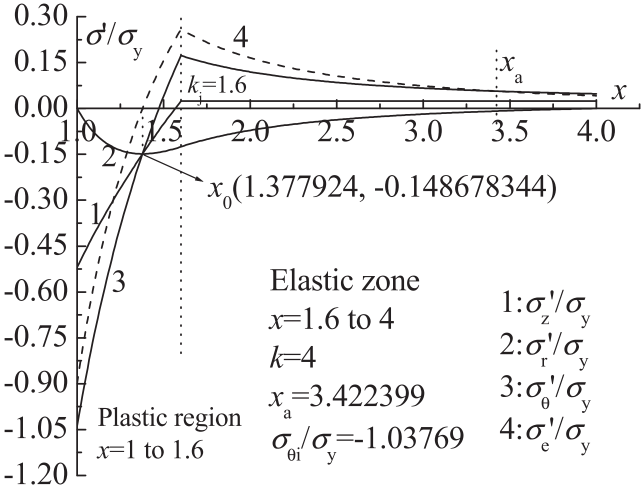

Fig. 3.1 – The typical distributions of the residual stresses.

When based on the maximum distortion strain energy theory, the components of residual stresses are $\frac{2}{\sqrt{3}}$ times as much as those based on the maximum shear stress theory, and as mentioned above, $\sigma_e^d = \frac{\sqrt{3}}{2}\sigma_e^s$, as a result, the equivalent residual stress based on the maximum distortion strain energy theory (Mises yield criterion) is the same as that based on the maximum shear stress theory (Tresca yield criterion).

At the inside surface, $x = r/r_i = 1$, then, from equations (3.2)–(3.5), the residual stresses on the inside surface are as follows.

$$\frac{\sigma'_{zi}}{\sigma_y} = -\frac{k^2 \ln k_j^2 - k_j^2 + 1}{\sqrt{3}(k^2 - 1)} \leq 0 \tag{3.10}$$

$$\sigma'_{ri} = 0 \tag{3.11}$$

$$\frac{\sigma'_{\theta i}}{\sigma_y} = -\frac{2}{\sqrt{3}} \frac{k^2 \ln k_j^2 - k_j^2 + 1}{k^2 - 1} = 2\frac{\sigma'_{zi}}{\sigma_y} \leq 0 \tag{3.12}$$

$$\frac{\sigma'_{ei}}{\sigma_y} = \frac{\sqrt{3}}{2}\left(\frac{\sigma'_{\theta i}}{\sigma_y} - \frac{\sigma'_{ri}}{\sigma_y}\right) = \frac{\sqrt{3}}{2}\frac{\sigma'_{\theta i}}{\sigma_y} = -\frac{k^2 \ln k_j^2 - k_j^2 + 1}{k^2 - 1} = \sqrt{3}\frac{\sigma'_{zi}}{\sigma_y} \leq 0 \tag{3.13}$$

At the inside surface, the absolute value of $\sigma'_{\theta\mathrm{i}}$ is the maximum while $\sigma'_{\mathrm{ri}} = 0$. Therefore, when $\left|\sigma'_{\mathrm{ei}}\right| = \sigma_{\mathrm{y}}$, there is $\left|\sigma'_{\theta\mathrm{i}}\right| = \sigma_{\mathrm{y}}$. Therefore, in the next chapter, we will study mechanical autofrettage technology by limiting circumferential residual stress based on the Mises yield criterion.

At the elastic-plastic juncture, $x = r_{\mathrm{j}}/r_{\mathrm{i}} = k_{\mathrm{j}}$, the residual stresses calculated by equations (3.2)–(3.5) are equal to those calculated by equations (3.6)–(3.9). Therefore, the residual stresses in the whole wall are continuous. Substituting $x = k_{\mathrm{j}}$ into equations (3.2)–(3.5) or equations (3.6)–(3.9) leads to the residual stresses at the elastic-plastic juncture

$$\frac{\sigma'_{\mathrm{zj}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{\sqrt{3}(k^2 - 1)} \geq 0 \tag{3.14}$$

$$\frac{\sigma'_{\mathrm{rj}}}{\sigma_{\mathrm{y}}} = \left(1 - \frac{k^2}{k_{\mathrm{j}}^2}\right) \frac{\sigma'_{\mathrm{zj}}}{\sigma_{\mathrm{y}}} \leq 0 \tag{3.15}$$

$$\frac{\sigma'_{\theta\mathrm{j}}}{\sigma_{\mathrm{y}}} = \left(1 + \frac{k^2}{k_{\mathrm{j}}^2}\right) \frac{\sigma'_{\mathrm{zj}}}{\sigma_{\mathrm{y}}} > \frac{\sigma'_{\mathrm{zj}}}{\sigma_{\mathrm{y}}} \geq 0 \tag{3.16}$$

$$0 \leq \frac{\sigma'_{\theta\mathrm{j}}}{\sigma_{\mathrm{y}}} < \frac{\sigma'_{\mathrm{ej}}}{\sigma_{\mathrm{y}}} = \sqrt{3}\,\frac{k^2}{k_{\mathrm{j}}^2}\frac{\sigma'_{\mathrm{zj}}}{\sigma_{\mathrm{y}}} = \frac{k^2(k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2)}{(k^2 - 1)k_{\mathrm{j}}^2} \leq 1 \tag{3.17}$$

Based on the maximum shear stress theory (Tresca yield criterion), $\sigma'_{\mathrm{z}} > 0$, σ'_{θ}, σ'_{e} and σ'_{z} monotonically increase as x increases in the plastic zone, σ'_{θ} and σ'_{e} monotonically decrease as x increases in the elastic zone, σ'_{z} remains a constant as x increases in the elastic zone, σ'_{r} monotonically increase as x increases in the elastic zone. At the elastic-plastic juncture, $\sigma'_{\theta\mathrm{j}}$, σ'_{ej}, and σ'_{zj} are the maximum within the whole wall (algebraic value, not absolute value). Regardless of the value of k_{j}, the equivalent residual stress at an elastic-plastic juncture (σ'_{ej}) is certainly tension and smaller than yield strength σ_{y}. Therefore, it is quite necessary to pay close attention to the residual stresses at the inside surface.

At the outside surface, $x = r_{\mathrm{o}}/r_{\mathrm{i}} = k$, then, from equations (3.6)–(3.9),

$$\frac{\sigma'_{\mathrm{zo}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{\sqrt{3}(k^2 - 1)} = \frac{\sigma'_{\mathrm{zj}}}{\sigma_{\mathrm{y}}} > 0 \tag{3.18}$$

$$\sigma'_{\mathrm{ro}} = 0 \tag{3.19}$$

$$0 < \sigma'_{\theta\mathrm{o}} = 2\sigma'_{\mathrm{zo}} < \sigma'_{\theta\mathrm{j}} > 0 \tag{3.20}$$

$$\frac{\sigma'_{\mathrm{eo}}}{\sigma_{\mathrm{y}}}=\frac{k_{\mathrm{j}}^2-1-\ln k_{\mathrm{j}}^2}{k^2-1}=\sqrt{3}\frac{\sigma'_{\mathrm{zo}}}{\sigma_{\mathrm{y}}}>0 \tag{3.21}$$

Within the whole wall, $\sigma'_{\mathrm{r}}\le 0,\sigma'_{\mathrm{eo}}<\sigma'_{\theta\mathrm{o}}$.

In the elastic region, letting $\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}}=\frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}}$ gives $x=\sqrt{\sqrt{3}-1}k=0.8556\ k=x_{\mathrm{a}}$, or x_{a} is the abscissa at which $\sigma'_{\mathrm{e}}=\sigma'_{\theta}$. Therefore, when $x<x_{\mathrm{a}}$, $\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}}>\frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}}$; when $x>x_{\mathrm{a}}$, $\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}}<\frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}}$.

2. Autofrettage pressure p_{a}

Autofrettage pressure is the internal pressure applied to a pressure vessel to cause a plastic depth k_{j} before it is put into production, which is just equation (3.1), or

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{2}{\sqrt{3}}\ln k_{\mathrm{j}}+\frac{k^2-k_{\mathrm{j}}^2}{\sqrt{3}k^2}$$

3. The location (defined as x_0) of the intersection of the three residual stress curves or the abscissa at which $\sigma'_{\mathrm{e}}=0$.

X_0 is the same as that based on the maximum shear stress theory (Tresca yield criterion), which is as follows

$$x_0=\frac{r}{r_{\mathrm{i}}}=\sqrt{\frac{k^2-k_{\mathrm{j}}^2+k^2\ln k_{\mathrm{j}}^2}{k^2-1}}<k_{\mathrm{j}} \tag{3.22}$$

The value of radial residual stresses at $x=x_0$ (σ'_{rxo}) is the minimum, which is

$$\begin{aligned}\frac{\sigma'_{\mathrm{rx0}}}{\sigma_{\mathrm{y}}}=\frac{\sigma'_{\mathrm{rmin}}}{\sigma_{\mathrm{y}}}=\frac{\sigma'_{\mathrm{zx0}}}{\sigma_{\mathrm{y}}}=\frac{\sigma'_{\theta\mathrm{x0}}}{\sigma_{\mathrm{y}}}&=\frac{2}{\sqrt{3}}\ln\frac{x_0}{k_{\mathrm{j}}}+\frac{\sigma'_{\mathrm{zj}}}{\sigma_{\mathrm{y}}}\\&=\frac{2}{\sqrt{3}}\ln\sqrt{\frac{k^2-k_{\mathrm{j}}^2+k^2\ln k_{\mathrm{j}}^2}{k_{\mathrm{j}}^2(k^2-1)}}+\frac{k_{\mathrm{j}}^2-1-\ln k_{\mathrm{j}}^2}{\sqrt{3}(k^2-1)}=y_0^{\mathrm{d}}\end{aligned} \tag{3.23}$$

y_0^{d} is $\frac{2}{\sqrt{3}}$ times of y_0^{s} in chapter 2.

In figures 3.1 and 2.1, $x_0=\sqrt{\frac{k^2-k_{\mathrm{j}}^2+k^2\ln k_{\mathrm{j}}^2}{k^2-1}}=1.377924<k_{\mathrm{j}}=1.6$. In figure 3.1, $y_0^{\mathrm{d}}=-0.148678344$, and in figure 2.1, $y_0^{\mathrm{s}}=-0.128759223$, $y_0^{\mathrm{d}}=\frac{2}{\sqrt{3}}y_0^{\mathrm{s}}$.

4. The location (defined as x_1) where $\sigma'_{\mathrm{z}}=0$

Although the residual stresses based on the maximum distortion strain energy theory are $\frac{2}{\sqrt{3}}$ times of the residual stresses based on the maximum shear stress theory, the abscissa, x_1, at which $\sigma'_{\mathrm{z}}=0$, is the same based on both strength theories, is as follows,

$$x_1 = \sqrt{\mathrm{e}^{\frac{1-k_\mathrm{j}^2+k^2\ln k_\mathrm{j}^2}{k^2-1}}} \tag{3.24}$$

In figures 3.1 and 2.1, $x_1 = 1.567273$.

5. Lamé equations

The components of the operation stresses in three directions, or the axial elastic stress $\sigma_\mathrm{z}^\mathrm{p}$, the radial elastic stress $\sigma_\mathrm{r}^\mathrm{p}$ and the circumferential elastic stress σ_θ^p, are the same as those based on the maximum shear stress theory (Tresca criterion), and the equivalent operation stress is as follows:

$$\frac{\sigma_\mathrm{e}^\mathrm{p}}{\sigma_\mathrm{y}} = \frac{\sqrt{3}}{2}\left(\frac{\sigma_\theta^\mathrm{p}}{\sigma_\mathrm{y}} - \frac{\sigma_\mathrm{r}^\mathrm{p}}{\sigma_\mathrm{y}}\right) = \frac{\sqrt{3}k^2}{x^2}\frac{\sigma_\mathrm{z}^\mathrm{p}}{\sigma_\mathrm{y}} = \frac{\sqrt{3}k^2}{x^2(k^2-1)}\frac{p}{\sigma_\mathrm{y}} \tag{3.25}$$

6. The total stresses

The total stresses include the residual stresses and the elastic stresses caused by internal pressure p.

The total stresses in general form within the plastic region ($1 \leq x \leq k_\mathrm{j}$)

$$\frac{\sigma_\mathrm{z}^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{1}{\sqrt{3}}\left[\frac{k_\mathrm{j}^2}{k^2} + \ln\frac{x^2}{k_\mathrm{j}^2} - \left(1 - \frac{k_\mathrm{j}^2}{k^2} + \ln k_\mathrm{j}^2\right)\frac{1}{k^2-1}\right] + \frac{1}{k^2-1}\frac{p}{\sigma_\mathrm{y}} \tag{3.26}$$

$$\frac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{1}{\sqrt{3}}\left[\frac{k_\mathrm{j}^2}{k^2} - 1 + \ln\frac{x^2}{k_\mathrm{j}^2} - \left(1 - \frac{k_\mathrm{j}^2}{k^2} + \ln k_\mathrm{j}^2\right)\frac{1}{k^2-1}\left(1 - \frac{k^2}{x^2}\right)\right] - \frac{k^2-x^2}{x^2(k^2-1)}\frac{p}{\sigma_\mathrm{y}} \tag{3.27}$$

$$\frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{1}{\sqrt{3}}\left[\frac{k_\mathrm{j}^2}{k^2} + 1 + \ln\frac{x^2}{k_\mathrm{j}^2} - \left(1 - \frac{k_\mathrm{j}^2}{k^2} + \ln k_\mathrm{j}^2\right)\frac{1}{k^2-1}\left(1 + \frac{k^2}{x^2}\right)\right] + \frac{k^2+x^2}{x^2(k^2-1)}\frac{p}{\sigma_\mathrm{y}} \tag{3.28}$$

$$\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} = 1 - \frac{k^2 - k_\mathrm{j}^2 + k^2\ln k_\mathrm{j}^2}{x^2(k^2-1)} + \frac{\sqrt{3}k^2}{x^2(k^2-1)}\frac{p}{\sigma_\mathrm{y}} \tag{3.29}$$

Setting $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} \leq 1$ obtains the load-bearing capability

$$\frac{p}{\sigma_\mathrm{y}} \leq \frac{2}{\sqrt{3}}\ln k_\mathrm{j} + \frac{k^2 - k_\mathrm{j}^2}{\sqrt{3}k^2} = \frac{p_\mathrm{a}}{\sigma_\mathrm{y}} \tag{3.30}$$

Setting $\frac{\mathrm{d}\left(\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}}\right)}{\mathrm{d}x} = \frac{k^2-k_\mathrm{j}^2+k^2\ln k_\mathrm{j}^2}{k^2-1}\frac{2}{x^3} - \frac{2\sqrt{3}k^2}{k^2-1}\frac{1}{x^3}\frac{p}{\sigma_\mathrm{y}} \geq 0$ also obtains equation (3.30).

Therefore, when $\frac{p}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}\ln k_\mathrm{j} + \frac{k^2-k_\mathrm{j}^2}{\sqrt{3}k^2} = \frac{p_\mathrm{a}}{\sigma_\mathrm{y}}$, $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} \equiv 1 =$ constant within the plastic region. Since $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{\sqrt{3}}{2}\left(\frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}} - \frac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}}\right) =$ constant and $\frac{\sigma_\mathrm{z}^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{1}{2}\left(\frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}} + \frac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}}\right)$, the three curves for $\sigma_\mathrm{z}^\mathrm{T}$, σ_θ^T, $\sigma_\mathrm{r}^\mathrm{T}$ are parallel to each other within the plastic region.

Setting $\frac{\mathrm{d}\left(\frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}}\right)}{\mathrm{d}x} \geq 0$ obtains

$$\frac{p}{\sigma_\mathrm{y}} \leq \frac{x^2(k^2-1)}{\sqrt{3}k^2} + \frac{k^2-k_\mathrm{j}^2+k^2\ln k_\mathrm{j}^2}{\sqrt{3}k^2} > \frac{p_\mathrm{a}}{\sigma_\mathrm{y}}$$

Setting $\frac{\mathrm{d}\left(\frac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}}\right)}{\mathrm{d}x} \geq 0$ obtains

$$\frac{p}{\sigma_\mathrm{y}} \geq \frac{k^2-k_\mathrm{j}^2+k^2\ln k_\mathrm{j}^2}{\sqrt{3}k^2} - \frac{x^2(k^2-1)}{\sqrt{3}k^2} = \frac{p_\mathrm{r}}{\sigma_\mathrm{y}} < \frac{p_\mathrm{a}}{\sigma_\mathrm{y}} \tag{3.31}$$

When $x = 1$, equation (3.31) becomes

$$\frac{p}{\sigma_\mathrm{y}} \geq \frac{2}{\sqrt{3}}\ln k_\mathrm{j} - \frac{k_\mathrm{j}^2-1}{\sqrt{3}k^2} = \frac{p_\mathrm{ri}}{\sigma_\mathrm{y}} \tag{3.32}$$

When $x = k_\mathrm{j}$, equation (3.31) becomes

$$\frac{p}{\sigma_\mathrm{y}} \geq \frac{2}{\sqrt{3}}\ln k_\mathrm{j} - \frac{k_\mathrm{j}^2-1}{\sqrt{3}k^2} = \frac{p_\mathrm{ro}}{\sigma_\mathrm{y}} < 0 \tag{3.33}$$

As long as the operation pressure p does not exceed autofrettage pressure p_a, $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}}$ and $\frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}}$ increase with the increase of x. Clearly, $\frac{\sigma_\mathrm{z}^\mathrm{T}}{\sigma_\mathrm{y}}$ always increases with the increase of x. When $\frac{p}{\sigma_\mathrm{y}}$ satisfies equation (3.31), $\frac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}}$ monotonously increases with the increase of x, otherwise, $\frac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}}$ does not increase monotonously with the increase of x in the plastic region. Especially, when $p = 0$, $\frac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}}$ is just $\frac{\sigma_\mathrm{r}'}{\sigma_\mathrm{y}}$, hence, when $x \leq x_0 = \sqrt{\frac{k^2-k_\mathrm{j}^2+k^2\ln k_\mathrm{j}^2}{k^2-1}}$, $\frac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}}$ decreases as x increases; when $x \geq x_0$, $\frac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}}$ increases as x increases. Besides, from equations (3.29) and (3.31), when $\frac{p}{\sigma_\mathrm{y}} = \frac{p_\mathrm{r}}{\sigma_\mathrm{y}}$, $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} = 0$. Since $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{\sqrt{3}}{2}\left(\frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}} - \frac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}}\right)$, when $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} = 0$, there must be $\frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}}$. Since $\sigma_\mathrm{z}^\mathrm{T} = \frac{\sigma_\theta^\mathrm{T}+\sigma_\mathrm{r}^\mathrm{T}}{2}$, when $\frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}}$, there must be $\frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{\sigma_\mathrm{z}^\mathrm{T}}{\sigma_\mathrm{y}}$. Therefore, if $p \leq p_\mathrm{r}$, $\frac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}}$ does not increase monotonously with the increase of x in the plastic region, the three curves for $\frac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}}$, $\frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}}$ and $\frac{\sigma_\mathrm{z}^\mathrm{T}}{\sigma_\mathrm{y}}$ have an intersection in the plastic region, at the intersection, $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} = 0$. In addition, it can be proved that at the intersection, $\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ gets the minimum, or $\left|\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\right|$ gets the maximum. The abscissa (x_0^{T}) of the intersection point of curves of total stresses is solved from equation (3.31) as follows.

$$x_0^{\mathrm{T}} = \sqrt{\frac{k^2 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2}{k^2 - 1} - \frac{\sqrt{3}k^2}{k^2 - 1}\frac{p}{\sigma_{\mathrm{y}}}} \tag{3.34}$$

However, in engineering practice, x_0^{T} usually does not exist between 1 and k. Usually, $x_0^{\mathrm{T}} < 1$.

The total stresses in general form within the elastic region ($k_{\mathrm{j}} \leq x \leq k$)

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{\sqrt{3}(k^2 - 1)} + \frac{1}{k^2 - 1}\frac{p}{\sigma_{\mathrm{y}}} = \text{constant} > 0 \tag{3.35}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = -\frac{k^2 - x^2}{x^2}\frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{\sqrt{3}(k^2 - 1)} - \frac{k^2 - x^2}{x^2(k^2 - 1)}\frac{p}{\sigma_{\mathrm{y}}} < 0 \tag{3.36}$$

$$\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k^2 + x^2}{x^2}\frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{\sqrt{3}(k^2 - 1)} + \frac{k^2 + x^2}{x^2(k^2 - 1)}\frac{p}{\sigma_{\mathrm{y}}} > 0 \tag{3.37}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{k^2 - 1}\frac{k^2}{x^2} + \frac{\sqrt{3}k^2}{x^2(k^2 - 1)}\frac{p}{\sigma_{\mathrm{y}}} > 0 \tag{3.38}$$

Within the whole wall, $\frac{\sigma_{\mathrm{r}}'}{\sigma_{\mathrm{y}}} < 0$ and $\frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} < 0$, then, $\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} < 0$ within the whole wall.

In the elastic region, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ decreases as x increases. Therefore, if $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \equiv 1$ within the plastic region, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} < 1$ within the elastic region.

In the elastic region, setting $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \geq \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$, we obtain

$$\left(\sqrt{3}k^2 - k^2 - x^2\right)\frac{p}{\sigma_{\mathrm{y}}} \geq -\frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{\sqrt{3}}\left(\sqrt{3}k^2 - k^2 - x^2\right)$$

Since $k_{\mathrm{j}}{}^2 - 1 > \ln k_{\mathrm{j}}$, $k_{\mathrm{j}}{}^2 - 1 - \ln k_{\mathrm{j}} > 0$. When $\sqrt{3}k^2 - k^2 - x^2 \geq 0$, or $x \leq \sqrt{\sqrt{3} - 1}k = x_{\mathrm{a}}$, meanwhile $\frac{p}{\sigma_{\mathrm{y}}} \geq -\frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{\sqrt{3}}$ (that's of course in practice), $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \geq \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$; When $\sqrt{3}k^2 - k^2 - x^2 \leq 0$, or $x \geq x_{\mathrm{a}}$, meanwhile $\frac{p}{\sigma_{\mathrm{y}}} \leq -\frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{\sqrt{3}}$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \geq \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$, but $\frac{p}{\sigma_{\mathrm{y}}} \leq -\frac{k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2}{\sqrt{3}}$ is impossible in practice, as a result, when $x \geq x_{\mathrm{a}}$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \leq \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$. Therefore, when $x = k$, $\frac{\sigma_{\mathrm{eo}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \leq \frac{\sigma_{\theta\mathrm{o}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ is absolute.

Obviously, x_a is also the abscissa at which $\sigma_\mathrm{e}^\mathrm{T} = \sigma_\theta^\mathrm{T}$.

Setting $\frac{\sigma_\mathrm{ej}^\mathrm{T}}{\sigma_\mathrm{y}} \ge \frac{\sigma_{\theta\mathrm{j}}^\mathrm{T}}{\sigma_\mathrm{y}}$ results in $\left(\sqrt{3}k^2 - k^2 - k_\mathrm{j}^2\right)\frac{p}{\sigma_\mathrm{y}} \ge -\frac{k_\mathrm{j}^2-1-\ln k_\mathrm{j}^2}{\sqrt{3}}\left(\sqrt{3}k^2 - k^2 - k_\mathrm{j}^2\right)$. When $\sqrt{3}k^2 - k^2 - k_\mathrm{j}^2 \ge 0$, or $k_\mathrm{j} \le \sqrt{\sqrt{3}-1}k = x_\mathrm{a}$, meanwhile $\frac{p}{\sigma_\mathrm{y}} \ge -\frac{k_\mathrm{j}^2-1-\ln k_\mathrm{j}^2}{\sqrt{3}}$ (that's of course in practice), $\frac{\sigma_\mathrm{ej}^\mathrm{T}}{\sigma_\mathrm{y}} \ge \frac{\sigma_{\theta\mathrm{j}}^\mathrm{T}}{\sigma_\mathrm{y}}$; When $\sqrt{3}k^2 - k^2 - k_\mathrm{j}^2 \le 0$, or $k_\mathrm{j} \ge x_\mathrm{a}$, meanwhile $\frac{p}{\sigma_\mathrm{y}} \le -\frac{k_\mathrm{j}^2-1-\ln k_\mathrm{j}^2}{\sqrt{3}}$ (minus pressure), $\frac{\sigma_\mathrm{ej}^\mathrm{T}}{\sigma_\mathrm{y}} \ge \frac{\sigma_{\theta\mathrm{j}}^\mathrm{T}}{\sigma_\mathrm{y}}$, but $\frac{p}{\sigma_\mathrm{y}} \le -\frac{k_\mathrm{j}^2-1-\ln k_\mathrm{j}^2}{\sqrt{3}}$ is impossible in practice, as a result, when $k_\mathrm{j} \ge x_\mathrm{a}$, $\frac{\sigma_\mathrm{ej}^\mathrm{T}}{\sigma_\mathrm{y}} \le \frac{\sigma_{\theta\mathrm{j}}^\mathrm{T}}{\sigma_\mathrm{y}}$. Therefore, when $x = k_\mathrm{j}$, $\frac{\sigma_\mathrm{ej}^\mathrm{T}}{\sigma_\mathrm{y}} \ge \frac{\sigma_{\theta\mathrm{j}}^\mathrm{T}}{\sigma_\mathrm{y}}$ or $\frac{\sigma_\mathrm{ej}^\mathrm{T}}{\sigma_\mathrm{y}} \le \frac{\sigma_{\theta\mathrm{j}}^\mathrm{T}}{\sigma_\mathrm{y}}$ is conditional and depends on $k_\mathrm{j} \le x_\mathrm{a}$ or $k_\mathrm{j} \ge x_\mathrm{a}$.

In fact, we can also make the following inference:

When $x < x_\mathrm{a}$, $\frac{\sigma_\mathrm{e}'}{\sigma_\mathrm{y}} > \frac{\sigma_\theta'}{\sigma_\mathrm{y}}$, when $x = x_\mathrm{a}$, $\frac{\sigma_\mathrm{e}'}{\sigma_\mathrm{y}} = \frac{\sigma_\theta'}{\sigma_\mathrm{y}}$, when $x > x_\mathrm{a}$, $\frac{\sigma_\mathrm{e}'}{\sigma_\mathrm{y}} < \frac{\sigma_\theta'}{\sigma_\mathrm{y}}$.

When $x < x_\mathrm{a}$, $\frac{\sigma_\mathrm{e}^\mathrm{p}}{\sigma_\mathrm{y}} > \frac{\sigma_\theta^\mathrm{p}}{\sigma_\mathrm{y}}$, when $x = x_\mathrm{a}$, $\frac{\sigma_\mathrm{e}^\mathrm{p}}{\sigma_\mathrm{y}} = \frac{\sigma_\theta^\mathrm{p}}{\sigma_\mathrm{y}}$, when $x > x_\mathrm{a}$, $\frac{\sigma_\mathrm{e}^\mathrm{p}}{\sigma_\mathrm{y}} < \frac{\sigma_\theta^\mathrm{p}}{\sigma_\mathrm{y}}$.

$\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{\sigma_\mathrm{e}'}{\sigma_\mathrm{y}} + \frac{\sigma_\mathrm{e}^\mathrm{p}}{\sigma_\mathrm{y}}$ and $\frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{\sigma_\theta'}{\sigma_\mathrm{y}} + \frac{\sigma_\theta^\mathrm{p}}{\sigma_\mathrm{y}}$. Therefore, there must be that when $x < x_\mathrm{a}$, $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} > \frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}}$; when $x = x_\mathrm{a}$, $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}}$; when $x > x_\mathrm{a}$, $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} < \frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}}$.

The variation tendencies of $\frac{\sigma_\mathrm{z}^\mathrm{T}}{\sigma_\mathrm{y}}$, $\frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}}$, $\frac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}}$ and $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}}$ are the same as that based on the maximum shear stress theory.

In order to summarize and validate the above-mentioned results, we present several figures for total stresses according to equations (3.26)–(3.29) and equations (3.35)–(3.38) as shown in figures 3.2–3.11.

(1) Add operation stresses to figure 3.1 to illustrate the total stresses as shown in figure 3.2. When $k = 4$ and $k_\mathrm{j} = 1.6$, $\frac{p}{\sigma_\mathrm{y}} = \frac{p_\mathrm{a}}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}\ln k_\mathrm{j} + \frac{k^2-k_\mathrm{j}^2}{\sqrt{3}k^2} = 1.027688$.

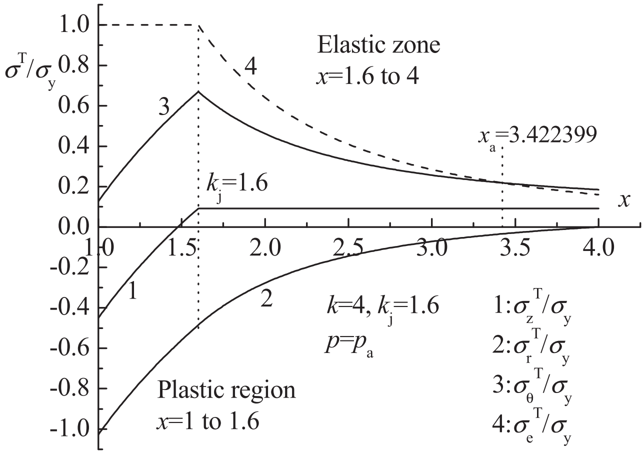

FIG. 3.2 – The total stresses when $k = 4$, $k_\mathrm{j} = 1.6$ and $p = p_\mathrm{a}$.

$x_{\mathrm{a}} = \sqrt{\sqrt{3}-1}k = 3.422399 > k_{\mathrm{j}}$. When $x < x_{\mathrm{a}}$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} > \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$; When $x > x_{\mathrm{a}}$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} < \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$; At $x = x_{\mathrm{a}}$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$.

Within the plastic region, curves 1, 2, and 3 are parallel to each other.

When $p = p_{\mathrm{a}}$, $\sigma_{\mathrm{e}}^{\mathrm{T}} \equiv \sigma_{\mathrm{y}}$ within the whole plastic region, and $\sigma_{\mathrm{e}}^{\mathrm{T}} < \sigma_{\mathrm{y}}$ within the elastic region.

There is no x_0^{T} between 1 and k.

Because $p = p_{\mathrm{a}}$, the three curves for $\sigma_{\mathrm{z}}^{\mathrm{T}}$, $\sigma_{\theta}^{\mathrm{T}}$, $\sigma_{\mathrm{r}}^{\mathrm{T}}$ are parallel to each other within the plastic region.

(2) When $k = 4$ and $k_{\mathrm{j}} = 3.422399 = x_{\mathrm{a}}$, $\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\ln k_{\mathrm{j}} + \frac{k^2-k_{\mathrm{j}}^2}{\sqrt{3}k^2} = 1.575377$.

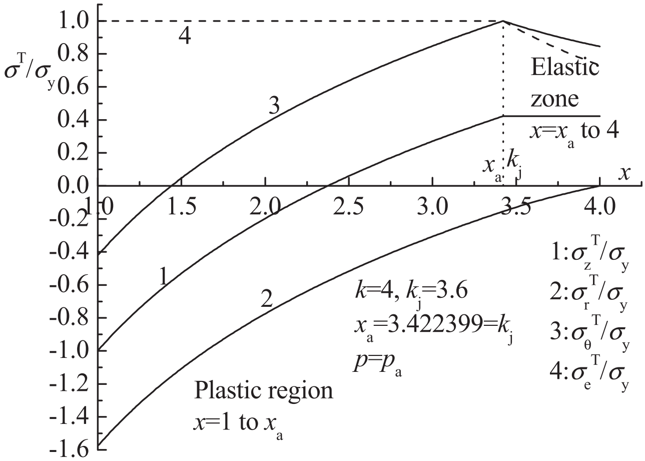

FIG. 3.3 – The total stresses when $k = 4$, $k_{\mathrm{j}} = x_{\mathrm{a}} = 3.422399$ and $p = p_{\mathrm{a}}$.

$x_{\mathrm{a}} = \sqrt{\sqrt{3}-1}k = 3.422399 = k_{\mathrm{j}}$. At $x = x_{\mathrm{a}} = k_{\mathrm{j}}$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$.

When $p = p_{\mathrm{a}}$, $\sigma_{\mathrm{e}}^{\mathrm{T}} \equiv \sigma_{\mathrm{y}}$ within the whole plastic region, and $\sigma_{\mathrm{e}}^{\mathrm{T}} < \sigma_{\mathrm{y}}$ within the elastic region.

There is no x_0^{T} between 1 and k.

Because $p = p_{\mathrm{a}}$, the three curves for $\sigma_{\mathrm{z}}^{\mathrm{T}}$, $\sigma_{\theta}^{\mathrm{T}}$, $\sigma_{\mathrm{r}}^{\mathrm{T}}$ are parallel to each other within the plastic region.

(3) When $k = 4$ and $k_{\mathrm{j}} = 3.6 > x_{\mathrm{a}}$, $\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\ln k_{\mathrm{j}} + \frac{k^2-k_{\mathrm{j}}^2}{\sqrt{3}k^2} = 1.588792$.

When $k_{\mathrm{j}} > x_{\mathrm{a}}$, the point at which $\sigma_{\mathrm{e}} = \sigma_{\theta}$ is within the plastic region, represented by x_{a}', and x_{a} does not exist. When $x < x_{\mathrm{a}}'$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} > \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$; When $x > x_{\mathrm{a}}'$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} < \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$; At $x = x_{\mathrm{a}}'$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$. At $x = k_{\mathrm{j}}$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} < \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$. For figure 3.4, $x_{\mathrm{a}}' = 3.46239 > x_{\mathrm{a}}$.

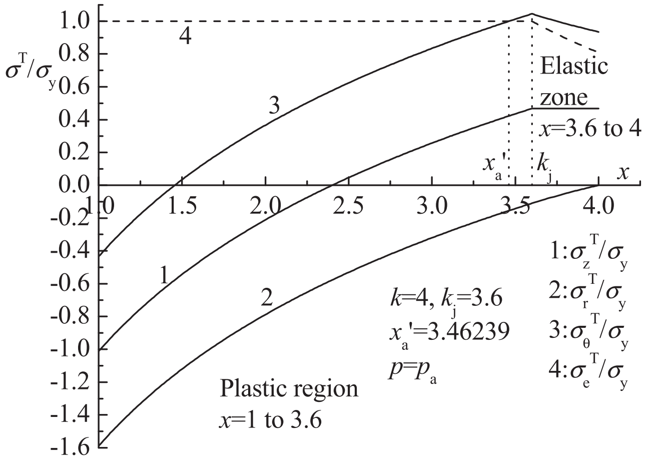

FIG. 3.4 – The total stresses when $k = 4$, $k_j = 3.6$ and $p = p_a$.

When $p = p_a$, $\sigma_e^T \equiv \sigma_y$ within the whole plastic region, and $\sigma_\theta^T > \sigma_y$ near k_j. There is no x_0^T between 1 and k.

Because $p = p_a$, the three curves for σ_z^T, σ_θ^T, σ_r^T are parallel to each other within the plastic region.

(4) When $k = 4$ and $k_j = 1.6$, $\frac{p}{\sigma_y} = 0.9 < \frac{p_a}{\sigma_y}$.

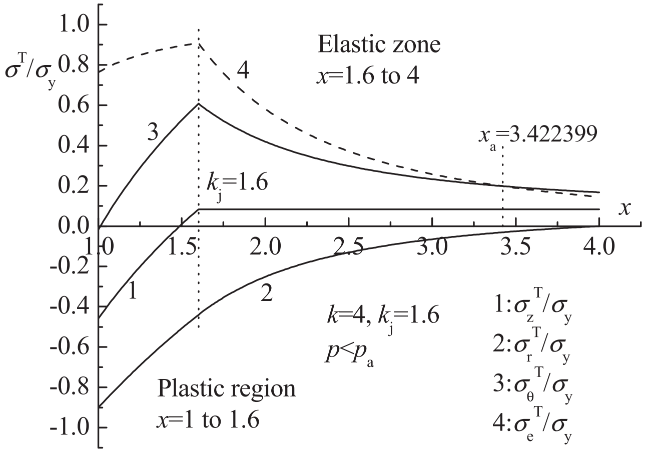

FIG. 3.5 – The total stresses when $k = 4$, $k_j = 1.6$ and $p < p_a$.

$x_a = \sqrt{\sqrt{3}-1}k = 3.422399 > k_j$. When $x < x_a$, $\frac{\sigma_e^T}{\sigma_y} > \frac{\sigma_\theta^T}{\sigma_y}$; When $x > x_a$, $\frac{\sigma_e^T}{\sigma_y} < \frac{\sigma_\theta^T}{\sigma_y}$; At $x = x_a$, $\frac{\sigma_e^T}{\sigma_y} = \frac{\sigma_\theta^T}{\sigma_y}$.

Within the whole wall, curves 1, 2, and 3 are not parallel to each other because $\sigma_{\mathrm{e}}^{\mathrm{T}}$ is not constant.

Since $p < p_{\mathrm{a}}$, $\sigma_{\mathrm{e}}^{\mathrm{T}} < \sigma_{\mathrm{y}}$ within the whole wall.

There is no x_0^{T} between 1 and k.

Because $p \neq p_{\mathrm{a}}$, the three curves for $\sigma_{\mathrm{z}}^{\mathrm{T}}$, $\sigma_{\theta}^{\mathrm{T}}$, $\sigma_{\mathrm{r}}^{\mathrm{T}}$ are not parallel to each other within the plastic region.

(5) When $k = 4$ and $k_{\mathrm{j}} = 1.6$, $\frac{p}{\sigma_{\mathrm{y}}} = 1.1 > \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$.

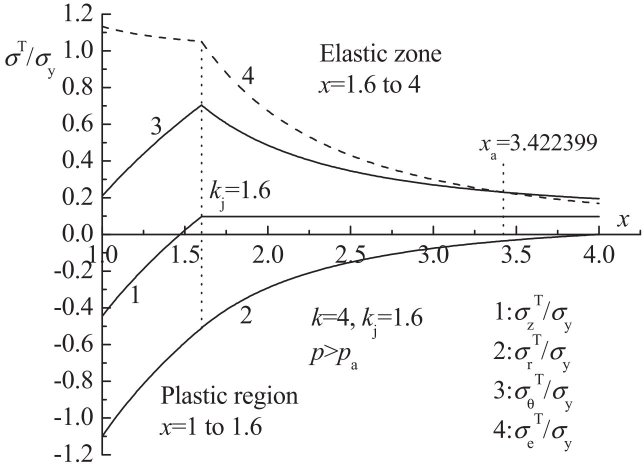

FIG. 3.6 – The total stresses when $k = 4$, $k_{\mathrm{j}} = 1.6$ and $p > p_{\mathrm{a}}$.

$x_{\mathrm{a}} = \sqrt{\sqrt{3} - 1}k = 3.422399 > k_{\mathrm{j}}$. When $x < x_{\mathrm{a}}$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} > \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$; When $x > x_{\mathrm{a}}$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} < \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$; At $x = x_{\mathrm{a}}$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$.

Within the whole wall, curves 1, 2, and 3 are not parallel to each other because $\sigma_{\mathrm{e}}^{\mathrm{T}}$ is not constant.

Since $p > p_{\mathrm{a}}$, $\sigma_{\mathrm{e}}^{\mathrm{T}} > \sigma_{\mathrm{y}}$ within the whole plastic region.

There is no x_0^{T} between 1 and k.

Because $p \neq p_{\mathrm{a}}$, the three curves for $\sigma_{\mathrm{z}}^{\mathrm{T}}$, $\sigma_{\theta}^{\mathrm{T}}$, $\sigma_{\mathrm{r}}^{\mathrm{T}}$ are not parallel to each other within the plastic region.

(6) When $k = 4$ and $k_{\mathrm{j}} = 3.6 > x_{\mathrm{a}}$, $\frac{p}{\sigma_{\mathrm{y}}} = 1.3 < \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$.

When $k_{\mathrm{j}} > x_{\mathrm{a}}$, x_{a}' is within the plastic region, and x_{a} does not exist. When $x < x_{\mathrm{a}}'$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} > \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$; When $x > x_{\mathrm{a}}'$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} < \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$; At $x = x_{\mathrm{a}}'$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 0.955529$. For figure 3.7, $x_{\mathrm{a}}' = 3.463761 > x_{\mathrm{a}}$. At $x = k_{\mathrm{j}}$, $\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1.001982$ and $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 0.958831$.

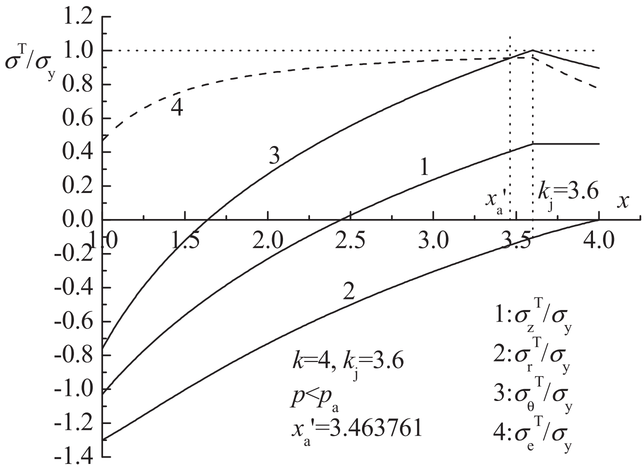

FIG. 3.7 – The total stresses when $k = 4$, $k_j = 3.6$ and $p < p_a$.

Within the whole wall, curves 1, 2, and 3 are not parallel to each other because σ_e^T is not constant.

Since $p < p_a$, $\sigma_e < \sigma_y$ is within the whole plastic region.

There is no x_0^T between 1 and k.

Because $p \neq p_a$, the three curves for σ_z^T, σ_θ^T, σ_r^T are not parallel to each other within the plastic region.

(7) When $k = 4$ and $k_j = 3.6 > x_a$, $\frac{p}{\sigma_y} = 1.7 > \frac{p_a}{\sigma_y}$.

When $k_j > x_a$, x_a' is within the plastic region, and x_a does not exist. When $x < x_a'$, $\frac{\sigma_e^T}{\sigma_y} > \frac{\sigma_\theta^T}{\sigma_y}$; When $x > x_a'$, $\frac{\sigma_e^T}{\sigma_y} < \frac{\sigma_\theta^T}{\sigma_y}$; At $x = x_a'$, $\frac{\sigma_e^T}{\sigma_y} = \frac{\sigma_\theta^T}{\sigma_y} = 1.0171435$. For figure 3.8, $x_a' = 3.461886 > x_a$. At $x = k_j$, $\frac{\sigma_\theta^T}{\sigma_y} = 1.061571$ and $\frac{\sigma_e^T}{\sigma_y} = 1.015853$.

Within the whole wall, curves 1, 2, and 3 are not parallel to each other because σ_e^T is not constant.

Since $p > p_a$, $\sigma_e^T > \sigma_y$ within the whole plastic region.

There is no σ_0^T between 1 and k.

Because $p \neq p_a$, the three curves for σ_z^T, σ_θ^T, σ_r^T are not parallel to each other within the plastic region.

In the above examples, since $p > p_{ri}$, $\frac{\sigma_r^T}{\sigma_y}$ increases monotonously with the increase of x in the plastic region.

(8) When $k = 4$ and $k_j = 1.65$, $\frac{p}{\sigma_y} = \frac{p_{ri}}{\sigma_y} = 0.5160901 < \frac{p_e}{\sigma_y} = 0.541266 < \frac{p_a}{\sigma_y} = 1.057356$.

$x_a = \sqrt{\sqrt{3} - 1}k = 3.422399 > k_j$. When $x < x_a$, $\frac{\sigma_e^T}{\sigma_y} > \frac{\sigma_\theta^T}{\sigma_y}$; When $x > x_a$, $\frac{\sigma_e^T}{\sigma_y} < \frac{\sigma_\theta^T}{\sigma_y}$;

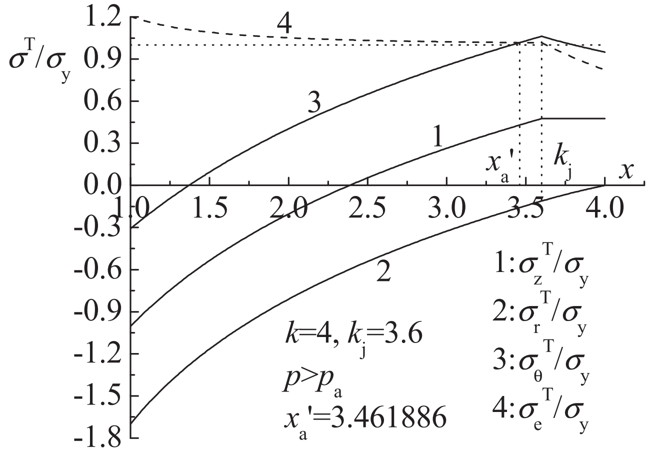

FIG. 3.8 – The total stresses when $k = 4$, $k_j = 3.6$ and $p > p_a$.

At $x = x_a$, $\frac{\sigma_e^T}{\sigma_y} = \frac{\sigma_\theta^T}{\sigma_y}$.

$\frac{\sigma_r^T}{\sigma_y}$ increases monotonously with the increase of x in the plastic region, but it is in a critical state, the intersection point of the three curves for total stresses is just on the vertical axis where $x_0^T = 1$. This is because $p = p_{ri}$. $\frac{\sigma_{ei}^T}{\sigma_y} = 0$ and $\frac{\sigma_r^T}{\sigma_y}$ gets the minimum. The coordinates of the intersection point c are c $(1, -\frac{p}{\sigma_y})$ = c $(1, -\frac{p_{ri}}{\sigma_y})$ = c $(1, -0.5160901)$.

Within the whole wall, curves 1, 2, and 3 are not parallel to each other because $p \neq p_a$, thus σ_e^T is not constant.

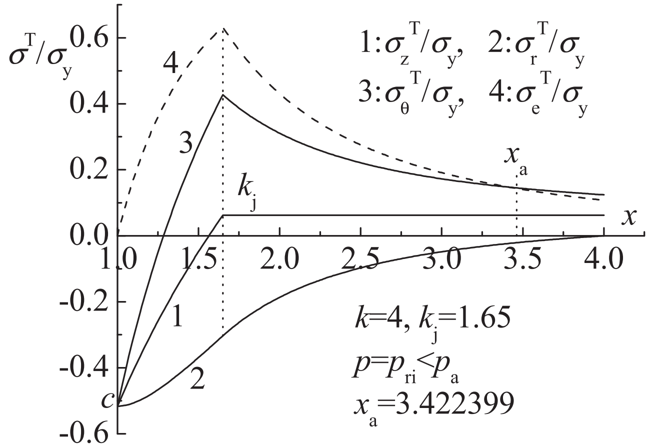

FIG. 3.9 – The total stresses when $k = 4$, $k_j = 1.65$ and $p = p_{ri}$.

Since $p < p_\mathrm{a}$, $\sigma_\mathrm{e}^\mathrm{T} < \sigma_\mathrm{y}$ is within the whole wall.

Because $p \neq p_\mathrm{a}$, the three curves for $\sigma_\mathrm{z}^\mathrm{T}$, σ_θ^T, and $\sigma_\mathrm{r}^\mathrm{T}$ are not parallel to each other within the plastic region.

(9) When $k = 4$ and $k_\mathrm{j} = 1.65$, $\frac{p}{\sigma_\mathrm{y}} = 0.3 < \frac{p'_\mathrm{ri}}{\sigma_\mathrm{y}} = 0.5160901 < \frac{p_\mathrm{e}}{\sigma_\mathrm{y}} = 0.541266 < \frac{p_\mathrm{a}}{\sigma_\mathrm{y}} = 1.057356$.

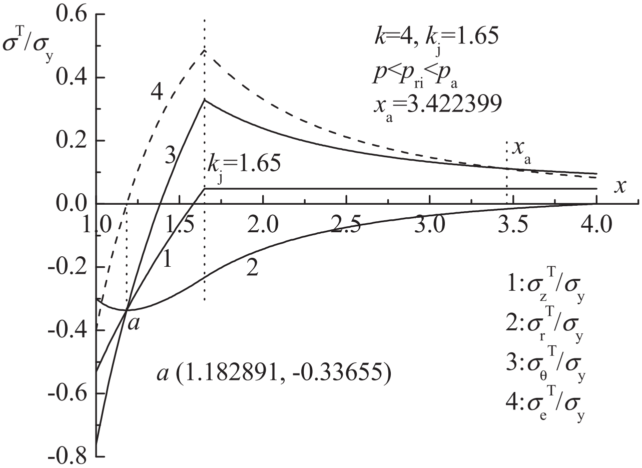

FIG. 3.10 – The total stresses when $k = 4$, $k_\mathrm{j} = 1.65$, and $p < p_\mathrm{ri}$.

$x_\mathrm{a} = \sqrt{\sqrt{3} - 1}k = 3.422399 > k_\mathrm{j}$. When $x < x_\mathrm{a}$, $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} > \frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}}$; When $x > x_\mathrm{a}$, $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} < \frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}}$; At $x = x_\mathrm{a}$, $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}}$.

Since $p < p_\mathrm{ri}$, the three curves for total stresses have an intersection point in the plastic region, and the coordinates of the intersection point a are a $\left(x_0^\mathrm{T}, -\frac{p}{\sigma_\mathrm{y}}\right)$ = a (1.182891, −0.3). When $x = x_0^\mathrm{T} = 1.182891$, $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} = 0$ and $\frac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}}$ gets the minimum (−0.3).

Within the whole wall, curves 1, 2, and 3 are not parallel to each other because $p \neq p_\mathrm{a}$, thus $\sigma_\mathrm{e}^\mathrm{T}$ is not constant.

Since $p < p_\mathrm{a}$, $\sigma_\mathrm{e} < \sigma_\mathrm{y}$ within the whole wall. Because $p \neq p_\mathrm{a}$, the three curves for $\sigma_\mathrm{z}^\mathrm{T}$, σ_θ^T, $\sigma_\mathrm{r}^\mathrm{T}$ are not parallel to each other within the plastic region.

(10) When $k = 4$ and $k_\mathrm{j} = 2$, $\frac{p_\mathrm{e}}{\sigma_\mathrm{y}} = 0.541266 < \frac{p}{\sigma_\mathrm{y}} = 0.6 < \frac{p_\mathrm{ri}}{\sigma_\mathrm{y}} = 0.692124 < \frac{p_\mathrm{a}}{\sigma_\mathrm{y}} = 1.23339$.

$x_\mathrm{a} = \sqrt{\sqrt{3} - 1}k = 3.422399 > k_\mathrm{j}$. When $x < x_\mathrm{a}$, $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} > \frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}}$; When $x > x_\mathrm{a}$, $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} < \frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}}$; At $x = x_\mathrm{a}$, $\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}}$.

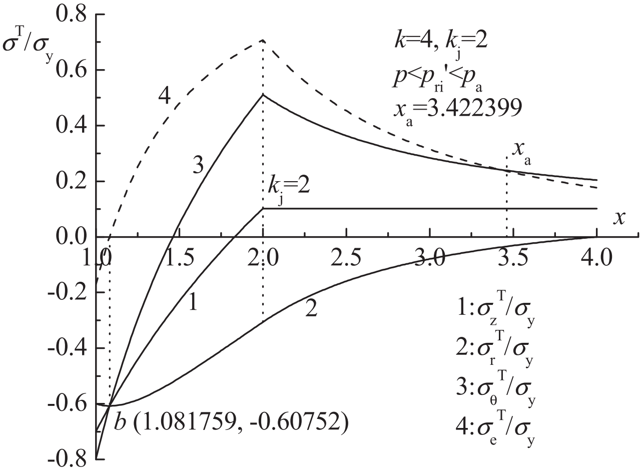

FIG. 3.11 – The total stresses when $k = 4$, $k_j = 2$, and p_{ri}.

Since $p < p_{ri}$, the three curves for total stresses have an intersection point in the plastic region, and the coordinates of the intersection point b are b $(x_0^T, -\frac{p}{\sigma_y})$ = b (1.182891, −0.6). When $x = x_0^T = 1.081759$, $\frac{\sigma_e^T}{\sigma_y} = 0$ and $\frac{\sigma_r^T}{\sigma_y}$ gets the minimum (−0.6).

Within the whole wall, curves 1, 2, and 3 are not parallel to each other because $p \neq p_a$, thus σ_e^T is not constant.

Since $p < p_a$, $\sigma_e < \sigma_y$ within the whole wall.

From the above we see that some stresses exceed the strength limit σ_y, therefore, it is necessary to seek more reasonable control methods and conditions.

3.1.2 The Critical Radius Ratio

For equation (3.13), setting $k_j = k$ leads to

$$\frac{\sigma'_{ei}}{\sigma_y} = 1 - \frac{k^2 \ln k^2}{k^2 - 1} \tag{3.39}$$

For equation (3.27), letting $\frac{\sigma'_{ei}}{\sigma_y} \geq -1$

$$\frac{k^2 \ln k}{k^2 - 1} \leq 1 \tag{3.40}$$

The solution of equation (3.40) is

$$k \leq k_c = 2.218\,457\,489\,916\,7\ldots \tag{3.41}$$

Therefore, when $k \leq k_c$, irrespective of k_j, even if $k_j = k$, $-\sigma_y \leq \sigma'_{ei} \leq 0$.

The results are the same as those based on the maximum shear stress theory (Tresca criterion), see equations (2.32) and (2.33). This is because the equivalent residual stress based on the maximum shear stress theory is the same as that based on the maximum distortion strain energy theory, although the components of residual stresses based on the two strength theories are different.

3.1.3 The Optimum Plastic Depth k_{j*} (k_j is Written as k_{j*})

For equation (3.13), setting $\frac{\sigma'_{\text{ei}}}{\sigma_{\text{y}}} \geq -1$ and writing k_{j} as $k_{\text{j}*}$ obtain the optimum plastic depth $k_{\text{j}*}$

$$k^2 \ln k_{\text{j}*}^2 - k^2 - k_{\text{j}*}^2 + 2 \leq 0 \quad \text{or} \quad k^2 \leq \frac{k_{\text{j}}^2 - 2}{\ln k_{\text{j}}^2 - 1} \quad (\mathrm{e}^{0.5} \leq k_{\text{j}*} \leq k_{\text{c}} \text{ and } k \geq k_{\text{c}}) \tag{3.42}$$

Equation (3.42) is just equation (2.46). Thus, it can be seen that although the components of residual stresses based on the maximum distortion strain energy theory (Mises's yield criterion) are different from those based on the maximum shear stress theory (Tresca yield criterion), the optimum plastic depth $k_{\text{j}*}$ is the same.

From equation (3.42), when $k \to \infty$, $k_{\text{j}*} = \sqrt{\mathrm{e}}$; letting $k_{\text{j}*} = k$ just results in equation (3.40).

When $k \geq k_{\text{c}}$, if $k_{\text{j}} \leq k_{\text{j}*}$, *i.e.*, the plastic depth determined by equation (3.42), $-\sigma_{\text{y}} \leq \sigma'_{\text{ei}} \leq 0$. When $k \leq k_{\text{c}}$, even if $k_{\text{j}} = k$, there must be that $-\sigma_{\text{y}} \leq \sigma'_{\text{ei}} \leq 0$. The optimum plastic depth $k_{\text{j}*}$ is illustrated in figure 3.12, which is the same as figure 2.9. From figure 3.12 and equation (3.42), we know that when $k \geq k_{\text{c}}$, as k increases, $k_{\text{j}*}$ decreases, this means that for $\sigma'_{\text{ei}} \geq -\sigma_{\text{y}}$, thicker pressure vessels need shallower plastic depth.

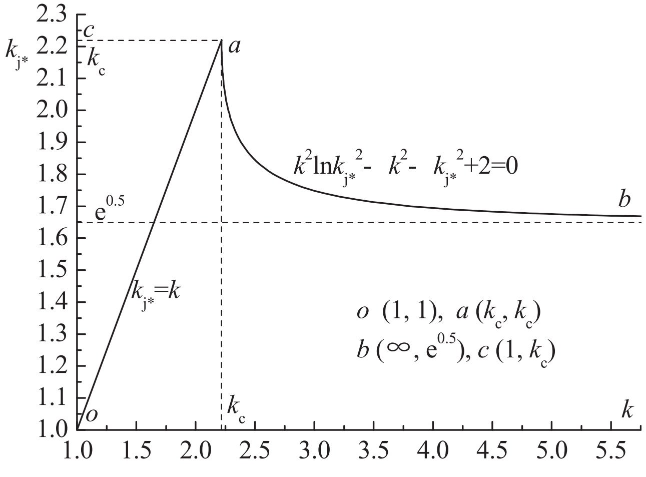

FIG. 3.12 – The optimum plastic depth $k_{\text{j}*}$.

In figure 3.12, point a is the intersection of straight line oa $(k_{\mathrm{j}*} = k)$ and curve ab $(k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 \leq 0)$, setting $k_{\mathrm{j}*} = k$ in the equation $(k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 \leq 0)$ just obtains $\frac{k^2 \ln k}{k^2-1} \leq 1$. The coordinates of point a are $a\,(k_{\mathrm{c}}, k_{\mathrm{c}})$; point b is at infinity; thus, the coordinates of point b are $b\,(\infty, \sqrt{\mathrm{e}})$.

In fact, $\frac{k^2 \ln k}{k^2-1} = 1$ is equivalent to $2p_{\mathrm{e}} = p_{\mathrm{y}}$, *i.e.*,

$$2\frac{k^2-1}{2k^2} = \ln k \,\text{or}\, 2\frac{k^2-1}{\sqrt{3}k^2} = \frac{2\ln k}{\sqrt{3}},$$

where p_{e} is the maximum elastic load-bearing capability or initial yield pressure, and p_{y} is the entire yield pressure.

Interestingly enough, letting $\frac{p_{\mathrm{ri}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\ln k_{\mathrm{j}} - \frac{k_{\mathrm{j}}^2-1}{\sqrt{3}k^2} = \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{\sqrt{3}k^2}$ just results in equation (3.42). When $k_{\mathrm{j}} < k_{\mathrm{j}*}$, $\frac{p_{\mathrm{ri}}}{\sigma_{\mathrm{y}}} < \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$; when $k_{\mathrm{j}} > k_{\mathrm{j}*}$, $\frac{p_{\mathrm{ri}}}{\sigma_{\mathrm{y}}} > \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$. Therefore, when $k_{\mathrm{j}} \leq k_{\mathrm{j}*}$, if $\frac{p}{\sigma_{\mathrm{y}}} > \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$, there must be $\frac{p}{\sigma_{\mathrm{y}}} > \frac{p_{\mathrm{ri}}}{\sigma_{\mathrm{y}}}$, thus, $\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ increases monotonously with the increase of x in the plastic region.

3.1.4 The Results When $k_j = k_{j^*}$ *or* $k^2 \ln k_{j^*}^2 - k^2 - k_{j^*}^2 + 2 = 0$

1. The components of residual stresses

Within the plastic region $(1 \leq x \leq k_{\mathrm{j}*})$

$$\frac{\sigma_{\mathrm{z}}'}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\ln x - \frac{1}{\sqrt{3}} \tag{3.43}$$

$$\frac{\sigma_{\mathrm{r}}'}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\left(\ln x + x^{-2} - 1\right) \tag{3.44}$$

$$\frac{\sigma_{\theta}'}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\left(\ln x - x^{-2}\right) \tag{3.45}$$

$$\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} = 1 - \frac{2}{x^2} \tag{3.46}$$

Within the elastic region $(k_{\mathrm{j}*} \leq x \leq k)$

$$\frac{\sigma_{\mathrm{z}}'}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}*}^2 - 2}{\sqrt{3}k^2} \tag{3.47}$$

$$\frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = \left(1 - \frac{k^2}{x^2}\right)\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} \tag{3.48}$$

$$\frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}} = \left(1 + \frac{k^2}{x^2}\right)\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} \tag{3.49}$$

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}*}^2 - 2}{x^2} > 0 \tag{3.50}$$

The complex formulas, or equations (3.2)–(3.9), are significantly simplified. On the inner surface, equations (3.43)–(3.46) become:

$$\frac{\sigma'_{\mathrm{zi}}}{\sigma_{\mathrm{y}}} = -\frac{1}{\sqrt{3}} \tag{3.51}$$

$$\sigma'_{\mathrm{ri}} = 0 \tag{3.52}$$

$$\frac{\sigma'_{\theta\mathrm{i}}}{\sigma_{\mathrm{y}}} = -\frac{2}{\sqrt{3}} < -1 \tag{3.53}$$

$$\frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}} = -1 \tag{3.54}$$

Therefore, if plastic depth $k_{\mathrm{j}} = k_{\mathrm{j}*}$, the residual stresses at the inner surface are always compressive and constant, which have nothing to do with k and k_{j}, or have nothing to do with the thickness and overstrain of a pressure vessel. The equivalent residual stress at inner surface just reaches $-\sigma_{\mathrm{y}}$. However, $|\sigma'_{\theta\mathrm{i}}|$ exceeds yield strength σ_{y}. This is inevitable, because $\sigma_{\mathrm{e}} = \frac{\sqrt{3}}{2}(\sigma_{\theta} - \sigma_{\mathrm{r}})$, at the inner surface, $\sigma'_{\mathrm{ri}} = 0$, then $\sigma'_{\mathrm{ei}} = \frac{\sqrt{3}}{2}\sigma'_{\theta\mathrm{i}}$, when $\frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}} = -1$, there must be $\frac{\sigma'_{\theta\mathrm{i}}}{\sigma_{\mathrm{y}}} = -\frac{2}{\sqrt{3}} < -1$. We will present the research results when σ'_{θ} is controlled in the next chapter (chapter 4).

On the elastic-plastic juncture, equations (3.43)–(3.46) become:

$$\frac{\sigma'_{\mathrm{zj}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\ln k_{\mathrm{j}*} - \frac{1}{\sqrt{3}} = \frac{k_{\mathrm{j}*}^2 - 2}{\sqrt{3}k^2} \tag{3.55}$$

$$\frac{\sigma'_{\mathrm{rj}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\left(\ln k_{\mathrm{j}*} + k_{\mathrm{j}*}^{-2} - 1\right) \tag{3.56}$$

$$\frac{\sigma'_{\theta\mathrm{j}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\left(\ln k_{\mathrm{j}*} - k_{\mathrm{j}*}^{-2}\right) \tag{3.57}$$

$$\frac{\sigma'_{\mathrm{ej}}}{\sigma_{\mathrm{y}}} = 1 - \frac{2}{k_{\mathrm{j*}}^2} \tag{3.58}$$

2. The optimum load-bearing capability

When the load exerted on the pressure vessel is just autofrettage pressure p_{a}, $\sigma'_{\mathrm{ei}} \equiv \sigma_{\mathrm{y}}$ within the whole plastic region irrespective of k and k_{j}. When $k \geq k_{\mathrm{c}}$ and $k_{\mathrm{j}} = k_{\mathrm{j*}}$, autofrettage pressure p_{a} (see equation (3.1)) becomes $\frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$. Then, when $k \geq k_{\mathrm{c}}$ and $k_{\mathrm{j}} = k_{\mathrm{j*}}$, the optimum load-bearing capability is

$$\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = 2\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2} \quad (k \geq k_c) \tag{3.59}$$

When $k \to \infty$, $\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = 2\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2} \to \frac{2}{\sqrt{3}}$, and $\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}\frac{k^2-1}{k^2} \to \frac{1}{\sqrt{3}}$.

When $k \leq k_{\mathrm{c}}$, $k_{\mathrm{j}} = k_{\mathrm{j*}} = k$, autofrettage pressure p_{a} or equation (3.1) becomes $\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\ln k$. Then, the optimum load-bearing capability is

$$\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\ln k = \frac{p_{\mathrm{y}}}{\sigma_{\mathrm{y}}} \quad (k \leq k_c) \tag{3.60}$$

The optimum load-bearing capability is illustrated as curve *oad* in figure 3.13, where the coordinates of some key points are marked.

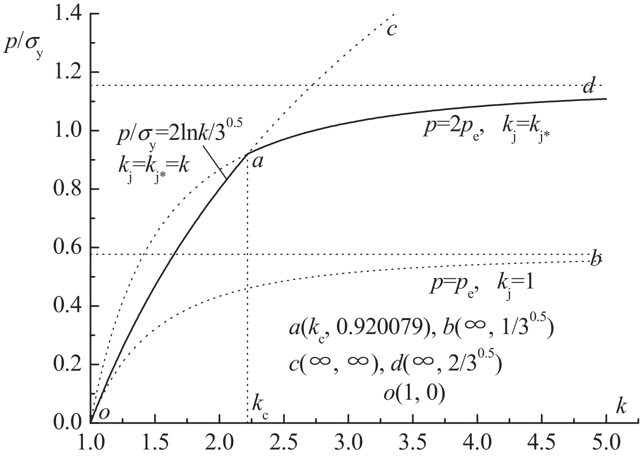

FIG. 3.13 – The optimum load-bearing capacity of an autofrettaged pressure vessel.

In figure 3.13, point a is the crossover point of curve oa $\left(\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\ln k = \frac{p_{\mathrm{y}}}{\sigma_{\mathrm{y}}}\right)$ and curve ad $\left(\frac{p}{\sigma_{\mathrm{y}}} = 2\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}\right)$, then, letting $\frac{2}{\sqrt{3}}\ln k = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$ results in $\frac{k^2\ln k}{k^2-1} = 1$, as is just equation (3.40). Therefore, the abscissa of point a is $k_{\rm c}$, and when $k = k_{\rm c}$, $\frac{2}{\sqrt{3}}\ln k = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2} = 0.920079\ldots$. When $k < k_{\rm c}$, $\frac{2}{\sqrt{3}}\ln k < \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$; When $k > k_{\rm c}$, $\frac{2}{\sqrt{3}}\ln k > \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$.

Points b and d are at infinity; thus, the coordinates of points b and d are $b\ (\infty, \frac{1}{\sqrt{3}})$ and $d\ (\infty, \frac{2}{\sqrt{3}})$, respectively.

3. The location (defined as x_0) of the intersection of the three residual stress curves or the abscissa at which $\sigma_e' = 0$, or x_0

When plastic depth $k_{\rm j}$=$k_{{\rm j}^*}$, equation (3.22) becomes

$$x_0 \equiv \sqrt{2} \tag{3.61}$$

In fact, for equation (3.46), letting $\sigma_e' = 0$ just results in $x = x_0 \equiv \sqrt{2}$. x_0 is a constant and is not related to k and $k_{\rm j}$.

The value of residual stresses when $x = x_0$ or the minimum radial residual stress is

$$y_0^{\rm d} = \frac{\sigma_{\rm r}'}{\sigma_{\rm y}} = \frac{\sigma_{\rm z}'}{\sigma_{\rm y}} = \frac{\sigma_{\theta}'}{\sigma_{\rm y}} = \frac{\sigma_{\rm rmin}'}{\sigma_{\rm y}} = \frac{\sigma_{\rm rx0}'}{\sigma_{\rm y}} \equiv -\frac{1-\ln 2}{\sqrt{3}} = \text{constant} \tag{3.62}$$

$y_0^{\rm d}$ is a constant and not related to k and $k_{\rm j}$. $y_0^{\rm d} = \frac{2}{\sqrt{3}}\, y_0^{\rm s}$.

That is to say, when plastic depth $k_{\rm j} = k_{{\rm j}^*}$, for any k, the three curves for the residual stress collect at the same point within the plastic region and the intersection is:

$$\left(\sqrt{2}, -\frac{1-\ln 2}{\sqrt{3}}\right) = (1.414214\ldots, -0.17716\ldots)$$

4. The location (x, defined as x_1) where $\sigma_{\rm z}' = 0$, or x_1

When plastic depth $k_{\rm j} = k_{{\rm j}^*}$, equation (3.24) becomes

$$x_1 \equiv \sqrt{\rm e} \tag{3.63}$$

In fact, for equation (3.43), letting $\sigma_{\rm z}' = 0$ just results in $x = x_1 \equiv \sqrt{\rm e}$. x_1 is a constant and is not related to k and $k_{\rm j}$.

When $k = 4$, from equation (3.42), $k_{{\rm j}^*} = 1.694172$, the residual stresses are illustrated in figure 3.14 to summarize and validate the above-mentioned results. The meaning of each curve and main parameters have been marked in the figure.

5. The components of total stresses when $k_{\rm j} \geq k_{\rm c}$, plastic depth $k_{\rm j} = k_{{\rm j}^*}$, and load $p = p_{\rm a} = 2p_{\rm e}$.

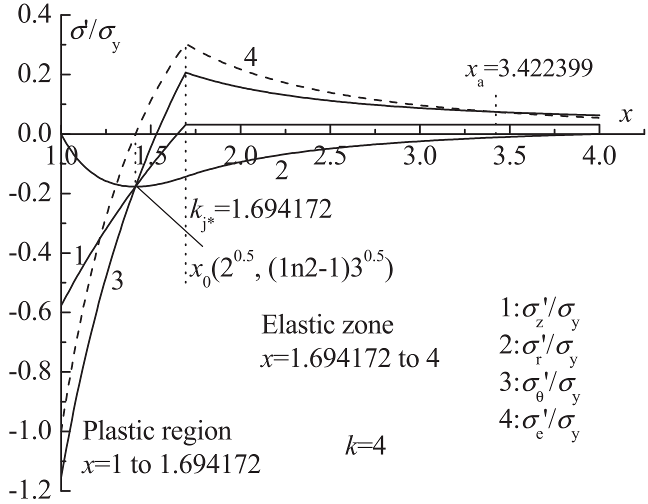

FIG. 3.14 – The residual stresses when $k_j = k_{j*}$.

Within the plastic region ($1 \leq x \leq k_{j*}$)

$$\frac{\sigma_z^T}{\sigma_y} = \frac{\ln x^2}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}k^2} \tag{3.64}$$

$$\frac{\sigma_r^T}{\sigma_y} = \frac{\ln x^2}{\sqrt{3}} - \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}k^2} \tag{3.65}$$

$$\frac{\sigma_\theta^T}{\sigma_y} = \frac{\ln x^2}{\sqrt{3}} + \frac{2}{\sqrt{3}k^2} \tag{3.66}$$

$$\frac{\sigma_e^T}{\sigma_y} \equiv 1 \tag{3.67}$$

Clearly, σ_z^T, σ_r^T, and σ_θ^T increase as x increases.

Setting $\frac{\sigma_{ri}^T}{\sigma_y} = -\frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}k^2} = -\frac{2}{\sqrt{3}}\frac{k^2-1}{k^2} = -\frac{p}{\sigma_y} \geq -1$ results in

$$k \leq \sqrt{2(2+\sqrt{3})} = \sqrt{(1+\sqrt{3})^2} = 1+\sqrt{3} = 2.732051 = k_r \tag{3.68}$$

When $k \leq k_r = 2.732051$, $|\sigma_{ri}^T|$ can not reach σ_y.
When $k = k_r = 2.732051$, $\frac{p}{\sigma_y} = 2\frac{p_e}{\sigma_y} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2} = 1$.

Setting $\frac{\sigma_{\theta\mathrm{i}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}k^2} \leq 1$ results in

$$k \geq \sqrt{\frac{2}{\sqrt{3}}} = 1.07457 = k_\theta \tag{3.69}$$

Therefore, usually $\sigma_\theta^{\mathrm{T}}$ can not reach σ_{y}.

For equations (3.66) and (3.67), setting $\sigma_{\mathrm{e}}^{\mathrm{T}} \leq \sigma_\theta^{\mathrm{T}}$ obtains

$$\begin{aligned} x &\geq \exp\left(\frac{\sqrt{3}k^2 - 2}{2k^2}\right) = \exp\left(\frac{\sqrt{3}}{2} - \frac{1}{k^2}\right) = \exp\left(\frac{k^2 - 1}{k^2} - \frac{2 - \sqrt{3}}{2}\right) \\ &= \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}} - \frac{2 - \sqrt{3}}{2}\right) = x_{\mathrm{z}} \end{aligned} \tag{3.70}$$

When $k = 1$, $x_{\mathrm{z}} = \exp\left(\frac{\sqrt{3}}{2} - 1\right) = 0.874612$; When $k = 2.6$, $x_{\mathrm{z}} = \exp\left(\frac{\sqrt{3}}{2} - \frac{1}{2.6\times2.6}\right) = 2.050526$; When $k = \infty$, $x_{\mathrm{z}} = \exp\left(\frac{\sqrt{3}}{2}\right) = 2.377443$.

Within the elastic region ($k_{\mathrm{j}*} \leq x \leq k$)

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}}^2}{\sqrt{3}k^2} = \text{constant} \tag{3.71}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = -\frac{k_{\mathrm{j}}^2}{\sqrt{3}}\left(\frac{1}{x^2} - \frac{1}{k^2}\right) \tag{3.72}$$

$$\frac{\sigma_\theta^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}}^2}{\sqrt{3}}\left(\frac{1}{x^2} + \frac{1}{k^2}\right) \tag{3.73}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}}^2}{x^2} \tag{3.74}$$

Clearly, $\sigma_{\mathrm{e}}^{\mathrm{T}}$ and $\sigma_\theta^{\mathrm{T}}$ decrease as x increases, and $\sigma_{\mathrm{r}}^{\mathrm{T}}$ increases as x increases.

Letting $\frac{\sigma_{\theta\mathrm{j}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}} + \frac{k_{\mathrm{j}}^2}{\sqrt{3}k^2} \leq 1$ leads to $k_{\mathrm{j}} \leq \sqrt{\sqrt{3} - 1}k = x_{\mathrm{a}}$. This shows again that if $k_{\mathrm{j}} \leq x_{\mathrm{a}}$, $\sigma_{\theta\mathrm{j}}^{\mathrm{T}} \leq \sigma_{\mathrm{ej}}^{\mathrm{T}}$.

In order for σ'_{ei} to be greater than σ_{y}, we have gotten $k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 \leq 0$ ($\mathrm{e}^{0.5} \leq k_{\mathrm{j}*} \leq k_{\mathrm{c}}$ and $k \geq k_{\mathrm{c}}$). If $k_{\mathrm{j}} > k_{\mathrm{j}*}$, or the actual value of plastic depth k_{j} is larger than the value calculated by $k^2 \ln k_{\mathrm{j}}^2 - k^2 - k_{\mathrm{j}}^2 + 2 = 0$, there will be $k^2 \ln k_{\mathrm{j}}^2 - k^2 - k_{\mathrm{j}}^2 + 2 > 0$. Then, substituting $k_{\mathrm{j}} = \sqrt{\sqrt{3} - 1}k$ into $k^2 \ln k_{\mathrm{j}}^2 - k^2 - k_{\mathrm{j}}^2 + 2$ obtains

$$k^2 \ln[(\sqrt{3} - 1)k^2] - \sqrt{3}k^2 + 2 \tag{3.75}$$

Equation (3.75) is plotted in figure 3.15.

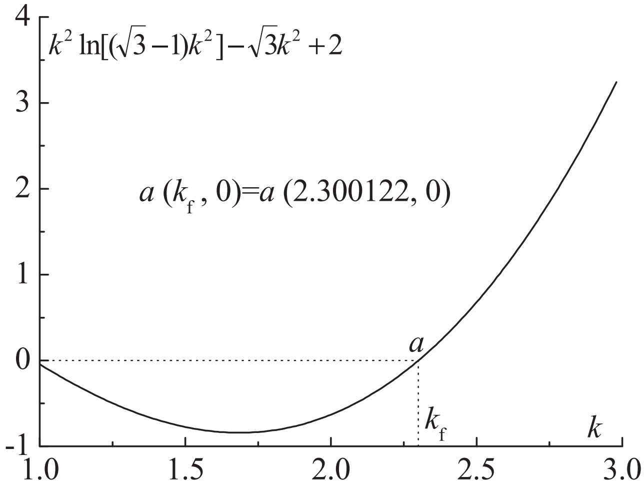

FIG. 3.15 – The graph of $k^2\ln[(\sqrt{3}-1)k^2]-\sqrt{3}k^2+2$.

Setting $k^2\ln[(\sqrt{3}-1)k^2]-\sqrt{3}k^2+2\geq 0$ (see figure 3.27) results in

$$k \geq 2.300122\ldots = k_f \tag{3.76}$$

That is to say, when $k > k_f$, $k_{j^*} < \sqrt{\sqrt{3}-1}k = k_j$, thus $\sigma^T_{\theta j} < \sigma_y$; when $k < k_f$, $k_{j^*} > \sqrt{\sqrt{3}-1}k = k_j$, thus $\sigma^T_{\theta j} > \sigma_y$. For example, when $k = 2.25$, $k_{j^*} = 2.046308$, $\sqrt{\sqrt{3}-1}k = 1.925099 < k_{j^*} = 2.046308$, $\frac{\sigma^T_{\theta j}}{\sigma_y} = 1.054898 > 1$; when $k = 2.5$, $k_{j^*} = 1.844363$, $\sqrt{\sqrt{3}-1}k = 2.138999 > k_{j^*} = 1.844363$, $\frac{\sigma^T_{\theta j}}{\sigma_y} = 0.891584 < 1$.

When $k^2\ln[(\sqrt{3}-1)k^2]-\sqrt{3}k^2+2=0$, or $k = k_f = 2.300122\ldots$, equation (3.70) becomes $x_z = x_a = k_{j^*} = 1.967984$.

Letting $\frac{\sigma^T_{\theta i}}{\sigma_y} = \frac{2}{\sqrt{3}k^2} \leq 1$ leads to $k \geq \sqrt{\frac{2}{\sqrt{3}}} = 1.07457\ldots = k_\theta$. Therefore, usually $\sigma^T_{\theta i}$ can not reach σ_y.

To confirm the above conclusions, we present several total stress figures as follows.

The total stresses for $k = 4 > k_r$, $k_j = k_{j^*}$ and $p = p_a = 2p_e$ are illustrated in figure 3.16.

The total stresses for $k = 2.25 < k_f$, $k_j = k_{j^*}$ and $p = p_a = 2p_e$ are illustrated in figure 3.17.

The total stresses for $k = k_r$, $k_j = k_{j^*}$ and $p = p_a = 2p_e$ are illustrated in figure 3.18.

The total stresses for $k = k_f$, $k_j = k_{j^*}$ and $p = p_a = 2p_e$ are illustrated in figure 3.19.

The meaning of each curve and main parameters have been marked in the figures.

When $k = k_\mathrm{r} = 2.732051$, from equation (3.42) or equation (2.34), $k_{\mathrm{j}*} = 1.785136$; When $k = k_\mathrm{f} = 2.300122$, from (3.42) or equation (2.34), $k_{\mathrm{j}*} = 1.967984$.

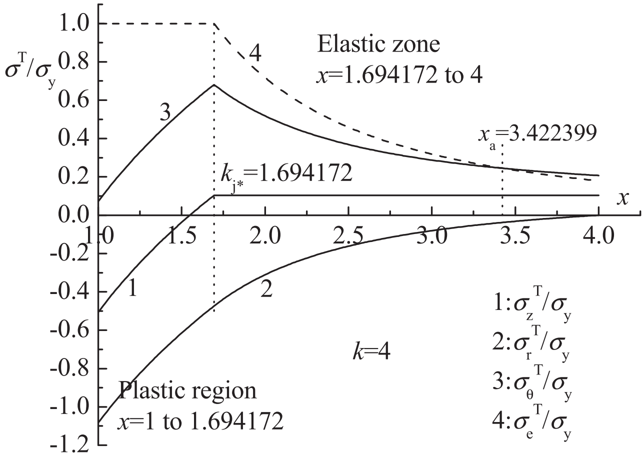

FIG. 3.16 – The total stresses with $k = 4$, $k_\mathrm{j} = k_{\mathrm{j}*}$ and $p = p_\mathrm{a} = 2p_\mathrm{e}$.

In figures 3.14 and 3.16, $x_\mathrm{a} = \sqrt{\sqrt{3}-1}k = 0.8556\ k = 3.422399 > k_{\mathrm{j}*}$. Therefore, x_a exists and x_z does not exist. In other words, if $k_\mathrm{j} < x_\mathrm{a}$, there is $\sigma_\theta^\mathrm{T} = \sigma_\mathrm{e}^\mathrm{T} = \sigma_\mathrm{y}$ in the elastic region, and when $\sigma_\theta^\mathrm{T} = \sigma_\mathrm{e}^\mathrm{T} = \sigma_\mathrm{y}$, $x = x_\mathrm{a}$, see figures 3.14 and 3.16; if $k_\mathrm{j} > x_\mathrm{a}$, there is $\sigma_\theta^\mathrm{T} = \sigma_\mathrm{e}^\mathrm{T} = \sigma_\mathrm{y}$ in the plastic region, and when $\sigma_\theta^\mathrm{T} = \sigma_\mathrm{e}^\mathrm{T} = \sigma_\mathrm{y}$, $x = x_\mathrm{z}$, see figure 3.17; if $k_\mathrm{j} = x_\mathrm{a}$, there is $\sigma_{\theta\mathrm{j}}^\mathrm{T} = \sigma_{\mathrm{ej}}^\mathrm{T} = \sigma_\mathrm{y}$ at the elastic-plastic juncture where $x = x_\mathrm{a} = x_\mathrm{z}$, see figure 3.19. Because $k = 4 > k_\mathrm{r}$, $p > \sigma_\mathrm{y}$, and $\sigma_{\mathrm{ri}}^\mathrm{T} < -\sigma_\mathrm{y}$. Because $k > k_\mathrm{f}$, $k_{\mathrm{j}*} < \sqrt{\sqrt{3}-1}k$, thus $\sigma_{\theta\mathrm{i}}^\mathrm{T} < \sigma_\mathrm{y}$.

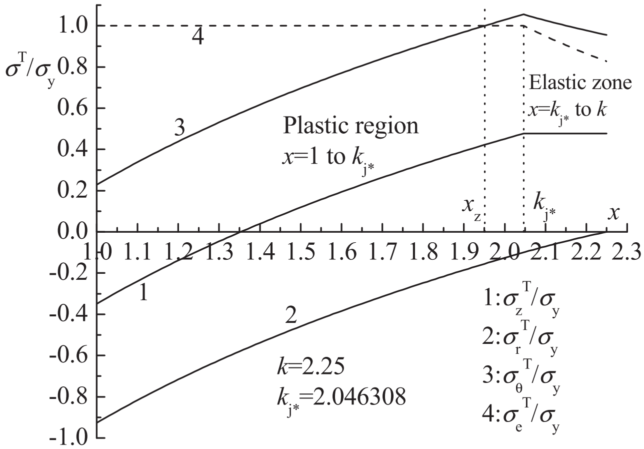

FIG. 3.17 – The total stresses with $k = 2.25$, $k_\mathrm{j} = k_{\mathrm{j}*}$ and $p = p_\mathrm{a} = 2p_\mathrm{e}$.

In figure 3.17, from equation (3.70), $x_z = 1.951298$; $k_{j*} = 2.046308 > x_a = 1.925099$, so, x_z exists and x_a does not exist. Because $k = 2.25 < k_r$, $p < \sigma_y$ and $\sigma_{ri}^T > -\sigma_y$. Because $k < k_f$, $k_{j*} > \sqrt{\sqrt{3} - 1}k$, thus $\sigma_{\theta j}^T > \sigma_y$.

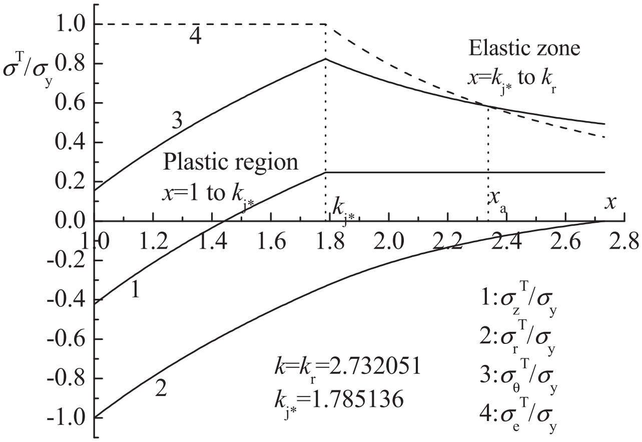

FIG. 3.18 – The total stresses with $k = k_r$, $k_j = k_{j*}$ and $p = p_a = 2p_e$.

In figure 3.18, $x_a = 2.337542$; $k_{j*} = 1.785136 < x_a$, so, x_a exists and x_z does not exist. Because $k = k_r$, $p = \sigma_y$ and $\sigma_{ri}^T = -\sigma_y$. Because $k > k_f$, $k_{j*} < \sqrt{\sqrt{3} - 1}k$, $\sigma_{ri}^T < \sigma_y$.

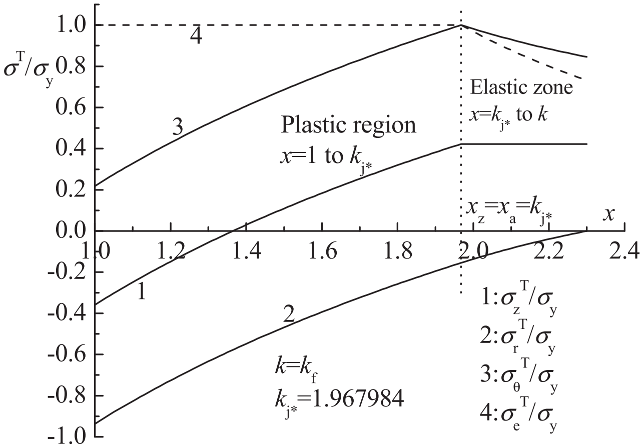

FIG. 3.19 – The total stresses with $k = k_f$, $k_j = k_{j*}$ and $p = p_a = 2p_e$.

In figure 3.19, since $k = k_f$, at $k_j = k_{j*}$, $\sigma_{\theta j}^T$ is just σ_{ej}^T, and they are just σ_y, and $x_z = x_a = k_{j*}$.

When based on the maximum distortion strain energy theory (Mises's yield criterion), sometimes $\frac{\sigma'_{\theta\mathrm{i}}}{\sigma_\mathrm{y}} = -\frac{2}{\sqrt{3}} < -1$ and $\sigma_\theta^\mathrm{T} > \sigma_\mathrm{y}$, we will solve the problem in the next chapter (chapter 4).

3.2 Mechanical Autofrettage Technology Under Entire Yield State

3.2.1 The Residual Stresses

As mentioned above, when $k \leq k_\mathrm{c}$, the plastic depth k_j can be k. When $k_\mathrm{j} = k$, or overstrain $\varepsilon = 100\%$, there is only a plastic zone and no elastic zone within the whole wall of a pressure vessel, then, the components of residual stresses, or equations (3.2)–(3.5) become equations (3.77)–(3.80).

Within the whole wall ($1 \leq x \leq k$)

$$\frac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} = \frac{1}{\sqrt{3}}\left(1 + \ln\frac{x^2}{k^2} - \frac{\ln k^2}{k^2-1}\right) \tag{3.77}$$

$$\frac{\sigma'_\mathrm{r}}{\sigma_\mathrm{y}} = \frac{1}{\sqrt{3}}\left(\ln\frac{x^2}{k^2} - \frac{\ln k^2}{k^2-1} + \frac{k^2\ln k^2}{(k^2-1)x^2}\right) \tag{3.78}$$

$$\frac{\sigma'_\theta}{\sigma_\mathrm{y}} = \frac{1}{\sqrt{3}}\left(2 + \ln\frac{x^2}{k^2} - \frac{\ln k^2}{k^2-1} - \frac{k^2\ln k^2}{(k^2-1)x^2}\right) \tag{3.79}$$

$$\frac{\sigma'_\mathrm{e}}{\sigma_\mathrm{y}} = 1 - \frac{k^2\ln k^2}{(k^2-1)x^2} \tag{3.80}$$

Clearly, σ'_z, σ'_e and σ'_θ increase as x increases.

$x_0 = \frac{r}{r_\mathrm{i}} = \sqrt{\frac{k^2 - k_\mathrm{j}^2 + k^2\ln k_\mathrm{j}^2}{k^2-1}}$ becomes $x_0 = \sqrt{\frac{k^2\ln k^2}{k^2-1}}$. When $x < x_0 = \sqrt{\frac{k^2\ln k^2}{k^2-1}}$, σ'_r decreases as x increases; When $x > x_0 = \sqrt{\frac{k^2\ln k^2}{k^2-1}}$, σ'_r increases as x increases.

At the inner surface, or $x = 1$, $\frac{\sigma'_\mathrm{ei}}{\sigma_\mathrm{y}} = 1 - \frac{k^2\ln k^2}{k^2-1}$, which is equation (3.27).

When $\frac{\sigma'_\mathrm{ei}}{\sigma_\mathrm{y}} = -1$, $\frac{k^2\ln k}{k^2-1} = 1$, therefore, under entire yield, or $k_\mathrm{j} = k$, for $\sigma'_\mathrm{ei} \geq -\sigma_\mathrm{y}$, there must be $k \leq k_\mathrm{c}$; and when $\frac{k^2\ln k}{k^2-1} = 1$, $\frac{\sigma'_{\theta\mathrm{i}}}{\sigma_\mathrm{y}} = -\frac{2}{\sqrt{3}}\frac{k^2\ln k}{k^2-1} < -1$, or when $\left|\sigma'_\mathrm{ei}\right| = \sigma_\mathrm{y}$, $\left|\sigma'_{\theta\mathrm{i}}\right| > \sigma_\mathrm{y}$. We will solve the problem in the next chapter (chapter 4).

Clearly, $\frac{\sigma'_\mathrm{eo}}{\sigma_\mathrm{y}} > 0$.

Figures 3.20 and 3.21 show the distributions of the residual stresses for $k = k_c$ and $k = 2$, respectively. The meaning of each curve and main parameters have been marked in the figures.

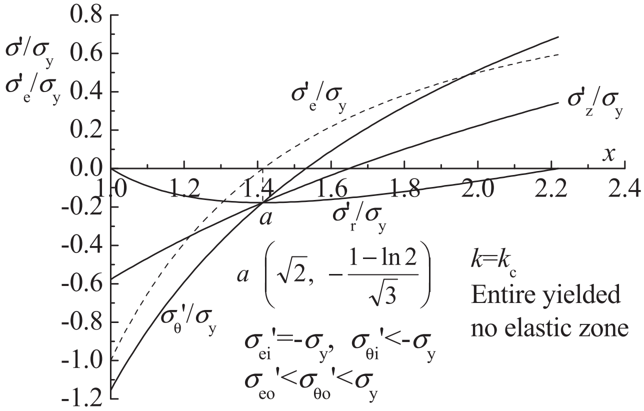

FIG. 3.20 – The distributions of the residual stresses for $k = k_c$ and $k_j = k$.

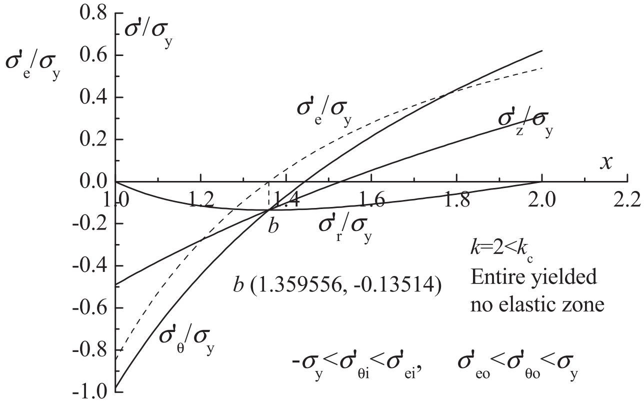

FIG. 3.21 – The distributions of the residual stresses for $k = 2$ and $k_j = k$.

Since $x_0 = \sqrt{\frac{k^2 \ln k^2}{k^2-1}}$ when $k_j = k$, if $k = k_c$, $\frac{k^2 \ln k}{k^2-1} = 1$, then, $x_0 = \sqrt{2}$ and $y_0^d = \frac{\sigma'_{zx0}}{\sigma_y} = \frac{\sigma'_{\theta x0}}{\sigma_y} = \frac{\sigma'_{rx0}}{\sigma_y} = -\frac{1-\ln 2}{\sqrt{3}}$. Therefore, the coordinate of the intersections of the three curves for the residual stresses in figure 3.20 is a$\left(\sqrt{2}, -\frac{1-\ln 2}{\sqrt{3}}\right)$. The ordinate of the intersections of the three curves for the residual stresses in figure 3.21 ($k \neq k_c$) is $\frac{\sigma'_{zx0}}{\sigma_y} = \frac{\sigma'_{\theta x0}}{\sigma_y} = \frac{\sigma'_{rx0}}{\sigma_y} = \frac{1}{\sqrt{3}} \ln \frac{\ln k^2}{k^2-1} + \frac{k^2-1-\ln k^2}{\sqrt{3}(k^2-1)} = -0.13514\ldots$ The abscissa of the intersections of the three curves for the residual stresses in figure 3.21 ($k \neq k_c$) is $x_0 = \sqrt{\frac{k^2 \ln k^2}{k^2-1}} = 1.359556\ldots$

3.2.2 The Total Stresses

The total stresses within the whole wall ($1 \leq x \leq k$) when the plastic depth $k_{\mathrm{j}} = k$.

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}\left(1 + \ln\frac{x^2}{k^2} - \frac{\ln k^2}{k^2-1}\right) + \frac{1}{k^2-1}\frac{p}{\sigma_{\mathrm{y}}} \tag{3.81}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}\left(\ln\frac{x^2}{k^2} - \frac{\ln k^2}{k^2-1} + \frac{k^2\ln k^2}{(k^2-1)x^2}\right) - \frac{k^2-x^2}{x^2(k^2-1)}\frac{p}{\sigma_{\mathrm{y}}} \tag{3.82}$$

$$\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}\left(2 + \ln\frac{x^2}{k^2} - \frac{\ln k^2}{k^2-1} - \frac{k^2\ln k^2}{(k^2-1)x^2}\right) + \frac{k^2+x^2}{x^2(k^2-1)}\frac{p}{\sigma_{\mathrm{y}}} \tag{3.83}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2\ln k^2}{(k^2-1)x^2} + \frac{\sqrt{3}k^2}{x^2(k^2-1)}\frac{p}{\sigma_{\mathrm{y}}} = 1 + \left(\frac{\sqrt{3}k^2}{k^2-1}\frac{p}{\sigma_{\mathrm{y}}} - \frac{k^2\ln k^2}{k^2-1}\right)\frac{1}{x^2} \tag{3.84}$$

From equation (3.84), when $\frac{p}{\sigma_{\mathrm{y}}} \leq \frac{2}{\sqrt{3}}\ln k = \frac{p_{\mathrm{y}}}{\sigma_{\mathrm{y}}}$, the larger x is, the larger $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ is. Setting $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \leq 1$ just obtains $\frac{p}{\sigma_{\mathrm{y}}} \leq \frac{2}{\sqrt{3}}\ln k = \frac{p_{\mathrm{y}}}{\sigma_{\mathrm{y}}}$. That is to say, when $k_{\mathrm{j}} = k$ and $k \leq k_{\mathrm{c}}$, the maximum load is entire yield pressure. When a pressure vessel is subjected to entire yield pressure $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\ln k = \frac{p_{\mathrm{y}}}{\sigma_{\mathrm{y}}}$, Lamé equations become

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{\ln k^2}{\sqrt{3}(k^2-1)} \tag{3.85}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = -\frac{k^2-x^2}{\sqrt{3}x^2(k^2-1)}\ln k^2 \tag{3.86}$$

$$\frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{k^2+x^2}{\sqrt{3}x^2(k^2-1)}\ln k^2 \tag{3.87}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{k^2\ln k^2}{(k^2-1)x} \tag{3.88}$$

Clearly, when $k_{\mathrm{j}} = k$ and $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\ln k = \frac{p_{\mathrm{y}}}{\sigma_{\mathrm{y}}}$, $\frac{\sigma_{\mathrm{ei}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{k^2\ln k^2}{(k^2-1)}$, even if $k \leq k_{\mathrm{c}}$, $\sigma_{\mathrm{ei}}^{\mathrm{p}}$ may exceed σ_{y} if autofrettage process is not performed.

When plastic depth $k_{\mathrm{j}} = k$ and $p = p_{\mathrm{y}}$, the total stresses are as follows

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\left(\frac{1}{2} + \ln x - \ln k\right) \tag{3.89}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}(\ln x - \ln k) \tag{3.90}$$

$$\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}(1 + \ln x - \ln k) \tag{3.91}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \equiv 1 \tag{3.92}$$

Setting $\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}(1 + \ln x - \ln k) = \frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1$ leads to

$$x = \exp\left(\ln k + \frac{\sqrt{3}}{2} - 1\right) = x_{\mathrm{b}} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}} - \frac{2-\sqrt{3}}{2}\right) = x_{\mathrm{z}} \tag{3.93}$$

When $x < x_{\mathrm{b}}$, $\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} < \frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1$; When $x > x_{\mathrm{b}}$, $\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} > \frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1$. When $p = p_{\mathrm{y}}$, equation (3.93) is just equation (1.25), or $x_{\mathrm{b}} = x_{\mathrm{z}}$.

$x_{\mathrm{b}} = \exp\left(\ln k + \frac{\sqrt{3}}{2} - 1\right) = \exp(\ln k)\exp\left(\frac{\sqrt{3}}{2} - 1\right) = \frac{k}{\exp\left(1-\frac{\sqrt{3}}{2}\right)} < k$. This shows that the curve $\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ is always intersected with the curve $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$, and the intersection point is between 1 and k. When $k = 1$, $x_{\mathrm{b}} = 0.866025$; When $k = 1$, $x_{\mathrm{b}} = k_{\mathrm{c}}$, $x_{\mathrm{b}} = 2.084483 < k_{\mathrm{c}}$. $x_{\mathrm{b}} = \exp\left(\ln k + \frac{\sqrt{3}}{2} - 1\right)$ is plotted in figure 3.22.

As x increases, $\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$, $\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$, and $\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ increase.

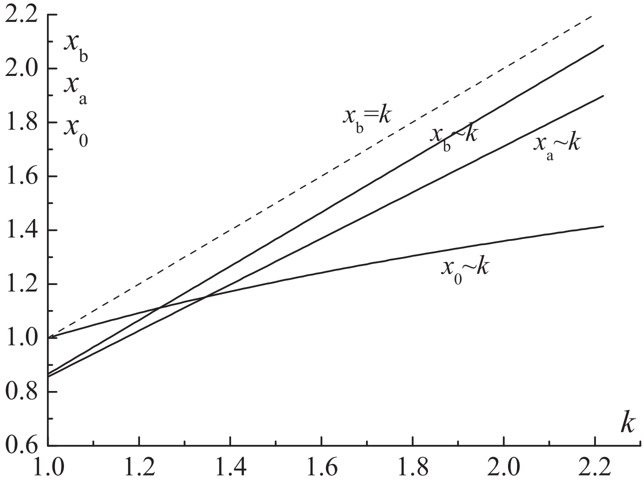

Fig. 3.22 – The image between k and x_{b}.

At the inner surface,

$$\frac{\sigma_{\mathrm{zi}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{2}{\sqrt{3}}\left(\frac{1}{2}-\ln k\right) \tag{3.94}$$

$$\frac{\sigma_{\mathrm{ri}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=-\frac{2}{\sqrt{3}}\ln k=-\frac{p}{\sigma_{\mathrm{y}}} \tag{3.95}$$

$$\frac{\sigma_{\theta\mathrm{i}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{2}{\sqrt{3}}(1-\ln k) \tag{3.96}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\equiv 1 \tag{3.97}$$

For $\frac{\sigma_{\mathrm{zi}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\geq -1$, there must be $k\leq \exp\left(\frac{1+\sqrt{3}}{2}\right)=3.91974$; for $\frac{\sigma_{\theta\mathrm{i}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\geq -1$, there must be $k\leq \exp\left(1+\frac{\sqrt{3}}{2}\right)=6.462559$; for $\frac{\sigma_{\mathrm{ri}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\geq -1$, there must be $k\leq \exp\left(\frac{\sqrt{3}}{2}\right)=2.377443$. The premise of $k_{\mathrm{j}}=k$ is $k\leq k_{\mathrm{c}}$, therefore, $\frac{\sigma_{\mathrm{zi}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\geq -1$, $\frac{\sigma_{\theta\mathrm{i}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\geq -1$ and $\frac{\sigma_{\mathrm{ri}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\geq -1$ are natural.

At the outer surface,

$$\frac{\sigma_{\mathrm{zo}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{1}{\sqrt{3}} \tag{3.98}$$

$$\frac{\sigma_{\mathrm{ro}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=0 \tag{3.99}$$

$$\frac{\sigma_{\theta\mathrm{o}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{2}{\sqrt{3}}>1 \tag{3.100}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\equiv 1 \tag{3.101}$$

From equation (3.83), at the outer surface,

$$\frac{\sigma_{\theta\mathrm{o}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{1}{\sqrt{3}}\left(2-\frac{2\ln k^2}{k^2-1}\right)+\frac{2}{k^2-1}\frac{p}{\sigma_{\mathrm{y}}} \tag{3.102}$$

Setting $\frac{\sigma_{\theta\mathrm{o}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\leq 1$ obtains

$$\frac{p}{\sigma_{\mathrm{y}}}\leq\frac{2}{\sqrt{3}}\ln k-\frac{2-\sqrt{3}}{2\sqrt{3}}\left(k^2-1\right)=\frac{p_{\mathrm{y}}^{-}}{\sigma_{\mathrm{y}}}<\frac{p_{\mathrm{y}}}{\sigma_{\mathrm{y}}} \tag{3.103}$$

Equation (3.103) is the maximum load for $\sigma_{\theta o}^{T} \le \sigma_y$.

From equation (3.103), it is known that when $k = k_r = \sqrt{2(2+\sqrt{3})} = 1+\sqrt{3} > k_c$, $\frac{p_y^-}{\sigma_y}$ reaches the maximum, $\frac{p_y^-}{\sigma_y} = 0.660535\ldots$, and when $k = k_r$, $\frac{p_y}{\sigma_y} = 1.160535$. When $k = k_c$, $\frac{p_y^-}{\sigma_y} = 0.616746\ldots$ and $\frac{p_y}{\sigma_y} = 0.920079\ldots. > \frac{p_y^-}{\sigma_y}$.

$\frac{p_y^-}{\sigma_y}$ is plotted in figure 3.23 along with $\frac{p_y}{\sigma_y}$ and $\frac{p_e}{\sigma_y}$. It is seen that $p_y^- > p_e$.

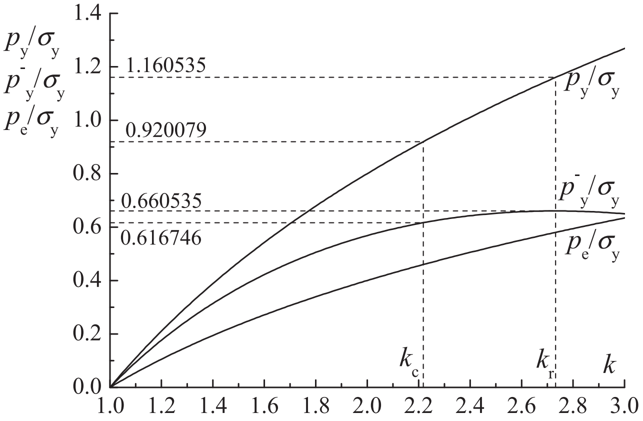

FIG. 3.23 – The allowable load for $\sigma_{\theta o}^{T} \le \sigma_y$.

When plastic depth $k_j = k$, $p = p_y$, and $k \le k_c$, the distributions of the total stresses are plotted in figures 3.24 and 3.25 for $k = k_c$ and $k = 2$, respectively. The meaning of each curve and coordinates of key points as well as the main parameters are marked in the figures.

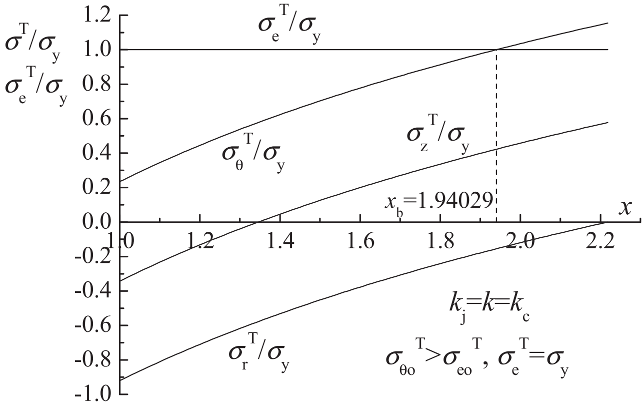

FIG. 3.24 – The distributions of the total stresses when $k_j = k$, $p = p_y$ and $k = k_c$.

In figures 3.24 and 3.25, after $x \geq x_{\mathrm{b}}$, $\sigma_{\theta}^{\mathrm{T}} \geq \sigma_{\mathrm{e}}^{\mathrm{T}} = 1$.

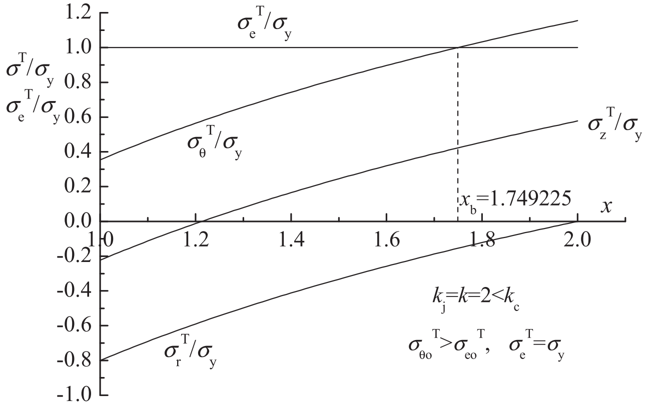

Fig. 3.25 – The distributions of the total stresses when $k_{\mathrm{j}} = k$, $p = p_{\mathrm{y}}$ and $k = 2$.

When $k_{\mathrm{j}} = k$, $p = p_{\mathrm{y}}^{-} = 0.616746\sigma_{\mathrm{y}}$ and $k = k_{\mathrm{c}}$, the distributions of the total stresses are plotted in figure 3.26.

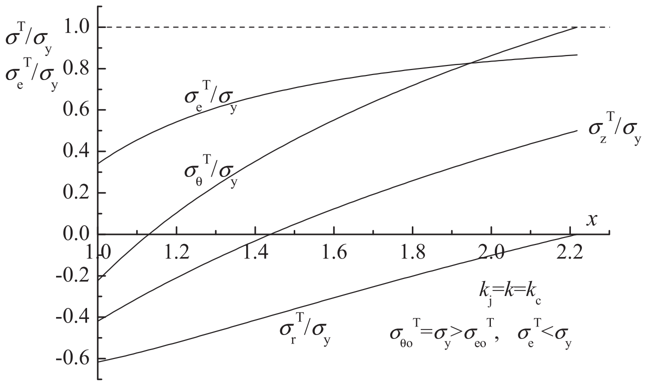

Fig. 3.26 – The distributions of the total stresses when $k_{\mathrm{j}} = k$, $p = p_{\mathrm{y}}^{-}$ and $k = k_{\mathrm{c}}$.

In figure 3.26, $\sigma_{\theta\mathrm{o}}^{\mathrm{T}} = \sigma_{\mathrm{y}}$, $\sigma_{\mathrm{e}}^{\mathrm{T}} < \sigma_{\mathrm{y}}$ because $p = p_{\mathrm{y}}^{-} < p_{\mathrm{y}}$.

When $k_{\mathrm{j}} = k$, $p = p_{\mathrm{y}}^{-} = 0.568327\sigma_{\mathrm{y}}$ and $k = 2$, the distributions of the total stresses are plotted in figure 3.27.

In figure 3.27, $\sigma_{\theta\mathrm{o}}^{\mathrm{T}} = \sigma_{\mathrm{y}}$, $\sigma_{\mathrm{e}}^{\mathrm{T}} < \sigma_{\mathrm{y}}$.

When $k_{\mathrm{j}} = k$, $p = 0.65\sigma_{\mathrm{y}} > p_{\mathrm{y}}^{-}$ and $k = k_{\mathrm{c}}$, the distributions of the total stresses are plotted in figure 3.28.

In figure 3.27, $\sigma_{\theta\mathrm{o}}^{\mathrm{T}} = 1.01696\sigma_{\mathrm{y}}$, $\sigma_{\mathrm{e}}^{\mathrm{T}} < \sigma_{\mathrm{y}}$.

When $k_{\mathrm{j}} = k$, $p = 0.6\sigma_{\mathrm{y}} < p_{\mathrm{y}}^{-}$ and $k = k_{\mathrm{c}}$, the distributions of the total stresses are plotted in figure 3.29.

In figure 3.29, $\sigma_{\theta\mathrm{o}}^{\mathrm{T}} = 0.991459\sigma_{\mathrm{y}}$, $\sigma_{\mathrm{e}}^{\mathrm{T}} < \sigma_{\mathrm{y}}$.

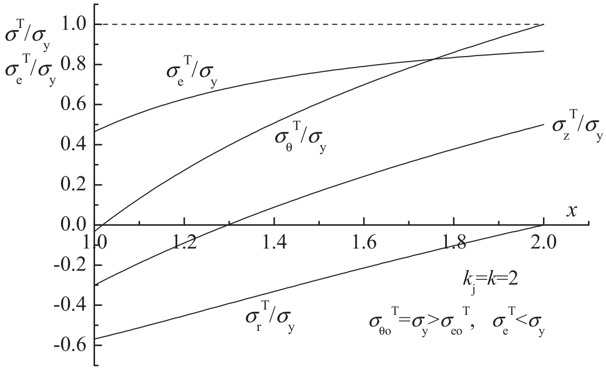

FIG. 3.27 – The distributions of the total stresses when $k_j = k$, $p = p_y^-$ and $k = 2$.

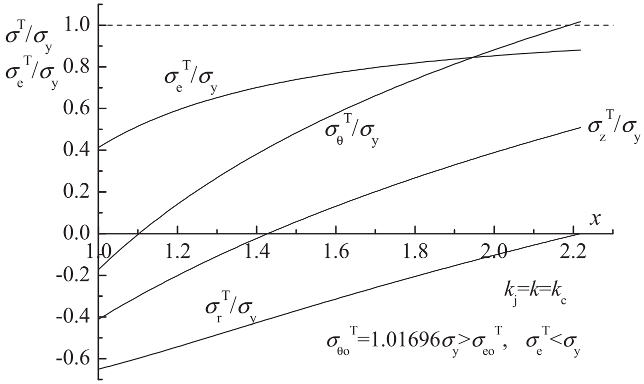

FIG. 3.28 – The distributions of the total stresses when $k_j = k$, $p > p_y^-$ and $k = k_c$.

Whether it is necessary that $\sigma_{\theta o}^T < \sigma_y$ in engineering practice can be determined by the engineers according to the results of this research.

All in all, the premise of $k_j = k$ is $k \leq k_c$.

In addition, when $k_j = k$, we have the following results.

(1) Autofrettage pressure

$$\frac{p_a}{\sigma_y} = \frac{2}{\sqrt{3}} \ln k$$

This is just the maximum load under the condition $k_j = k$;

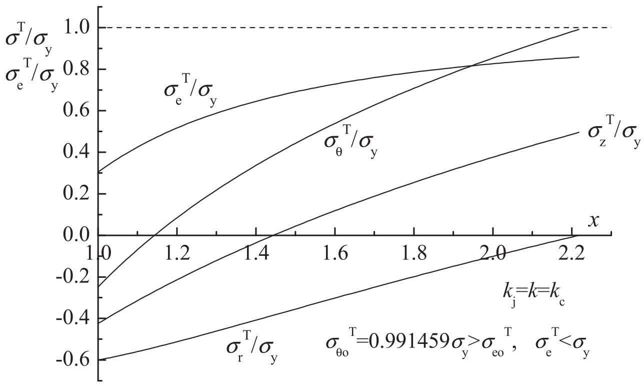

FIG. 3.29 – The distributions of the total stresses when $k_{\mathrm{j}} = k$, $p < p_{\mathrm{y}}^{-}$ and $k = k_{\mathrm{c}}$.

(2) The abscissa of the intersection of the three residual stress curves or the abscissa at which $\sigma'_{\mathrm{e}} = 0$

$$x_0 = \sqrt{\frac{k^2 \ln k^2}{k^2 - 1}} < k$$

When $k = k_{\mathrm{c}}$, $x_0 = \sqrt{2}$.

(3) The value of residual stresses when $x = x_0$ or the minimum stresses are

$$\frac{\sigma'_{\mathrm{zx0}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\theta\mathrm{x0}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{rx0}}}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}} \ln \frac{\ln k^2}{k^2 - 1} + \frac{k^2 - 1 - \ln k^2}{\sqrt{3}(k^2 - 1)}$$

(4) The abscissa at which $\sigma'_{\mathrm{z}} = 0$, or x_1 is

$$x_1 = \sqrt{\mathrm{e}^{\frac{1 - k^2 + k^2 \ln k^2}{k^2 - 1}}}$$

3.3 Mechanical Autofrettage Technology with Radius of Elastic-Plastic Juncture Being Arithmetic Mean Radius of Inside Radius and Outside Radius

The overstrain is defined as

$$\varepsilon = \frac{r_{\mathrm{j}} - r_{\mathrm{i}}}{r_{\mathrm{o}} - r_{\mathrm{i}}} \tag{3.104}$$

When the radius of the plastic region is an arithmetic mean value of the internal and external radius of the cylindrical pressure vessel, or

$$r_j = \frac{r_i + r_o}{2} \quad \text{or} \quad k_j = \frac{k+1}{2} \tag{3.105}$$

the overstrain $\varepsilon = 50\%$.

Figure 3.12 is the relationship between k_j and k, so we can add $k_j = \frac{k+1}{2}$ into this figure as straight line *ofb*, as is shown in figure 3.30.

In figure 3.30,

(1) curve *afgd* is for $k^2 \ln k_j{}^2 - k^2 - k_j{}^2 + 2 = 0$, point *a* is an intersection of curve *afgd*, straight line *oac* ($k_j = k$) and curve *oan*; point *f* is an intersection of curve *afgd* and straight line *ofb*; point *g* is an intersection of curve *afgd* and curve *ogh*; point *d* is at infinity, its coordinates are $d\,(\infty, \sqrt{\mathrm{e}})$.

(2) straight line *oac* is for $k_j = k$ (entirely yielded), point *c* is at infinity, and its coordinates are $c\,(\infty, \infty)$.

(3) straight line *ofb* is for $k_j = \frac{k+1}{2}$, point *b* is at infinity, its coordinates are $b\,(\infty, \infty)$.

(4) curve *ogh* is for $k_j = \sqrt{k}$ (see below), point *h* is at infinity, and its coordinates are $h\,(\infty, \infty)$.

(5) curve *oan* is for $k_j = \exp\left(\frac{k^2-1}{k^2}\right)$ (see below), point *n* is at infinity, and its coordinates are $n\,(\infty, \mathrm{e})$.

(6) curve *om* is for $k_j = \exp\left(\frac{k^2-1}{2k^2}\right)$ (see below), point *m* is at infinity, and its coordinates are $m\,(\infty, \sqrt{\mathrm{e}})$.

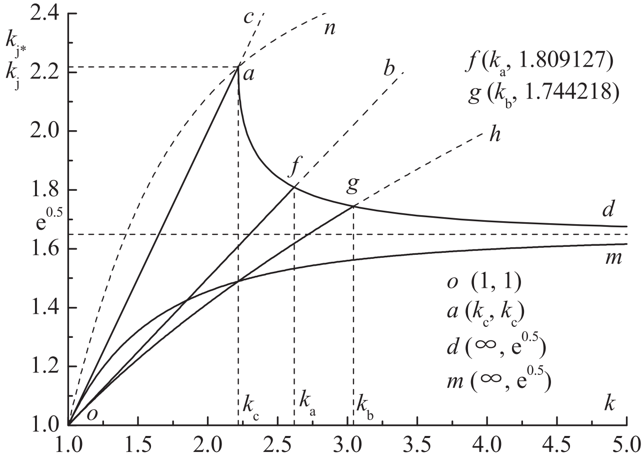

FIG. 3.30 – The relationship between k_j and k.

Substituting equation (3.105) into equation (3.2)–(3.5) obtains the residual stresses within the plastic region

$$\frac{\sigma_z'}{\sigma_y}=\frac{1}{\sqrt{3}}\left[\frac{(k+1)^2}{4k^2}+\ln\frac{4x^2}{(k+1)^2}-\left(1-\frac{(k+1)^2}{4k^2}+\ln\frac{(k+1)^2}{4}\right)\frac{1}{k^2-1}\right] \tag{3.106}$$

$$\frac{\sigma_r'}{\sigma_y}=\frac{1}{\sqrt{3}}\left[\frac{(k+1)^2}{4k^2}-1+\ln\frac{4x^2}{(k+1)^2}-\left(1-\frac{(k+1)^2}{4k^2}+\ln\frac{(k+1)^2}{4}\right)\frac{1}{k^2-1}\left(1-\frac{k^2}{x^2}\right)\right] \tag{3.107}$$

$$\frac{\sigma_\theta'}{\sigma_y}=\frac{1}{\sqrt{3}}\left[\frac{(k+1)^2}{4k^2}+1+\ln\frac{4x^2}{(k+1)^2}-\left(1-\frac{(k+1)^2}{4k^2}+\ln\frac{(k+1)^2}{4}\right)\frac{1}{k^2-1}\left(1+\frac{k^2}{x^2}\right)\right] \tag{3.108}$$

$$\frac{\sigma_e'}{\sigma_y}=1-\frac{k^2-\frac{(k+1)^2}{4}+k^2\ln\frac{(k+1)^2}{4}}{k^2-1}\frac{1}{x^2} \tag{3.109}$$

Substituting equation (3.105) into equations (3.6)–(3.9) obtains the residual stresses within the elastic region

$$\frac{\sigma_z'}{\sigma_y}=\frac{\frac{(k+1)^2}{4}-1-\ln\frac{(k+1)^2}{4}}{\sqrt{3}(k^2-1)}>0 \tag{3.110}$$

$$\frac{\sigma_r'}{\sigma_y}=\left(1-\frac{k^2}{x^2}\right)\frac{\sigma_z'}{\sigma_y}<0 \tag{3.111}$$

$$\frac{\sigma_\theta'}{\sigma_y}=\left(1+\frac{k^2}{x^2}\right)\frac{\sigma_z'}{\sigma_y}>0 \tag{3.112}$$

$$\frac{\sigma_e'}{\sigma_y}=\frac{\frac{(k+1)^2}{4}-1-\ln\frac{(k+1)^2}{4}}{k^2-1}\frac{k^2}{x^2}>0 \tag{3.113}$$

When $k=2$, $k_j=1.5$. The residual stresses within the whole wall are plotted in figure 3.31, where the meaning of each curve and the main parameters are marked.

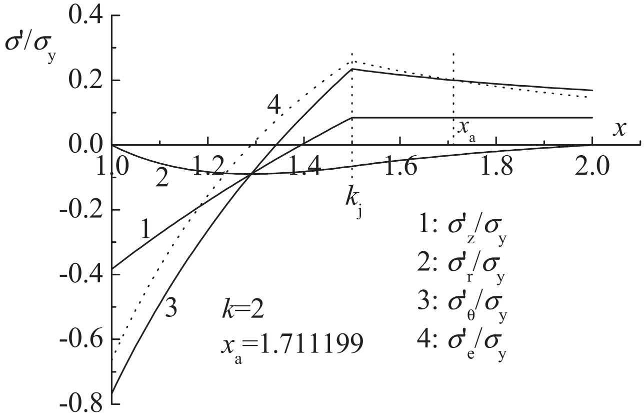

FIG. 3.31 – The residual stresses within the whole wall when $k = 2$, $k_j = 1.5$.

When $k = 3$, $k_j = 2$. The residual stresses within the whole wall are plotted in figure 3.32, where the meaning of each curve and the main parameters are marked.

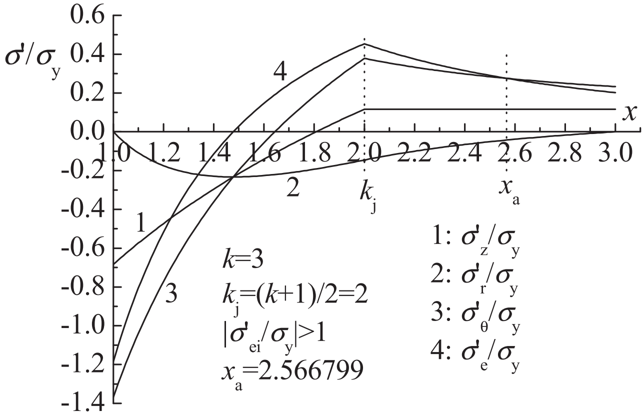

FIG. 3.32 – The residual stresses within the whole wall when $k = 3$, $k_j = 2$.

Based on the maximum shear stress theory, letting $\sigma'_{\mathrm{ei}} \geq -\sigma_{\mathrm{y}}$ obtain

$$5k^2 + 2k - 7 - 4k^2 \ln \frac{(k+1)^2}{4} = c \geq 0 \tag{3.114}$$

Equation (3.114) is just equation (2.90), therefore, when $k \leq k_{\mathrm{a}} = 2.618253$, $\left|\sigma'_{\mathrm{ei}}\right| \leq \sigma_{\mathrm{y}}$. k_{a} is just the abscissa of the intersection point (f) of curves *ob* and *ad* in figure 3.30.

When $k = k_a$, $k_j = 1.809126\ldots$, the distribution of residual stresses within the whole wall is plotted in figure 3.33. The meaning of each curve and the main parameters are marked in the figure.

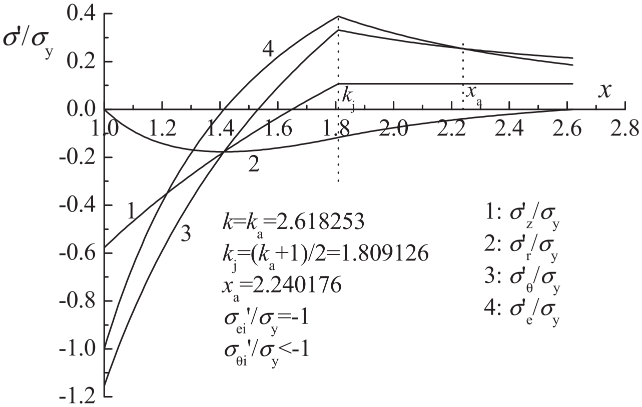

Fig. 3.33 – The residual stresses within the whole wall when $k = k_a$.

Converting the longitudinal coordinates of figure 3.30 into $\frac{k_j}{k}$ results in figure 3.34 so as to analyze the ratio of k_j to k. When the plastic depth $k_j = k$, $\frac{k_j}{k} = 1$; when $k_j = \frac{k+1}{2}$, $\frac{k_j}{k} = \frac{k+1}{2k}$; when $k_j = \sqrt{k}$, $k_j = \frac{1}{\sqrt{k}}$.

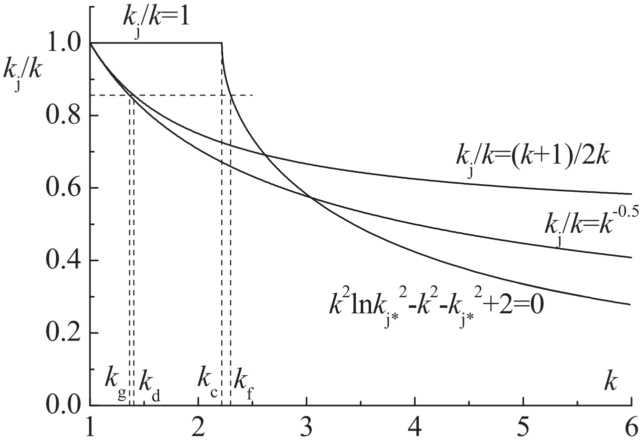

Fig. 3.34 – The relationship between $\frac{k_j}{k}$ and k.

As mentioned above, when $k_{\mathrm{j}} \leq \sqrt{\sqrt{3}-1}k = x_{\mathrm{a}}$, $\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \geq \frac{\sigma_{\theta\mathrm{j}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$; when $k_{\mathrm{j}} \geq x_{\mathrm{a}}$, $\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \leq \frac{\sigma_{\theta\mathrm{j}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$. Then, letting $k_{\mathrm{j}} = \frac{k+1}{2} \leq \sqrt{\sqrt{3}-1}k$ results in

$$k \geq \frac{1}{\sqrt{\sqrt{3}-1}-1} = 1.406076 = k_{\mathrm{d}} \tag{3.115}$$

k_{d} is marked in figure 3.34. That is to say, when $k_{\mathrm{j}} = \frac{k+1}{2}$, if $k \geq k_{\mathrm{d}}$, $k_{\mathrm{j}} \leq \sqrt{\sqrt{3}-1}k$, accordingly $\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \geq \frac{\sigma_{\theta\mathrm{j}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$.

The equivalent stress of the total stresses

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2 - \frac{(k+1)^2}{4} + k^2 \ln\frac{(k+1)^2}{4}}{k^2-1}\frac{1}{x^2} + \frac{\sqrt{3}k^2}{x^2(k^2-1)}\frac{p}{\sigma_{\mathrm{y}}} \tag{3.116}$$

Setting $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \leq 1$ gives

$$\frac{p}{\sigma_{\mathrm{y}}} \leq \frac{1}{\sqrt{3}} - \frac{(k+1)^2}{4\sqrt{3}k^2} + \frac{2}{\sqrt{3}}\ln\frac{k+1}{2} \tag{3.117}$$

Accordingly, the maximum load-bearing capacity of a pressure vessel is $\frac{p}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}} - \frac{(k+1)^2}{4\sqrt{3}k^2} + \frac{2}{\sqrt{3}}\ln\frac{k+1}{2}$ when $k_{\mathrm{j}} = \frac{k+1}{2}$.

Substituting $k_{\mathrm{j}} = \frac{k+1}{2}$ into equation (3.1) also results in equation (3.117).

Letting $\frac{p}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}} - \frac{(k+1)^2}{4\sqrt{3}k^2} + \frac{2}{\sqrt{3}}\ln\frac{k+1}{2} \leq 1$ obtains

$$k \leq 2.654911\ldots = k_{\mathrm{z}} \tag{3.118}$$

When $k \leq k_{\mathrm{z}}$, $p \leq \sigma_{\mathrm{y}}$ and $\sigma_{\mathrm{ri}}^{\mathrm{T}} \geq -\sigma_{\mathrm{y}}$; When $k \geq k_{\mathrm{z}}$, $p \geq \sigma_{\mathrm{y}}$ and $\sigma_{\mathrm{ri}}^{\mathrm{T}} \leq -\sigma_{\mathrm{y}}$. When $k \leq k_{\mathrm{a}}$, there must be $k \leq k_{\mathrm{z}}$.

When based on the maximum distortion strain energy theory, if $k \leq k_{\mathrm{c}}$, $k_{\mathrm{j}} = k$, and the maximum load-bearing capacity of a pressure vessel is the entire yield load $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\ln k$, that is plotted in figure 3.35 as curve *oa*; and if $k \geq k_{\mathrm{c}}$, k_{j} is calculated by $k^2 \ln k_{\mathrm{j}^*}^2 - k^2 - k_{\mathrm{j}^*}^2 + 2 = 0$, and the maximum load-bearing capacity of a pressure vessel is $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$, that is plotted in figure 3.35 as curve *ad*. Point *a* is the intersection point of the curve $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\ln k$ and curve $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$. The coordinates of point *a* are *a* (k_{c}, 0.920079).

Letting $\frac{p}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}} - \frac{(k+1)^2}{4\sqrt{3}k^2} + \frac{2}{\sqrt{3}}\ln\frac{k+1}{2} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$ just leads to $5k^2 + 2k - 7 - 4k^2 \ln\frac{(k+1)^2}{4} = 0$, which is just equation (3.114), or equation (2.90). $\frac{p}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}} - \frac{(k+1)^2}{4\sqrt{3}k^2} + \frac{2}{\sqrt{3}}\ln\frac{k+1}{2}$ is also plotted in figure 3.35 as curve *ob*. The coordinates of the intersection point of the curve *ob* and the curve *ad* are *f* (k_{a}, 0.986264). In figure 3.35, if $k \geq k_{\mathrm{a}}$, such as in the section *fb*, $\sigma_{\mathrm{e}}^{\mathrm{T}} > \sigma_{\mathrm{y}}$.

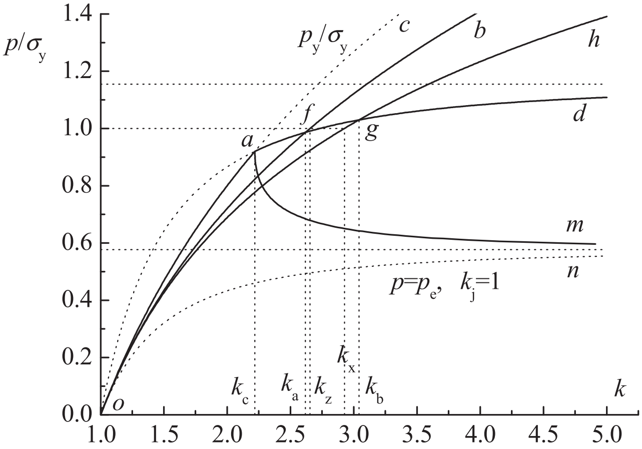

Fig. 3.35 – Comparison of load-bearing capacity.

For comparison, the maximum elastic load or initial yield load $\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}=\frac{1}{\sqrt{3}}\frac{k^2-1}{k^2}$ is also plotted in figure 3.35 as dash curve *on*. As k increases, point n and m gradually coincide to one point (the meanings of point m and curve *am* will be explained below.).

$$\frac{2}{\sqrt{3}}\ln k-\left(\frac{1}{\sqrt{3}}-\frac{(k+1)^2}{4\sqrt{3}k^2}+\frac{2}{\sqrt{3}}\ln\frac{k+1}{2}\right)=\frac{(k-1)^2(7k-1)}{2\sqrt{3}k^2(6k+2)}\geq 0$$

This means that the load-bearing capacity at $k_{\mathrm{j}}=\frac{k+1}{2}$ is smaller than $\frac{p_{\mathrm{y}}}{\sigma_{\mathrm{y}}}=\frac{2}{\sqrt{3}}\ln k$, as is seen from figure 3.35. This is because $k_{\mathrm{j}*}>\frac{k+1}{2}$, or the plastic depth when $k_{\mathrm{j}}=\frac{k+1}{2}$ is smaller than that when $k_{\mathrm{j}}=k_{\mathrm{j}*}$.

From figure 3.35, we can also see that when $k\leq k_{\mathrm{a}}$, $\frac{1}{\sqrt{3}}-\frac{(k+1)^2}{4\sqrt{3}k^2}+\frac{2}{\sqrt{3}}\ln\frac{k+1}{2}<\frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$; when $k\geq k_{\mathrm{a}}$, $\frac{1}{\sqrt{3}}-\frac{(k+1)^2}{4\sqrt{3}k^2}+\frac{2}{\sqrt{3}}\ln\frac{k+1}{2}>\frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$. However, when $k\geq k_{\mathrm{a}}$, $\left|\sigma'_{\mathrm{ei}}\right|>\sigma_{\mathrm{y}}$ or reverse yield will occur due to $k_{\mathrm{j}}=\frac{k+1}{2}$.

In addition, when $k_{\mathrm{j}}=\frac{k+1}{2}$, autofrettage pressure $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{1}{\sqrt{3}}-\frac{(k+1)^2}{4\sqrt{3}k^2}+\frac{2}{\sqrt{3}}\ln\frac{k+1}{2}=\frac{p}{\sigma_{\mathrm{y}}}$, which is just equation (3.117). This is because the general equation of autofrettage pressure $\frac{p_a}{\sigma_{\mathrm{y}}}=\frac{2}{\sqrt{3}}\ln k_{\mathrm{j}}+\frac{k^2-k_{\mathrm{j}}^2}{\sqrt{3}k^2}$ is obtained just by setting $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=1$ in a plastic region.

The total stresses $\sigma^{\mathrm{T}}=\sigma'+\sigma^{\mathrm{P}}$, $\sigma_{\mathrm{e}}^{\mathrm{T}}=\sigma'_{\mathrm{e}}+\sigma_{\mathrm{e}}^{\mathrm{p}}$.

When $k_{\mathrm{j}}=\frac{k+1}{2}=\sqrt{\sqrt{3}-1}k=x_{\mathrm{a}}$, or $k=k_{\mathrm{d}}$, equation (3.70) becomes $x_{\mathrm{z}}=x_{\mathrm{a}}=k_{\mathrm{j}}$.

The total stresses for $k=1.406076=k_{\mathrm{d}}$, $k_{\mathrm{j}}=\frac{k+1}{2}=1.203038$ and $\frac{p}{\sigma_{\mathrm{y}}}=\frac{1}{\sqrt{3}}-\frac{(k+1)^2}{4\sqrt{3}k^2}+\frac{2}{\sqrt{3}}\ln\frac{k+1}{2}$ are illustrated in figure 3.36.

The total stresses for $k = 1.2 < k_{\mathrm{d}}$, $k_{\mathrm{j}} = \frac{k+1}{2} = 1.1$ and $\frac{p}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}} - \frac{(k+1)^2}{4\sqrt{3}k^2} + \frac{2}{\sqrt{3}} \ln \frac{k+1}{2}$ are illustrated in figure 3.37.

The total stresses for $k = 1.5 > k_{\mathrm{d}}$, $k_{\mathrm{j}} = \frac{k+1}{2} = 1.25$ and $\frac{p}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}} - \frac{(k+1)^2}{4\sqrt{3}k^2} + \frac{2}{\sqrt{3}} \ln \frac{k+1}{2}$ are illustrated in figure 3.38.

The meaning of each curve and mweain parameters are marked in the figures.

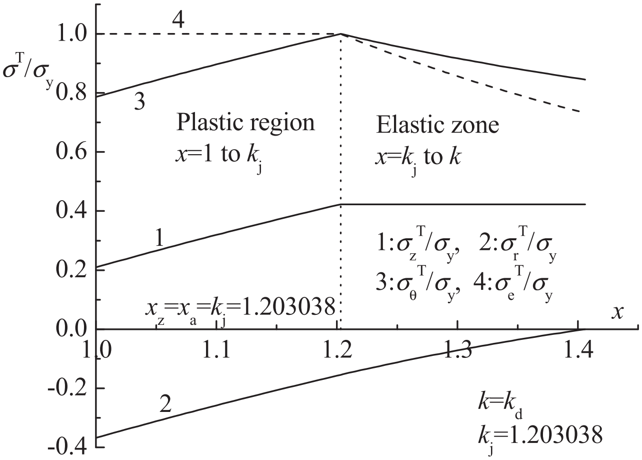

FIG. 3.36 – The total stresses for $k = k_{\mathrm{d}}$, $k_{\mathrm{j}} = \frac{k+1}{2}$ and $\frac{p}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}} - \frac{(k+1)^2}{4\sqrt{3}k^2} + \frac{2}{\sqrt{3}} \ln \frac{k+1}{2}$.

In figure 3.36, since $k = k_{\mathrm{d}}$, at $x = k_{\mathrm{j}}$, $\sigma_{\theta\mathrm{j}}^{\mathrm{T}}$ is just $\sigma_{\mathrm{e}}^{\mathrm{T}}$, and is just σ_{y}, and $x_{\mathrm{z}} = x_{\mathrm{a}} = k_{\mathrm{j}}$.

In figure 3.37, from equation (3.70), $x_{\mathrm{z}} = 1.042057$; $k_{\mathrm{j}} = 1.1 > x_{\mathrm{a}} = 1.02672$, so, x_{z} exists and x_{a} does not exist. Because $k = 1.2 < k_{\mathrm{z}}$, $p < \sigma_{\mathrm{y}}$ and $\sigma_{\mathrm{ri}}^{\mathrm{T}} > -\sigma_{\mathrm{y}}$. Because $k < k_{\mathrm{d}}$, $k_{\mathrm{j}} > \sqrt{\sqrt{3} - 1}k$, thus $\sigma_{\theta\mathrm{j}}^{\mathrm{T}} > \sigma_{\mathrm{y}}$.

In figure 3.38, $x_{\mathrm{a}} = 1.2834$; $k_{\mathrm{j}} = 1.25 < x_{\mathrm{a}}$, so, x_{a} exists and x_{z} does not exist. Because $k < k_{\mathrm{z}}$, $p < \sigma_{\mathrm{y}}$ and $\sigma_{\mathrm{ri}}^{\mathrm{T}} > -\sigma_{\mathrm{y}}$. Because $k > k_{\mathrm{d}}$, $k_{\mathrm{j}} < \sqrt{\sqrt{3} - 1}k$, thus $\sigma_{\theta\mathrm{j}}^{\mathrm{T}} < \sigma_{\mathrm{y}}$.

The total stresses for $k = 2.8 > k_{\mathrm{z}}$, $k_{\mathrm{j}} = \frac{k+1}{2} = 1.9$, and $\frac{p}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}} - \frac{(k+1)^2}{4\sqrt{3}k^2} + \frac{2}{\sqrt{3}} \ln \frac{k+1}{2} = 1.052653$ are illustrated in figure 3.39.

The total stresses for $k = 2.654911 = k_{\mathrm{z}}$, $k_{\mathrm{j}} = \frac{k+1}{2} = 1.8274555$ and $\frac{p}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}} - \frac{(k+1)^2}{4\sqrt{3}k^2} + \frac{2}{\sqrt{3}} \ln \frac{k+1}{2} = 1$ are illustrated in figure 3.40.

The meaning of each curve and main parameters have been marked in the figures.

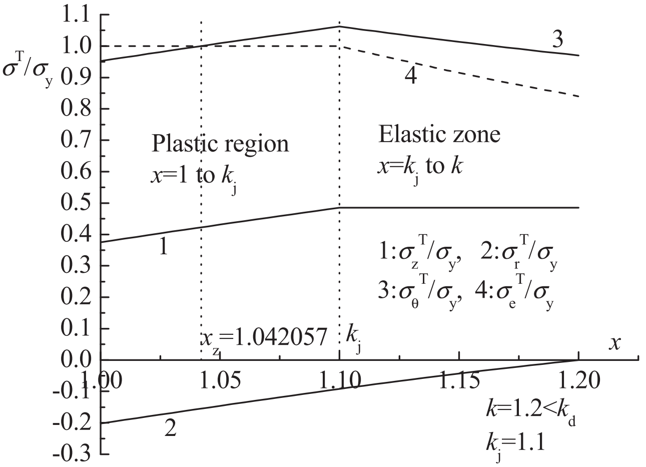

FIG. 3.37 – The total stresses for $k = 1.2 < k_{\mathrm{d}}$, $k_{\mathrm{j}} = \frac{k+1}{2}$ and $\frac{p}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}} - \frac{(k+1)^2}{4\sqrt{3}k^2} + \frac{2}{\sqrt{3}}\ln\frac{k+1}{2}$.

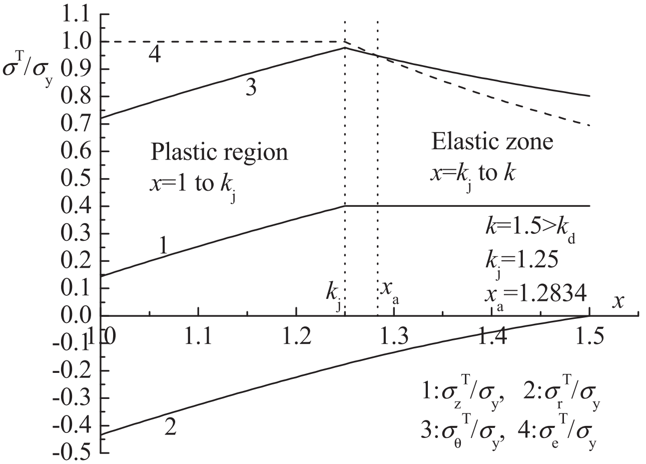

FIG. 3.38 – The total stresses with $k = 1.5 > k_{\mathrm{d}}$, $k_{\mathrm{j}} = \frac{k+1}{2}$ and $\frac{p}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}} - \frac{(k+1)^2}{4\sqrt{3}k^2} + \frac{2}{\sqrt{3}}\ln\frac{k+1}{2}$.

In figure 3.39, $x_{\mathrm{a}} = 2.395679$; $k_{\mathrm{j}} = 1.9 < x_{\mathrm{a}}$, so, x_{a} exists and x_{z} does not exist. Because $k > k_{\mathrm{z}}$, $p > \sigma_{\mathrm{y}}$ and $\sigma_{\mathrm{ri}}^{\mathrm{T}} = -1.05265 < -\sigma_{\mathrm{y}}$. Because $k > k_{\mathrm{d}}$, $k_{\mathrm{j}} < \sqrt{\sqrt{3}-1}k$, thus $\sigma_{\theta\mathrm{j}}^{\mathrm{T}} < \sigma_{\mathrm{y}}$.

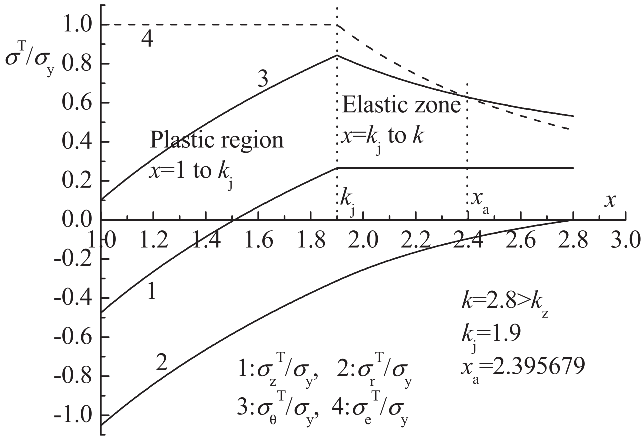

FIG. 3.39 – The total stresses with $k = 2.8 > k_z$, $k_j = \frac{k+1}{2}$ and $\frac{p}{\sigma_y} = \frac{1}{\sqrt{3}} - \frac{(k+1)^2}{4\sqrt{3}k^2} + \frac{2}{\sqrt{3}}\ln\frac{k+1}{2}$.

In figure 3.40, $x_a = 2.271541$; $k_j = 1.8274555 < x_a$, so, x_a exists and x_z does not exist. Because $k > k_d$, $k_j < \sqrt{\sqrt{3}-1}k$, $\sigma_{\theta j}^T < \sigma_y$. Since $k = k_z$, $p = \sigma_y$ and $\sigma_{ri}^T\sigma_{ri}{}^T = -\sigma_y$.

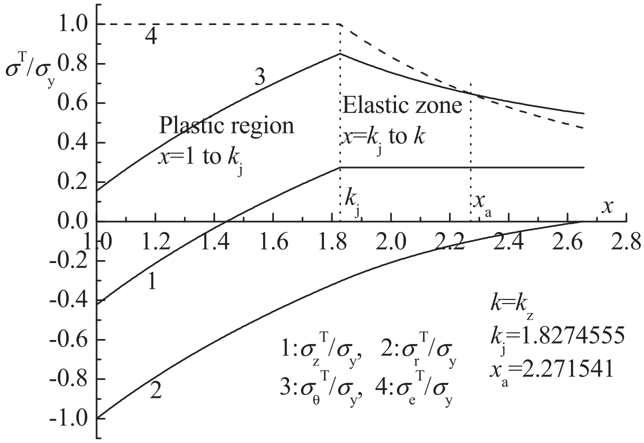

FIG. 3.40 – The total stresses with $k = 2.654911 = k_z$, $k_j = \frac{k+1}{2} = 1.8274555$ and $p = \sigma_y$.

3.4 The Solutions with Radius of Elastic-Plastic Juncture Being Geometrical Mean Radius of Inside Radius and Outside Radius

If the radius of the plastic region is a geometric mean value of the inner radius r_i and outer radius r_j of the pressure vessel, or

$$r_\mathrm{j} = \sqrt{r_\mathrm{i} r_\mathrm{o}} \tag{3.119}$$

then

$$k_\mathrm{j} = \sqrt{k} \tag{3.120}$$

Adding $k_\mathrm{j} = \sqrt{k}$ into figure 3.30 obtains curve *ogh*. It is seen that only section *og* on curve *oh* can ensure that $\left|\sigma'_\mathrm{ei}\right| \leq \sigma_\mathrm{y}$.

Substituting equation (3.120) into equations (3.2)–(3.5) obtains the residual stresses within the plastic region

$$\frac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} = \frac{1}{\sqrt{3}}\left(\ln\frac{x^2}{k} + \frac{k-1-\ln k}{k^2-1}\right) \tag{3.121}$$

$$\frac{\sigma'_\mathrm{r}}{\sigma_\mathrm{y}} = \frac{1}{\sqrt{3}}\left[\ln\frac{x^2}{k} + \frac{1}{k} - 1 + \left(\frac{1}{k} - 1 - \ln k\right)\frac{1}{k^2-1}\left(1 - \frac{k^2}{x^2}\right)\right] \tag{3.122}$$

$$\frac{\sigma'_\theta}{\sigma_\mathrm{y}} = \frac{1}{\sqrt{3}}\left[\ln\frac{x^2}{k} + \frac{1}{k} + 1 + \left(\frac{1}{k} - 1 - \ln k\right)\frac{1}{k^2-1}\left(1 + \frac{k^2}{x^2}\right)\right] \tag{3.123}$$

$$\frac{\sigma'_\mathrm{e}}{\sigma_\mathrm{y}} = 1 - \frac{k^2 - k + k^2\ln k}{k^2-1}\frac{1}{x^2} \tag{3.124}$$

Substituting equation (3.120) into equations (3.6)–(3.9) obtains the residual stresses within the elastic region

$$\frac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} = \frac{k-1-\ln k}{\sqrt{3}(k^2-1)} = \mathrm{a} \quad \text{constant} > 0 \tag{3.125}$$

$$\frac{\sigma'_\mathrm{r}}{\sigma_\mathrm{y}} = \left(1 - \frac{k^2}{x^2}\right)\frac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} < 0 \tag{3.126}$$

$$\frac{\sigma'_\theta}{\sigma_\mathrm{y}} = \left(1 + \frac{k^2}{x^2}\right)\frac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} > 0 \tag{3.127}$$

$$\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} = \frac{k-1-\ln k}{k^2-1}\frac{k^2}{x^2} > 0 \tag{3.128}$$

When $k = 3$, $k_{\mathrm{j}} = \sqrt{k} = 1.732051\ldots$. The residual stresses within the whole wall are plotted in figure 3.41, where $x_{\mathrm{a}} = 2.566799$. The meaning of each curve and the main parameters are marked in the figure.

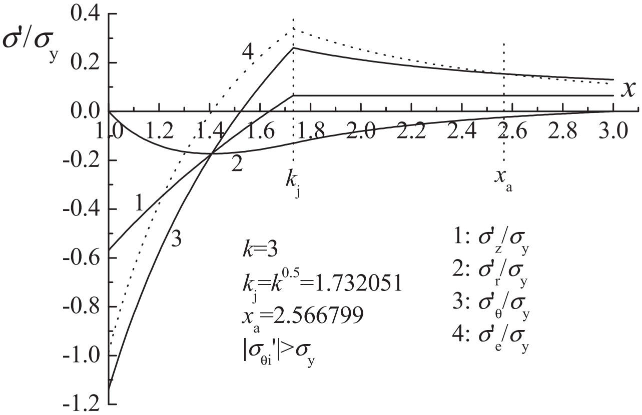

FIG. 3.41 – The residual stresses within the whole wall when $k = 3$.

When $k = 3.5$, $k_{\mathrm{j}} = \sqrt{k} = 1.870829\ldots$, $x_{\mathrm{a}} = 2.994599$. The residual stresses within the whole wall are plotted in figure 3.42, in which it is shown that $\left|\sigma_{\mathrm{ei}}'\right| > \sigma_{\mathrm{y}}$ when $k = 3.5$.

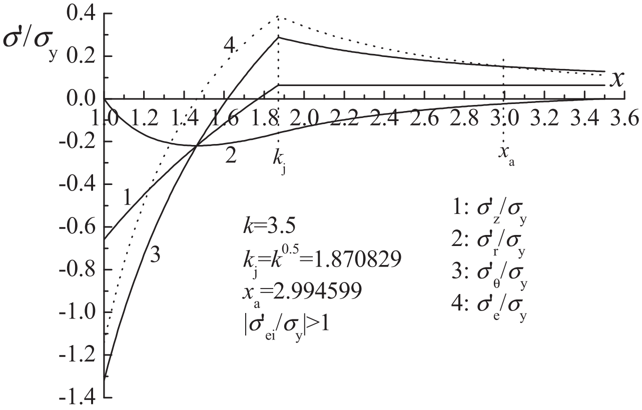

FIG. 3.42 – The residual stresses within the whole wall when $k = 3.5$.

Based on the maximum shear stress theory, when $k \leq k_{\mathrm{b}} = 3.042297$, $\left|\sigma'_{\mathrm{ei}}\right| \leq \sigma_{\mathrm{y}}$. When $k = k_{\mathrm{b}}$, $k_{\mathrm{j}} = 1.744218\ldots$, $x_{\mathrm{a}} = 2.602988\ldots$, the residual stresses within the whole wall are plotted in figure 3.43. The meaning of each curve and the main parameters are marked in the figure.

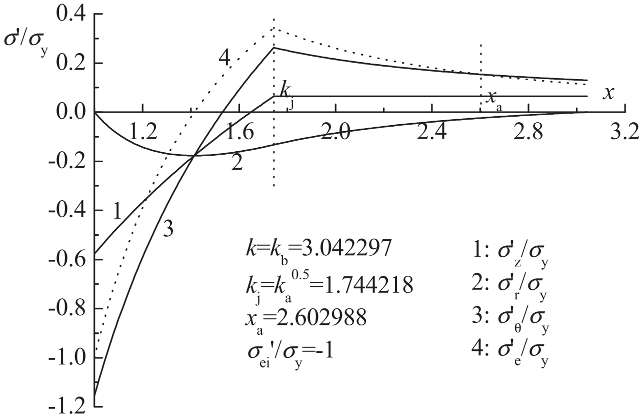

FIG. 3.43 – The residual stresses within the whole wall when $k = k_{\mathrm{b}}$.

In figure 3.43, σ'_{ei} is just $-\sigma_{\mathrm{y}}$.

k_{b} is just the abscissa of the intersection point (g) of curves *oh* and *ad*.

As based on the maximum shear stress theory, in figure 3.30, when $k_{\mathrm{j}} = \sqrt{k}$, if $k \geq k_{\mathrm{b}}$, such as in the *gh* section, the pressure vessel will undergo reverse yield when processed with autofrettage. From figure 3.30, it is known that when $k < k_{\mathrm{b}}$, the plastic depth $k_{\mathrm{j}} = \sqrt{k}$ is smaller than $k_{\mathrm{j}*}$. Nevertheless, this may be at the expense of reducing load-bearing capacity.

When $k_{\mathrm{j}} \leq \sqrt{\sqrt{3}-1}k = x_{\mathrm{a}}$, $\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \geq \frac{\sigma_{\theta\mathrm{j}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$; when $k_{\mathrm{j}} \geq x_{\mathrm{a}}$, $\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \leq \frac{\sigma_{\theta\mathrm{j}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$. Letting $k_{\mathrm{j}} = \sqrt{k} \leq \sqrt{\sqrt{3}-1}k$ results in

$$k \geq \frac{1}{\sqrt{3}-1} = 1.366025 = k_{\mathrm{g}} \tag{3.129}$$

k_{g} is plotted in figure 3.34. When $k \geq k_{\mathrm{g}}$, $k_{\mathrm{j}} \leq \sqrt{\sqrt{3}-1}k$, accordingly $\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \geq \frac{\sigma_{\theta\mathrm{j}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$.

The equivalent stress of the total stresses

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2 - k + k^2 \ln k}{k^2 - 1}\frac{1}{x^2} + \frac{\sqrt{3}k^2}{x^2(k^2-1)}\frac{p}{\sigma_{\mathrm{y}}} \tag{3.130}$$

Setting $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \leq 1$ gives

$$\frac{p}{\sigma_{\mathrm{y}}} \leq \frac{k-1}{\sqrt{3}k} + \frac{1}{\sqrt{3}}\ln k \tag{3.131}$$

Accordingly, the maximum load-bearing capacity of a pressure vessel is $\frac{p}{\sigma_\mathrm{y}} = \frac{k-1}{\sqrt{3}k} + \frac{1}{\sqrt{3}}\ln k$ when $k_\mathrm{j} = \sqrt{k}$.

Effectively, substituting $k_\mathrm{j} = \sqrt{k}$ into equation (3.1) also results in equation (3.131).

Letting $\frac{p}{\sigma_\mathrm{y}} = \frac{k-1}{\sqrt{3}k} + \frac{1}{\sqrt{3}}\ln k \leq 1$ obtains

$$k \leq 2.926388\ldots = k_{\mathrm{x}1} \tag{3.132}$$

When $k \leq k_{\mathrm{x}1}$, $p \leq \sigma_\mathrm{y}$, thus $\frac{\sigma_\mathrm{ri}^\mathrm{T}}{\sigma_\mathrm{y}} \geq -1$.

Letting $\frac{k-1}{\sqrt{3}k} + \frac{1}{\sqrt{3}}\ln k = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$ just leads to $k^2+k-2-k^2\ln k = 0$, or equation (2.106). $\frac{p}{\sigma_\mathrm{y}} = \frac{k-1}{\sqrt{3}k} + \frac{1}{\sqrt{3}}\ln k$ is also plotted in figure 3.35 as curve *oh*. The coordinates of the intersection point of the curve *oh* and the curve *ad* are *g* (k_b, 1.029943228). In figure 3.35, if $k \geq k_\mathrm{b}$, such as in the section *gh*, $\sigma_\mathrm{e}^\mathrm{T} > \sigma_\mathrm{y}$.

$$\frac{1}{\sqrt{3}} - \frac{(k+1)^2}{4\sqrt{3}k^2} + \frac{2}{\sqrt{3}}\ln\frac{k+1}{2} - \left(\frac{k-1}{\sqrt{3}k} + \frac{1}{\sqrt{3}}\ln k\right)$$

$$= \frac{(\sqrt{k}-1)^3(15k\sqrt{k}+11k+5\sqrt{k}+1)}{4\sqrt{3}k^2(k+1+2\sqrt{k})} \geq 0$$

This means that the load-bearing capacity when $k_\mathrm{j} = \sqrt{k}$ is smaller than that when $k_\mathrm{j} = \frac{k+1}{2}$, as is seen in figure 3.35. This is because $\frac{k+1}{2} > \sqrt{k}$, or the plastic depth when $k_\mathrm{j} = \frac{k+1}{2}$ is larger than that when $k_\mathrm{j} = \sqrt{k}$.

From figure 3.35, we can also see that when $k \leq k_\mathrm{b}$, $\frac{k-1}{\sqrt{3}k} + \ln\sqrt{k} < \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$; when $k \geq k_\mathrm{b}$, $\frac{k-1}{\sqrt{3}k} + \ln\sqrt{k} > \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$. However, when $k \geq k_\mathrm{b}$, $\left|\sigma_\mathrm{ei}'\right| > \sigma_\mathrm{y}$ or reverse yield will occur due to $k_\mathrm{j} = \sqrt{k}$.

The total stresses when $k = 1.366025 = k_\mathrm{g}$, $k_\mathrm{j} = \sqrt{k} = 1.1687707$ and $\frac{p}{\sigma_\mathrm{y}} = \frac{k-1}{\sqrt{3}k} + \frac{1}{\sqrt{3}}\ln k$ are illustrated in figure 3.44.

The total stresses when $k = 1.2 < k_\mathrm{g}$, $k_\mathrm{j} = \sqrt{k} = 1.0954451$ and $\frac{p}{\sigma_\mathrm{y}} = \frac{k-1}{\sqrt{3}k} + \frac{1}{\sqrt{3}}\ln k$ are illustrated in figure 3.45.

The total stresses when $k = 1.5 > k_\mathrm{g}$, $k_\mathrm{j} = \sqrt{k} = 1.2247449$ and $\frac{p}{\sigma_\mathrm{y}} = \frac{k-1}{\sqrt{3}k} + \frac{1}{\sqrt{3}}\ln k$ are illustrated in figure 3.46.

The meaning of each curve and main parameters have been marked in the figures.

In figure 3.44, since $k = k_\mathrm{g}$, at $x = k_\mathrm{j}$, $\sigma_{\theta\mathrm{j}}^\mathrm{T}$ is just $\sigma_\mathrm{e}^\mathrm{T}$, and they are just σ_y, and $x_\mathrm{z} = x_\mathrm{a} = k_\mathrm{j}$.

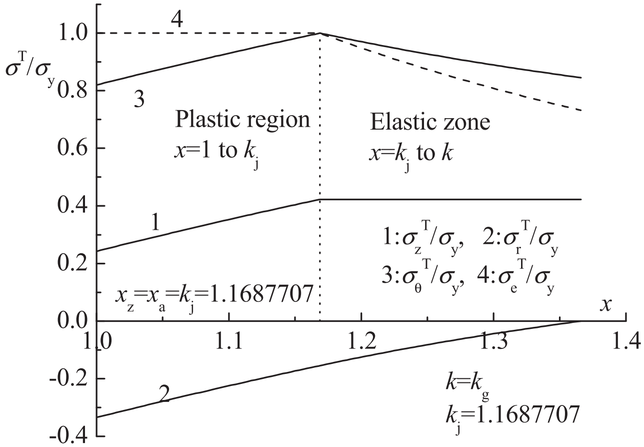

FIG. 3.44 – The total stresses with $k = k_g$, $k_j = \sqrt{k}$ and $\frac{p}{\sigma_y} = \frac{k-1}{\sqrt{3}k} + \frac{1}{\sqrt{3}}\ln k$.

In figure 3.45, from equation (3.70), $x_z = 1.041352$; $k_j = 1.0954451 > x_a = 1.02672$, so, x_z exists and x_a does not exist. Because $k = 1.2 < k_{x1}$, $p < \sigma_y$ and $\sigma'_{ri} > -\sigma_y$. Because $k < k_g$, $k_j > \sqrt{\sqrt{3}-1}k$, thus $\sigma^T_{\theta j} > \sigma_y$.

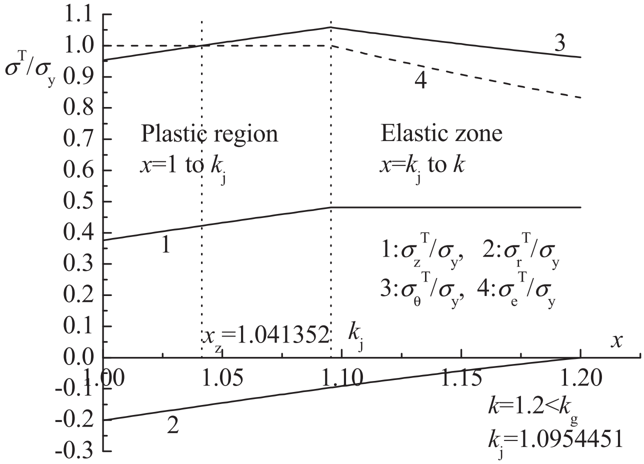

FIG. 3.45 – The total stresses with $k = 1.2 < k_g$, $k_j = \sqrt{k} = 1.0954451$ and $\frac{p}{\sigma_y} = \frac{k-1}{\sqrt{3}k} + \frac{1}{\sqrt{3}}\ln k$.

In figure 3.46, $x_a = 1.2834$; $k_j = 1.2247449 < x_a$, so, x_a exists and x_z does not exist. Because $k < k_{x1}$, $p < \sigma_y$ and $\sigma^T_{ri} > -\sigma_y$. Because $k > k_d$, $k_j < \sqrt{\sqrt{3}-1}k$, thus $\sigma^T_{\theta j} < \sigma_y$.

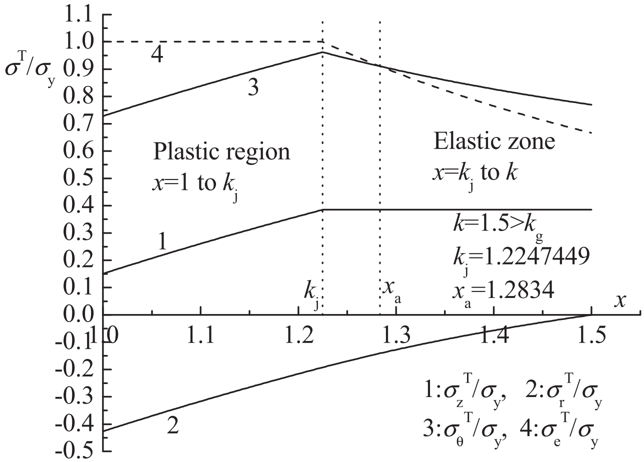

FIG. 3.46 – The total stresses with $k = 1.5 > k_g$, $k_j = \sqrt{k} = 1.2247449$ and $\frac{p}{\sigma_y} = \frac{k-1}{\sqrt{3}k} + \frac{1}{\sqrt{3}}\ln k$.

In addition, when $k_j = \sqrt{k}$, we have the following results.

(1) Autofrettage pressure

$$\frac{p_a}{\sigma_y} = \frac{k-1}{\sqrt{3}k} + \frac{1}{\sqrt{3}}\ln k = \frac{p}{\sigma_y}$$

which is just equation (3.131). As mentioned above, this is because the general equation of autofrettage pressure $\frac{p_a}{\sigma_y} = \frac{2}{\sqrt{3}}\ln k_j + \frac{k^2-k_j^2}{\sqrt{3}k^2}$ is obtained just by setting $\frac{\sigma_e^T}{\sigma_y} = 1$ in the plastic region, and $\frac{p}{\sigma_y} = \frac{p_a}{\sigma_y}$ also results in $\frac{\sigma_e^T}{\sigma_y} = 1$.

(2) The abscissa of the intersection of the three residual stress curves or the abscissa at which $\sigma'_e = 0$, x_0, is the same as based on the maximum shear stress theory, or

$$x_0 = \sqrt{\frac{k^2 - k + k^2\ln k}{k^2 - 1}} < k$$

(3) The value of residual stresses when $x = x_0$ or the minimum stresses are

$$y_0^d = y_0^d = \frac{\sigma'_{zx0}}{\sigma_y} = \frac{\sigma'_{\theta x0}}{\sigma_y} = \frac{\sigma'_{rx0}}{\sigma_y} = \frac{1}{\sqrt{3}}\ln\frac{k-1+k\ln k}{k^2-1} + \frac{k-1-\ln k}{\sqrt{3}(k^2-1)}.$$

(4) The abscissa at which $\sigma'_z = 0$, or x_1 is the same as based on the maximum shear stress theory, which is

$$x_1 = \sqrt{e^{\frac{1-k+k^2\ln k}{k^2-1}}}$$

3.5 The Solutions with Minimum Equivalent Total Stress on Elastic-Plastic Juncture

On the elastic-plastic juncture, $x = r/r_\mathrm{i} = k_\mathrm{j}$,

$$\frac{\sigma_\mathrm{ej}^\mathrm{T}}{\sigma_\mathrm{y}} = 1 - \frac{1}{k^2-1}\left(\frac{k^2}{k_\mathrm{j}^2} - 1 + \frac{2k^2}{k_\mathrm{j}^2}\ln k_\mathrm{j}\right) + \frac{\sqrt{3}k^2}{(k^2-1)k_\mathrm{j}^2}\frac{p}{\sigma_\mathrm{y}} \tag{3.133}$$

Setting $\frac{\mathrm{d}\left(\frac{\sigma_\mathrm{ej}^\mathrm{T}}{\sigma_\mathrm{y}}\right)}{\mathrm{d}k_\mathrm{j}} = 0$ results in

$$k_\mathrm{j} = \frac{r_\mathrm{j}}{r_\mathrm{i}} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right) \quad \text{or} \quad \varepsilon = \frac{\exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right) - 1}{k-1} \tag{3.134}$$

If $k_\mathrm{j} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right)$, the equivalent total stress at the elastic-plastic juncture $\sigma_\mathrm{ej}^\mathrm{T}$ is the minimum when k_j is between 1 and k, and the minimum is

$$\frac{\sigma_\mathrm{ej}^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{k^2}{k^2-1}\frac{\exp\left(\sqrt{3}\frac{p}{\sigma_\mathrm{y}}\right) - 1}{\exp\left(\sqrt{3}\frac{p}{\sigma_\mathrm{y}}\right)} = \frac{k^2}{k^2-1}\frac{k_\mathrm{j}^2 - 1}{k_\mathrm{j}^2} \tag{3.135}$$

When $\frac{p}{\sigma_\mathrm{y}} = \frac{p_\mathrm{e}}{\sigma_\mathrm{y}} = \frac{k^2-1}{\sqrt{3}k^2}$, from equation (3.134),

$$k_\mathrm{j} = \exp\left(\frac{k^2-1}{2k^2}\right) \tag{3.136}$$

Therefore, as based on the maximum shear stress theory in the last chapter, when the load is initial yield pressure p_e, the plastic depth determined by equations (3.136) or (3.134) cannot exceed radius ratio k.

When $\frac{p}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}} = \frac{k^2-1}{k^2}$, from equation (3.134),

$$k_\mathrm{j} = \exp\left(\frac{k^2-1}{k^2}\right) \tag{3.137}$$

Letting $k_\mathrm{j} = \exp\left(\frac{k^2-1}{k^2}\right) \leq k$ also obtains $\frac{k^2\ln k}{k^2-1} \geq 1$. We have known that when $k \geq k_\mathrm{c} = 2.2184574899167\ldots$, the equation $\frac{k^2\ln k}{k^2-1} \geq 1$ is tenable. Therefore, based on the maximum distortion strain energy theory, when $\frac{p}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}} = \frac{k^2-1}{k^2}$, the plastic depth calculated by equations (3.134) or (3.137) can not exceed radius ratio k if $k > k_\mathrm{c}$; otherwise, or $k < k_\mathrm{c}$, when $\frac{p}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}} = \frac{k^2-1}{k^2}$, the plastic depth calculated by equations (3.134) or (3.137) is larger than radius ratio k.

For $k < k_\mathrm{c}$, in equation (3.134), the setting $k_\mathrm{j} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_\mathrm{y}}\right) \leq k$ leads to

$$\frac{p}{\sigma_\mathrm{y}} \leq \frac{2}{\sqrt{3}}\ln k = \frac{p_\mathrm{y}}{\sigma_\mathrm{y}}$$

Setting $\frac{p}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2} = \frac{2}{\sqrt{3}}\ln k$ just obtains equation (3.40) but " ≤ " changed into " = ", or $\frac{k^2 \ln k}{k^2-1} = 1$. This indicates that when $k > k_\mathrm{c}$, $\ln k > \frac{k^2-1}{k^2}$, when $\frac{p}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$, $k_\mathrm{j} < k$, but the maximum load is unlikely $\frac{p}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$; when $k < k_\mathrm{c}$, $\ln k < \frac{k^2-1}{k^2}$, the maximum load is $\frac{p}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}\ln k$, when $\frac{p}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}\ln k$, $k_\mathrm{j} = k \left(\text{and } k_j \text{ is smaller than } \exp\left(\frac{k^2-1}{k^2}\right)\right)$, when $k = k_\mathrm{c}$, $\ln k = \frac{k^2-1}{k^2}$, $k_\mathrm{j} = k = k_\mathrm{c}$, the maximum load is $\frac{p}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}\frac{k_\mathrm{c}^2-1}{k_\mathrm{c}^2} = \frac{2}{\sqrt{3}}\ln k_\mathrm{c}$.

Equation (3.137) is equivalent to $\frac{k^2 \ln k_\mathrm{j}}{k^2-1} = 1$. Clearly, when $k = k_\mathrm{c}$, there is that $k_\mathrm{j} = k = k_\mathrm{c}$, that is to say, when $k = k_\mathrm{c}$, the plastic depth calculated by equation (3.137) is equal to the plastic depth calculated by equation (3.42). According to equation (3.137), with the increase of k, k_j increase, and according to equation (3.42), with the increase of k, k_j decreases. Therefore, when $k > k_\mathrm{c}$, the plastic depth calculated by equation (3.137) is certainly larger than the plastic depth calculated by equation (3.42).

According to equations (3.136) and (3.42), when $k = \infty$, $k_\mathrm{j} = \sqrt{\mathrm{e}}$, k_j calculated by equation (3.136) increases with the increase of k, and k_j calculated by equation equation (3.42) decreases with the increase of k. Therefore, k_j calculated by equation (3.42) is always larger than k_j calculated by equation (3.136).

$k_\mathrm{j} = \exp\left(\frac{k^2-1}{2k^2}\right)$ is plotted in figure 3.30 as curve *om*, and $k_\mathrm{j} = \exp\left(\frac{k^2-1}{k^2}\right)$ is plotted in figure 3.30 as dash curve *oan*.

Substituting equation (3.134) into equation (3.2) into equation (3.5) obtains the residual stresses within the plastic region

$$\frac{\sigma_\mathrm{z}'}{\sigma_\mathrm{y}} = \frac{1}{\sqrt{3}}\left[\ln x^2 + \frac{\exp(\sqrt{3}\frac{p}{\sigma_\mathrm{y}}) - 1}{k^2-1} - \frac{\sqrt{3}k^2}{k^2-1}\frac{p}{\sigma_\mathrm{y}}\right] \tag{3.138}$$

$$\frac{\sigma_\mathrm{r}'}{\sigma_\mathrm{y}} = \frac{1}{\sqrt{3}}\left[\frac{\exp(\sqrt{3}\frac{p}{\sigma_\mathrm{y}})}{k^2} - 1 + \ln x^2 - \sqrt{3}\frac{p}{\sigma_\mathrm{y}} - \left(1 - \frac{\exp(\sqrt{3}\frac{p}{\sigma_\mathrm{y}})}{k^2} + \sqrt{3}\frac{p}{\sigma_\mathrm{y}}\right)\frac{1}{k^2-1}\left(1 - \frac{k^2}{x^2}\right)\right] \tag{3.139}$$

$$\frac{\sigma_\theta'}{\sigma_\mathrm{y}} = \frac{1}{\sqrt{3}}\left[\frac{\exp(\sqrt{3}\frac{p}{\sigma_\mathrm{y}})}{k^2} + 1 + \ln x^2 - \sqrt{3}\frac{p}{\sigma_\mathrm{y}} - \left(1 - \frac{\exp(\sqrt{3}\frac{p}{\sigma_\mathrm{y}})}{k^2} + \sqrt{3}\frac{p}{\sigma_\mathrm{y}}\right)\frac{1}{k^2-1}\left(1 + \frac{k^2}{x^2}\right)\right] \tag{3.140}$$

$$\frac{\sigma_\mathrm{e}'}{\sigma_\mathrm{y}} = 1 - \frac{k^2 - \exp\left(\sqrt{3}\frac{p}{\sigma_\mathrm{y}}\right) + \sqrt{3}k^2\frac{p}{\sigma_\mathrm{y}}}{k^2-1}\frac{1}{x^2} \tag{3.141}$$

Substituting equation (3.134) into equation (3.6) into equation (3.9) obtains the residual stresses within the elastic region

$$\frac{\sigma_z'}{\sigma_y} = \frac{\exp\left(\sqrt{3}\frac{p}{\sigma_y}\right) - 1 - \sqrt{3}\frac{p}{\sigma_y}}{\sqrt{3}(k^2-1)} = \text{a} \quad \text{constant} > 0 \tag{3.142}$$

$$\frac{\sigma_r'}{\sigma_y} = \left(1 - \frac{k^2}{x^2}\right)\frac{\sigma_z'}{\sigma_y} < 0 \tag{3.143}$$

$$\frac{\sigma_\theta'}{\sigma_y} = \left(1 + \frac{k^2}{x^2}\right)\frac{\sigma_z'}{\sigma_y} > 0 \tag{3.144}$$

$$\frac{\sigma_e'}{\sigma_y} = \frac{\exp\left(\sqrt{3}\frac{p}{\sigma_y}\right) - 1 - \sqrt{3}\frac{p}{\sigma_y}}{k^2-1}\frac{k^2}{x^2} > 0 \tag{3.145}$$

When $k = 2$, $\frac{p}{\sigma_y} = \frac{k^2-1}{\sqrt{3}k^2} = 0.433012702$, $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right) = 1.454991$, $x_a = 1.711199$. The residual stresses within the whole wall are plotted in figure 3.47. The meaning of each curve and the main parameters are marked in the figure.

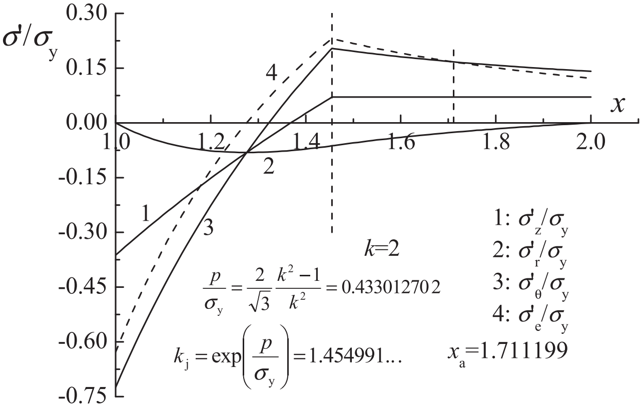

FIG. 3.47 – The residual stresses within the whole wall when $\frac{p}{\sigma_y} = \frac{k^2-1}{\sqrt{3}k^2} = 0.433012702$.

When $k = 2$, $\frac{p}{\sigma_y} = \frac{2}{\sqrt{3}}\ln k = 0.8003774...$, $k_j = k = 2$. The whole wall is plastic, the residual stresses within the whole wall are plotted in figure 3.48. The meaning of each curve and the main parameters are marked in the figure.

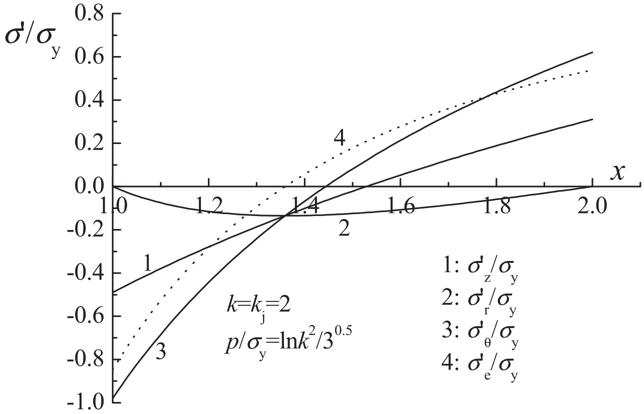

FIG. 3.48 – The residual stresses within the whole wall when $k_j = k = 2$.

When $k = 3$, $\frac{p}{\sigma_y} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2} = 1.0264005\ldots$, $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right) = 2.432425\ldots$, $x_a = 2.566799\ldots$. The residual stresses within the whole wall are plotted in figure 3.49. The meaning of each curve and the main parameters are marked in the figure.

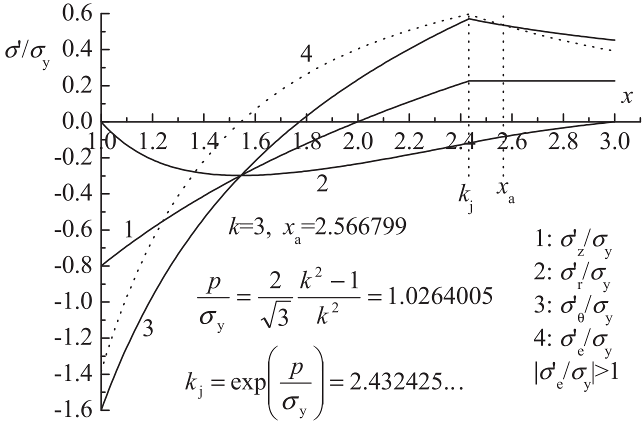

FIG. 3.49 – The residual stresses within the whole wall when $\frac{p}{\sigma_y} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2} = 1.0264005$.

When $k = 3$, $\frac{p}{\sigma_y} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2} = 1.0264005$, $\frac{\sigma'_{ei}}{\sigma_y} < -1$. According to $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right)$, we can not directly determine whether the equivalent residual stress on the inner surface σ'_{ei} is greater than strength limit σ_y. When $k \geq k_c$, in order to avoid reversed yield, for equation (3.141), setting $\frac{\sigma'_{ei}}{\sigma_y} \geq -1$ obtains

$$\exp\left(\sqrt{3}\,\frac{p}{\sigma_{\mathrm{y}}}\right)+k^2-\sqrt{3}k^2\,\frac{p}{\sigma_{\mathrm{y}}}-2\geq 0 \quad \text{or} \quad k^2\leq\frac{\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right)-2}{\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}-1} \tag{3.146}$$

According to equations (3.126) and (2.128), for the same k, the load-bearing capacity based on the maximum distortion strain energy theory is $\frac{2}{\sqrt{3}}$ times of that based on the maximum shear stress theory. When based on the maximum distortion strain energy theory, $k_{\mathrm{j}}=\frac{r_{\mathrm{j}}}{r_{\mathrm{i}}}=\exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right)$. Therefore, according to equation (3.134), the plastic depths are the same based on both the theories for the same k. For example, when $k=4$, $\frac{p}{\sigma_{\mathrm{y}}}=\frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}=1.082532$, $k_{\mathrm{j}}=\exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right)=2.553589$, this is the plastic depth based on the maximum distortion strain energy theory; when $k=4$, $\frac{p}{\sigma_{\mathrm{y}}}=\frac{k^2-1}{k^2}=0.9375$, $k_{\mathrm{j}}=\exp\left(\frac{p}{\sigma_{\mathrm{y}}}\right)=2.553589$, this is the plastic depth based on the maximum shear stress theory. It can be seen that the plastic depth based on both theories are the same.

When $k\geq k_{\mathrm{c}}$, the load-bearing capacity $\frac{p}{\sigma_{\mathrm{y}}}$ is determined by equation (3.146), as is shown by curve *am* in figure 3.35. From equation (3.146), the definition domain of k^2 is $\frac{p}{\sigma_{\mathrm{y}}}\geq\frac{1}{\sqrt{3}}=p_{\mathrm{b}}$ and $\frac{p}{\sigma_{\mathrm{y}}}\leq\frac{1}{\sqrt{3}}\ln 2=p_{\mathrm{c}}$. The function $k^2=\frac{\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right)-2}{\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}-1}$ is plotted in figure 3.50. There are two branches for the graphs of the function: curve *og* and curve *bah*. In addition, the function $k_{\mathrm{j}}=\exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right)$ is plotted in figure 3.50 as the solid curve *odai* and the function $\frac{p}{\sigma_{\mathrm{y}}}=\frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$ is plotted in figure 3.50 as the dash curve *ofac*. In figure 3.50, curve *og* corresponds to the definition domain of $\frac{p}{\sigma_{\mathrm{y}}}\leq\frac{1}{\sqrt{3}}\ln 2$. Since $k<1$ on curve *og*, it is of no practical significance; curve *bah* corresponds to the definition domain of $\frac{p}{\sigma_{\mathrm{y}}}>\frac{1}{\sqrt{3}}$. For the function $k^2=\frac{\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right)-2}{\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}-1}$, setting $\frac{\mathrm{d}k^2}{\mathrm{d}\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right)}=0$ gives

$$\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\exp\left(\sqrt{3}\,\frac{p}{\sigma_{\mathrm{y}}}\right)-\exp\left(\sqrt{3}\,\frac{p}{\sigma_{\mathrm{y}}}\right)+1=0 \tag{3.147}$$

Solving equation (3.147) obtains the horizontal ordinate of point a, that is the minimum value of *bah*.

$$\frac{p}{\sigma_{\mathrm{y}}}=0.920079 \tag{3.148}$$

Substituting $\frac{p}{\sigma_{\mathrm{y}}}=0.920079$ into $k^2=\frac{\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right)-2}{\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}-1}$ leads to $k=k_{\mathrm{c}}$.

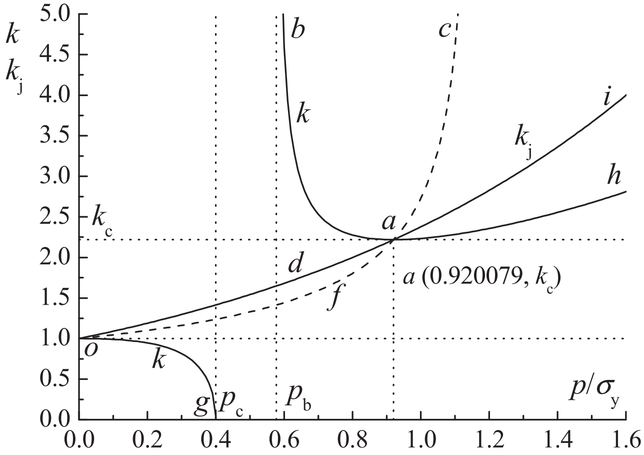

FIG. 3.50 – The relationship between k (k_j) and $\frac{p}{\sigma_y}$.

Letting $k^2 = \frac{\exp\left(\sqrt{3}\frac{p}{\sigma_y}\right)-2}{\sqrt{3}\frac{p}{\sigma_y}-1} = k_j^2 \exp\left(\sqrt{3}\,\frac{p}{\sigma_y}\right)$ also obtains equation (3.147). When $k = k_j$, substituting $k^2 = k_j^2 \exp\left(\sqrt{3}\,\frac{p}{\sigma_y}\right)$ into $\frac{p}{\sigma_y} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$ obtains equation (3.147) again. This shows that the three curves *bah*, *ofac* and *odai* intersect at one point. This point is just the minimum value point a of curve *bah*. From figure 3.50, when $\frac{p}{\sigma_y} < 0.920079...$, $k > k_c$, and k is larger than k_j calculated by $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right)$, and k is larger than the value determined by $\frac{p}{\sigma_y} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$. This is reasonable and makes sense; when $\frac{p}{\sigma_y} > 0.920079...$, $k > k_c$, and k is smaller than k_j determined by $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right)$, this is unreasonable and makes no sense. Curve *odac* can ensure $k > k_j$, but cannot ensure $\sigma'_{ei} \geq -\sigma_y$. Therefore, when $k > k_c$, the meaningful control curve is curve *ba*, which is also curve *am* in figures 3.35 or 2.23. If the design conditions are controlled by curve *ba* when $k > k_c$, there are both $k_j < k$ and $\sigma'_{ei} \geq -\sigma_y$. For example, when $k = 3$, from equation (3.146), $\frac{p}{\sigma_y} = 0.64516....$ When $k = 3$ and $\frac{p}{\sigma_y} = 0.64516...$, $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right) = 1.748442...$ The residual stresses within the whole wall are plotted in figure 3.51. The meaning of each curve and the main parameters are marked in the figure.

Compared with figure 3.49, the residual stresses are greatly improved.

In fact, when $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right)$, equations (3.147) and (3.146) are equivalent. When $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right)$ (but $k > k_j$), equation (3.146) becomes equation (3.42), or $k^2 \ln k_j^2 - k^2 - k_j^2 + 2 \leq 0$.

Therefore, curve *ad* in figure 3.30 is also the relation between k and k_j under the case that $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right)$ and $\sigma'_{ei} = -\sigma_y$. However, when the load-bearing capacity

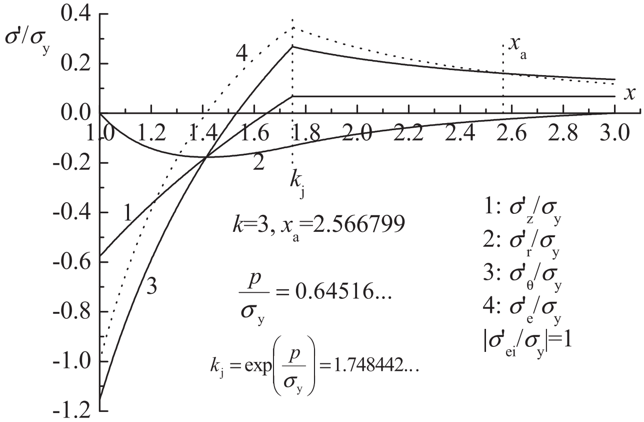

FIG. 3.51 – The residual stresses when $k = 3$, $\frac{p}{\sigma_\mathrm{y}} = 0.64516\ldots$

$\frac{p}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$, $\sigma_\mathrm{ej} \equiv \sigma_\mathrm{y}$ within the whole plastic region; when the load-bearing capacity $\frac{p}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}\ln k_\mathrm{j}$ $\left(\text{equivalent to } k_\mathrm{j} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_\mathrm{y}}\right)\right)$ where $k^2 \ln k_\mathrm{j}^2 - k^2 - k_\mathrm{j}^2 + 2 \leq 0$, $\sigma_\mathrm{ej} < \sigma_\mathrm{y}$.

Equation (3.146) can ensure $k_\mathrm{j} < k$ and $\sigma'_\mathrm{ei} \geq -\sigma_\mathrm{y}$, but under the premise of ensuring $k_\mathrm{j} < k$ and $\sigma'_\mathrm{ei} \geq -\sigma_\mathrm{y}$ as well as σ_ej being the minimum, the load-bearing capacity $\frac{p}{\sigma_\mathrm{y}}$ calculated by $k_\mathrm{j} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_\mathrm{y}}\right)$ may not be the most satisfactory. From equations (3.146) or (3.42), the greater k is, the smaller k_j is, or the smaller $\frac{p}{\sigma_\mathrm{y}}$ is, because $k_\mathrm{j} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_\mathrm{y}}\right)$. When $k = \infty$, $k_\mathrm{j} = \mathrm{e}^{0.5}$, hence $\frac{p}{\sigma_\mathrm{y}} = \frac{1}{\sqrt{3}}$, that is merely the maximum elastic load with $k = \infty$. Besides, when $k_\mathrm{j} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_\mathrm{y}}\right)$, from figure 3.35, the load-bearing capacity under the case that $k > k_\mathrm{c}$ is unexpectedly smaller than that under the case that $k < k_\mathrm{c}$. So, if it is not necessary, $k_\mathrm{j} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_\mathrm{y}}\right)$ is not a satisfactory solution.

When $\exp\left(\sqrt{3}\frac{p}{\sigma_\mathrm{y}}\right) + k^2 - \sqrt{3}k^2\frac{p}{\sigma_\mathrm{y}} - 2 \geq 0$ (equation (3.146)), equation (3.141) become

$$\frac{\sigma'_\mathrm{e}}{\sigma_\mathrm{y}} = 1 - \frac{k^2 - \exp\left(\sqrt{3}\frac{p}{\sigma_\mathrm{y}}\right) + \sqrt{3}k^2\frac{p}{\sigma_\mathrm{y}}}{k^2-1}\frac{1}{x^2} = 1 - \frac{2}{x^2} \tag{3.149}$$

When $k_\mathrm{j} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_\mathrm{y}}\right)$, autofrettage pressure

$$\frac{p_\mathrm{a}}{\sigma_\mathrm{y}} = \frac{p}{\sigma_\mathrm{y}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}k^2}\exp\left(\sqrt{3}\frac{p}{\sigma_\mathrm{y}}\right) \tag{3.150}$$

When $\frac{p}{\sigma_\mathrm{y}} \leq \frac{2}{\sqrt{3}}\ln k$, $\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}k^2}\exp\left(\sqrt{3}\frac{p}{\sigma_\mathrm{y}}\right) \geq 0$. Therefore, $\frac{p_\mathrm{a}}{\sigma_\mathrm{y}} \geq \frac{p}{\sigma_\mathrm{y}}$.

When $k_{\mathrm{j}} = k = k_{\mathrm{c}}$, $k_{\mathrm{j}}^2 = \exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right) = k^2$, then, from equation (3.150), $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}} \ln k_{\mathrm{c}} = 0.920079$.

$\frac{\mathrm{d}(p_{\mathrm{a}}/\sigma_{\mathrm{y}})}{\mathrm{d}(p/\sigma_{\mathrm{y}})} = 1 - \frac{1}{k^2}\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right)$. Setting $\frac{d(p_{\mathrm{a}}/\sigma_{\mathrm{y}})}{d(p/\sigma_{\mathrm{y}})} = 0$ gives $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}} \ln k = \frac{p_{\mathrm{y}}}{\sigma_{\mathrm{y}}}$. $\frac{\mathrm{d}^2(p_{\mathrm{a}}/\sigma_{\mathrm{y}})}{\mathrm{d}(p/\sigma_{\mathrm{y}})^2} = -\frac{\sqrt{3}}{k^2}\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right) < 0$. Consequently, when $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}} \ln k$, $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$ reaches its maximum value, which is $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}} \ln k$. It should be noted that only if $k \leq k_{\mathrm{c}}$, can we have $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}} \ln k$. For $k \geq k_{\mathrm{c}}$, from equation (3.146), we obtain $\frac{p}{\sigma_{\mathrm{y}}} - \frac{1}{\sqrt{3}k^2}\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right) = \frac{k^2-2}{\sqrt{3}k^2}$. Substituting this equation into equation (3.150) gives

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2} \tag{3.151}$$

Equation (3.30) indicates that in general, when the load-bearing capacity $\frac{p}{\sigma_{\mathrm{y}}}$ equals autofrettage pressure, or $\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$, it is the optimum. In this case, $\sigma_{\mathrm{e}}^{\mathrm{T}} \equiv \sigma_{\mathrm{y}}$ in the whole plastic region. So, when $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$, the load-bearing capacity is also $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$. In the case that $k_{\mathrm{j}} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right)$ and $k \geq k_{\mathrm{c}}$, when $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$, if $\frac{p}{\sigma_{\mathrm{y}}}$ is controlled by equation (3.146), $\frac{p}{\sigma_{\mathrm{y}}} < \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$, and $\sigma_{\mathrm{ej}}^{\mathrm{T}} < \sigma_{\mathrm{y}}$. For example, when $k = 3$, from equation (3.146) $\frac{p}{\sigma_{\mathrm{y}}} = 0.64516$, and from equation (3.151) $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = 1.0264$, from equation (3.135) $\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 0.756998$.

When $k \leq k_{\mathrm{c}}$, the optimum load-bearing capacity is $\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}} \ln k$, and in this case $\sigma_{\mathrm{e}}^{\mathrm{T}}\sigma_{\mathrm{e}}^{\mathrm{T}} \equiv \sigma_{\mathrm{y}}$ within the whole plastic region.

When $k = \infty$, from equation (3.151), $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}$; from equation (3.146), $\frac{p}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}$, when $\frac{p}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}$, from equation (3.150), we also have $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}$. When $k = \infty$, equation (3.150) become

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{p}{\sigma_{\mathrm{y}}} + \frac{1}{\sqrt{3}} \tag{3.152}$$

For equation (3.150), when $k \leq k_{\mathrm{c}}$, $\frac{p}{\sigma_{\mathrm{y}}}$ is from 0 to $\frac{2}{\sqrt{3}} \ln k_{\mathrm{c}}$; when $k \geq k_{\mathrm{c}}$, $\frac{p}{\sigma_{\mathrm{y}}}$ is from $\frac{2}{\sqrt{3}} \ln k_{\mathrm{c}}$ to the value calculated by equation (3.146). When $\frac{p}{\sigma_{\mathrm{y}}} = 0$, $k_{\mathrm{j}} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right) = 1$. When the pressure vessel is at the initial yield stage, the pressure applied to the pressure vessel is initial yield load $\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{\sqrt{3}k^2}$. For a pressure vessel with a certain radius ratio k ($\geq k_{\mathrm{c}}$), the maximum load for σ_{ej} to reach the minimum is the value calculated by equation (3.126). When $\frac{p}{\sigma_{\mathrm{y}}}$ equals the value calculated by equation (3.146), $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = 2\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$. For the convenience of engineering applications, some data calculated by equation (3.146) are listed in table 3.1.

Equation (3.150) is plotted in figure 3.52. In figure 3.52, curve *ab* is the connection of the ends of the curves expressed by equation (3.150), which shows the maximum load-bearing capacity $\frac{p}{\sigma_y}$ and the maximum autofrettage pressure $\frac{p_a}{\sigma_y}$ for $k \geq k_c$. Curve *ab* is derived as follows.

From equation (3.146) we can find k^2 expressed by $\frac{p}{\sigma_y}$, substituting k^2 into equation (3.151) we obtain $\frac{p_a}{\sigma_y}$, as a result the equation of curve *ab* in figure 3.52 is generated, that is equation (3.153).

$$\frac{p_a}{\sigma_y} = \frac{2}{\sqrt{3}} \frac{\exp\left(\sqrt{3}\frac{p}{\sigma_y}\right) - \sqrt{3}\frac{p}{\sigma_y} - 1}{\exp\left(\sqrt{3}\frac{p}{\sigma_y}\right) - 2} \tag{3.153}$$

TAB. 3.1 – Numerical value from $\exp\left(\sqrt{3}\frac{p}{\sigma_y}\right) + k^2 - \sqrt{3}k^2\frac{p}{\sigma_y} - 2 \geq 0$.

k	$\frac{p}{\sigma_y}$	$\frac{p_a}{\sigma_y}$	k	$\frac{p}{\sigma_y}$	$\frac{p_a}{\sigma_y}$
2.3	0.781825	0.936421	3.3	0.628887	1.048668
2.4	0.734722	0.954232	3.4	0.624871	1.054813
2.5	0.706831	0.969948	3.5	0.621348	1.060439
2.6	0.687494	0.983887	3.6	0.618235	1.065604
2.7	0.673047	0.996306	3.8	0.61299	1.074735
2.8	0.66176	1.007417	4	0.608751	1.082532
2.9	0.652662	1.0174	4.5	0.601082	1.097678
3	0.645159	1.026401	5	0.595999	1.108513
3.1	0.638863	1.034545	10	0.581613	1.143154
3.2	0.633503	1.041937	∞	0.57735	1.154701

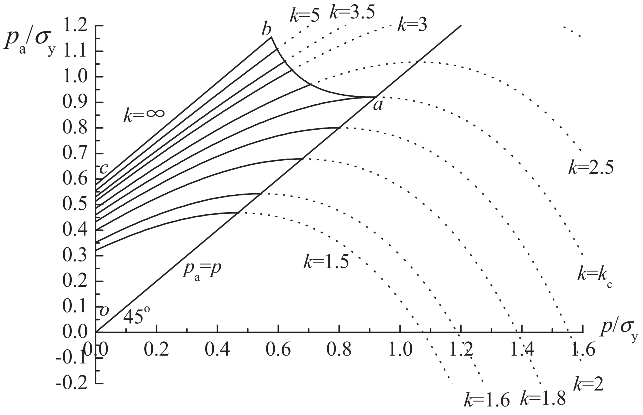

FIG. 3.52 – The graph of $\frac{p_a}{\sigma_y} = \frac{p}{\sigma_y} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}k^2}\exp\left(\sqrt{3}\frac{p}{\sigma_y}\right)$.

We have known that when $k = k_\mathrm{c}$, $\frac{p_\mathrm{a}}{\sigma_\mathrm{y}} = \frac{p}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}\ln k = \frac{2}{\sqrt{3}}\ln k_\mathrm{c}$ and $\frac{k_\mathrm{c}^2 \ln k_\mathrm{c}}{k_\mathrm{c}^2-1} = 1$ or $k_\mathrm{c}{}^2-1 = k_\mathrm{c}{}^2\ln k_\mathrm{c}$. Equation (3.153) is equivalent to $\frac{p_\mathrm{a}}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}\frac{k_\mathrm{c}^2-2\ln k_\mathrm{c}-1}{k_\mathrm{c}^2-2} = \frac{2}{\sqrt{3}}\frac{k_\mathrm{c}^2\ln k_\mathrm{c}-2\ln k_\mathrm{c}}{k_\mathrm{c}^2-2} = \frac{2}{\sqrt{3}}\frac{k_\mathrm{c}^2-2}{k_\mathrm{c}^2-2}\ln k_\mathrm{c} = \frac{2}{\sqrt{3}}\ln k_\mathrm{c}$. This confirms the above conclusion.

The coordinate of point a in figure 3.52 is a $(\frac{2}{\sqrt{3}}\ln k_\mathrm{c}, \frac{2}{\sqrt{3}}\ln k_\mathrm{c})$.

As mentioned above, the meaningful definition domain of k^2 is $\frac{p}{\sigma_\mathrm{y}} \geq \frac{1}{\sqrt{3}}$, and when $\frac{p}{\sigma_\mathrm{y}} = \frac{1}{\sqrt{3}}$, $k^2 = \infty$. Consequently from equation (3.150)–(3.153), $\frac{p_\mathrm{a}}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}$, or the coordinate of point b in figure 3.52 is b $(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}})$.

The coordinate of point c in figure 3.52 is c $(0, \frac{1}{\sqrt{3}})$.

The practically significant curves for a relationship between $\frac{p}{\sigma_\mathrm{y}}$ and $\frac{p_\mathrm{a}}{\sigma_\mathrm{y}}$ are located in a quadrilateral *abco* in figure 3.52, where the straight line *bc* is parallel to the straight line *ao*.

For $k \leq k_\mathrm{c}$, the ends of the curves described by equation (3.150) are on the straight line *ao* in figure 3.52, where $\frac{p_\mathrm{a}}{\sigma_\mathrm{y}} = \frac{p}{\sigma_\mathrm{y}}$. Thereupon the included angle between the straight line *ao* and the abscissa axis or vertical axis is "45°".

In figure 3.52, the dash curves have no practical significance, they are merely an extension of solid curves, or only the elongation of the mathematical relation expressed by equation (3.150).

The total stresses are expressed as follows

$$\begin{cases} \dfrac{\sigma_\mathrm{z}^\mathrm{T}}{\sigma_\mathrm{y}} = \dfrac{\sigma_\mathrm{z}'}{\sigma_\mathrm{y}} + \dfrac{\sigma_\mathrm{z}^\mathrm{p}}{\sigma_\mathrm{y}} \\ \dfrac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}} = \dfrac{\sigma_\mathrm{r}'}{\sigma_\mathrm{y}} + \dfrac{\sigma_\mathrm{r}^\mathrm{p}}{\sigma_\mathrm{y}} \\ \dfrac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}} = \dfrac{\sigma_\theta'}{\sigma_\mathrm{y}} + \dfrac{\sigma_\mathrm{e}^\mathrm{p}}{\sigma_\mathrm{y}} \\ \dfrac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} = \dfrac{\sigma_\mathrm{e}'}{\sigma_\mathrm{y}} + \dfrac{\sigma_\mathrm{e}^\mathrm{p}}{\sigma_\mathrm{y}} \end{cases} \tag{3.154}$$

If the pressure contained in a pressure vessel does not exceed the initial yield pressure, or $p \leq p_\mathrm{e}$, all stresses will not exceed the yield strength σ_y. Generally, autofrettage is not necessary. However, if autofrettage is performed, the stresses will be evened and cut down. Between 1 and k, if $k_\mathrm{j} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_\mathrm{y}}\right)$ $p = p_\mathrm{e}$, $\left(\text{when } p = p_\mathrm{e},\ k_\mathrm{j} = \exp\left(\frac{k^2-1}{2k^2}\right)\right)$, the equivalent total stress at the elastic-plastic juncture $\sigma_\mathrm{ej}^\mathrm{T}$ is the minimum, the minimum is equation (3.135).

When $p = p_\mathrm{e}$, $k_\mathrm{j} = \exp\left(\frac{k^2-1}{2k^2}\right)$, then, equation (3.135) becomes

$$\frac{\sigma_\mathrm{ej}^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{k^2}{k^2-1}\,\frac{\exp\left(\sqrt{3}\frac{p}{\sigma_\mathrm{y}}\right)-1}{\exp\left(\sqrt{3}\frac{p}{\sigma_\mathrm{y}}\right)} = \frac{k^2}{k^2-1}\,\frac{\exp\left(\frac{k^2-1}{k^2}\right)-1}{\exp\left(\frac{k^2-1}{k^2}\right)} \tag{3.155}$$

Several figures are presented to show the above-mentioned conclusions.

For $k = 3$, $\frac{p}{\sigma_y} = \frac{k^2-1}{\sqrt{3}k^2} = \frac{p_e}{\sigma_y} = 0.5132$, $k_j = \exp\left(\frac{k^2-1}{2k^2}\right) = 1.559623$, the operation stresses, residual stresses and total stresses are shown in figures 3.53–3.55, respectively.

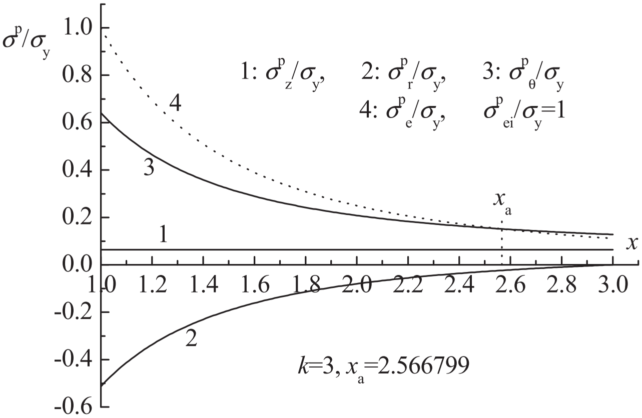

FIG. 3.53 – The distribution of the operation stresses for $k = 3$, $p = p_e$ and $k_j = \exp\left(\frac{k^2-1}{2k^2}\right)$.

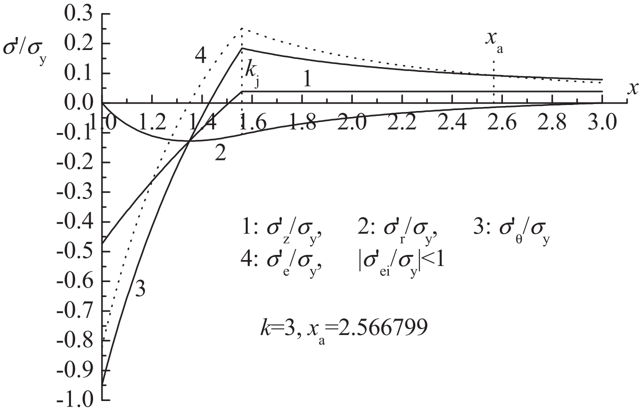

FIG. 3.54 – The distribution of the residual stresses for $k = 3$, $p = p_e$ and $k_j = \exp\left(\frac{k^2-1}{2k^2}\right)$.

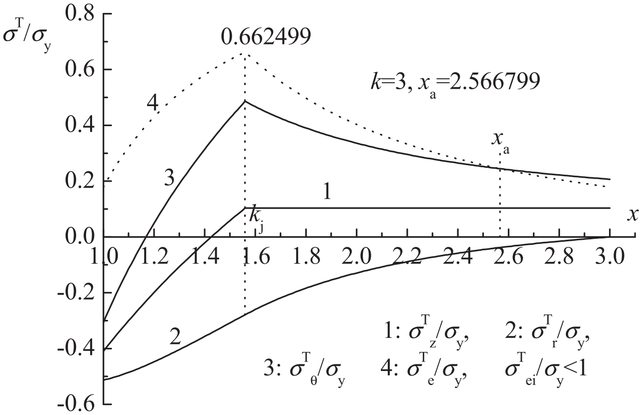

FIG. 3.55 – The distribution of the total stresses for $k = 3$, $p = p_e$ and $k_j = \exp\left(\frac{k^2-1}{2k^2}\right)$.

In figure 3.55, $\frac{\sigma^T_{ej}}{\sigma_y} = 0.662499$.

For $k = 3$, $\frac{p}{\sigma_y} = \frac{k^2-1}{\sqrt{3}k^2} = \frac{p_e}{\sigma_y} = 0.5132$, $k_j = 1.4 < \exp\left(\frac{k^2-1}{2k^2}\right)$, the total stresses are shown in figure 3.56.

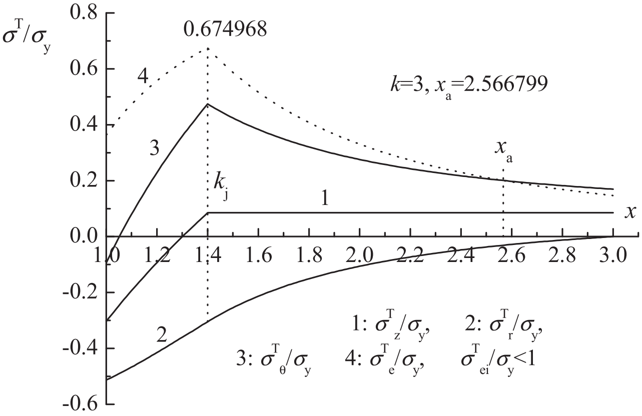

FIG. 3.56 – The distribution of the total stresses for $k = 3$, $p = p_e$ and $k_j = 1.4$.

In figure 3.56, $\frac{\sigma^T_{ej}}{\sigma_y} = 0.674968 > 0.662499$.

For $k = 3$, $\frac{p}{\sigma_y} = \frac{k^2-1}{\sqrt{3}k^2} = \frac{p_e}{\sigma_y} = 0.5132$, $\exp\left(\frac{k^2-1}{2k^2}\right) < k_j = 1.7 < k_{j*} = 1.748442$, the total stresses are shown in figure 3.57.

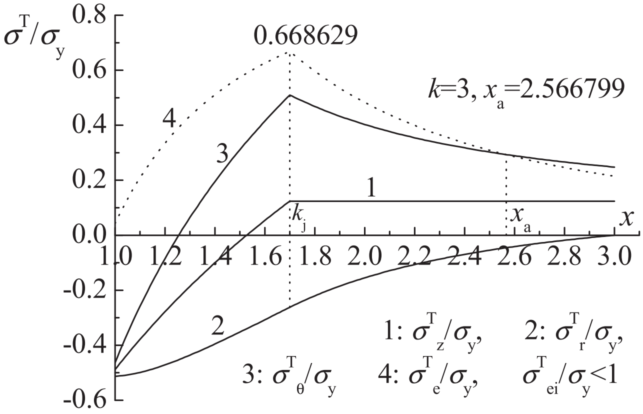

FIG. 3.57 – The distribution of the total stresses for $k = 3$, $p = p_e$ and $k_j = 1.7$.

In figure 3.57, $\frac{\sigma^T_{ej}}{\sigma_y} = 0.668629 > 0.662499$.

For $k = 3$, from equation (3.146), $\frac{p}{\sigma_y} = 0.64516 > \frac{p_e}{\sigma_y}$, $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right) = k_{j*} = 1.748442$, the total stresses are shown in figure 3.58.

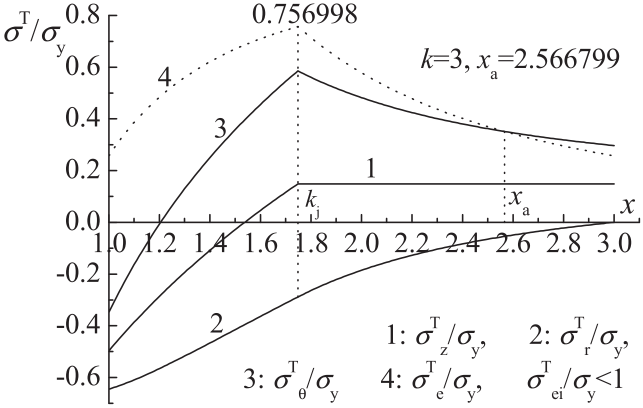

FIG. 3.58 – The distribution of the total stresses for $k = 3$, $p = 0.64516$ and $k_j = k_{j*}$.

In figure 3.58, $\frac{\sigma^T_{ej}}{\sigma_y} = 0.756998$.

For $k = 3$, from equation (3.146), $\frac{p}{\sigma_y} = 0.64516 > \frac{p_e}{\sigma_y}$, $k_j = 1.6 < k_{j*}$, the total stresses are shown in figure 3.59.

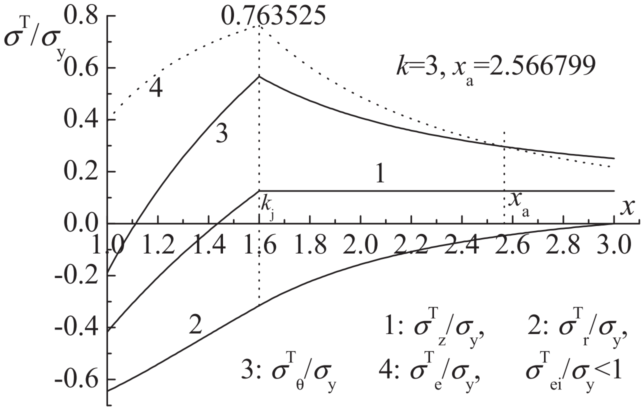

FIG. 3.59 – The distribution of the total stresses for $k = 3$, $p = 0.64516$ and $k_j = 1.6$.

In figure 3.59, $\frac{\sigma^T_{ej}}{\sigma_y} = 0.763525 > 0.756998$.

For $k = 3$, from equation (3.146), $\frac{p}{\sigma_y} = 0.64516 > \frac{p_e}{\sigma_y}$, $k_j = 1.8 > k_{j^*}$, the total stresses are shown in figure 3.60.

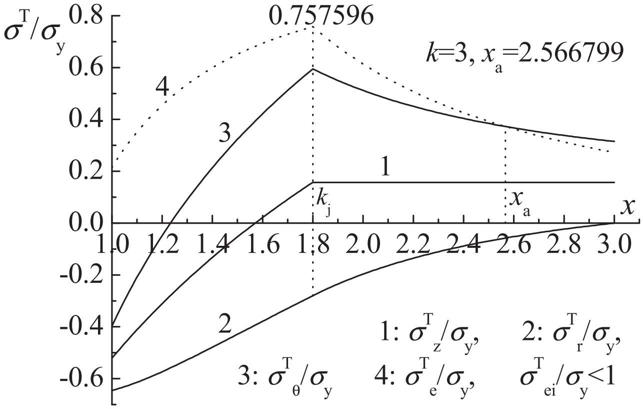

FIG. 3.60 – The distribution of the total stresses for $k = 3$, $p = 0.64516$ and $k_j = 1.8$.

In figure 3.60, $\frac{\sigma^T_{ej}}{\sigma_y} = 0.757596 > 0.756998$.

3.6 Comparison Between the Three Cases

So far, we have investigated autofrettage under four cases: $k_j = k$, $k_j = \frac{k+1}{2}$, $k_j = \sqrt{k}$ and $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right)$. $k_j = k$ is for $k < k_c$, and when $k < k_c$, $\frac{p}{\sigma_y} = \frac{2}{\sqrt{3}}\ln k$, that is equivalent to $k = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right)$. Therefore, when $k < k_{\mathrm{c}}$, $k_{\mathrm{j}} = k$ is equivalent to $k_{\mathrm{j}} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right)$. Then, we do a comparison between three cases: $k_{\mathrm{j}} = \frac{k+1}{2}$, $k_{\mathrm{j}} = \sqrt{k}$ and $k_{\mathrm{j}} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right)$.

3.6.1 $k = 2.5 > k_c$

1. $k_{\mathrm{j}} = \frac{k+1}{2}$

$$k_{\mathrm{j}} = \frac{k+1}{2} \triangleq k_{\mathrm{j}1} = 1.75$$

The equivalent residual stress is as follows.

From equation (3.109), in the plastic region,

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2 - \frac{(k+1)^2}{4} + k^2 \ln \frac{(k+1)^2}{4}}{k^2 - 1}\frac{1}{x^2} = 1 - \frac{1.939561}{x^2}$$

This is the same as that based on the maximum shear stress theory (see last chapter).

From equation (3.113), in the elastic region,

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{\frac{(k+1)^2}{4} - 1 - \ln\frac{(k+1)^2}{4}}{k^2 - 1}\frac{k^2}{x^2} = \frac{1.122939}{x^2}$$

This is the same as that based on the maximum shear stress theory.

From equation (3.117), the load-bearing capacity,

$$\frac{p}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}} - \frac{(k+1)^2}{4\sqrt{3}k^2} + \frac{2}{\sqrt{3}}\ln\frac{k+1}{2} = 0.940637$$

The load-bearing capacity is $\frac{2}{\sqrt{3}}$ times of that based on the maximum shear stress theory.

From equation (1.10), the equivalent operation stress,

$$\frac{\sigma^{\mathrm{p}}_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{\sqrt{3}k^2}{x^2(k^2-1)}\frac{p}{\sigma_{\mathrm{y}}} = \frac{1.939561}{x^2}$$

The load-bearing capacity is different from that based on the maximum shear stress theory, but the equivalent operation stress is the same as that based on the maximum shear stress theory.

From equations (3.29) or (3.154), the equivalent total stress,

$$\frac{\sigma^{\mathrm{T}}_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} + \frac{\sigma^{\mathrm{p}}_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$$

Hence, in the plastic region,

$$\frac{\sigma_{\rm e}^{\rm T}}{\sigma_{\rm y}} \equiv 1$$

In elastic region

$$\frac{\sigma_{\rm e}^{\rm T}}{\sigma_{\rm y}} = \frac{3.0625}{x^2}$$

The equivalent total stress is the same as that based on the maximum shear stress theory.

The autofrettage pressure

$$\frac{p_{\rm a}}{\sigma_{\rm y}} = \frac{p}{\sigma_{\rm y}}$$

2. $k_{\rm j} = \sqrt{k}$

$$k_{\rm j} = k^{0.5} \triangleq k_{\rm j2} = 1.581139$$

The equivalent residual stress is as follows.

From equation (3.124), in the plastic region,

$$\frac{\sigma_{\rm e}'}{\sigma_{\rm y}} = 1 - \frac{k^2 - k + k^2 \ln k}{k^2 - 1}\frac{1}{x^2} = 1 - \frac{1.805108}{x^2}$$

From equation (3.128), in the elastic region

$$\frac{\sigma_{\rm e}'}{\sigma_{\rm y}} = \frac{k - 1 - \ln k}{k^2 - 1}\frac{k^2}{x^2} = \frac{0.694892}{x^2}$$

The equivalent residual stress is the same as that based on the maximum shear stress theory.

From equation (3.131), the load-bearing capacity.

$$\frac{p}{\sigma_{\rm y}} = \frac{k-1}{\sqrt{3}k} + \frac{1}{\sqrt{3}}\ln k = 0.875431$$

The load-bearing capacity is $\frac{2}{\sqrt{3}}$ times of that based on the maximum shear stress theory.

From equation (1.10), the equivalent operation stress

$$\frac{\sigma_{\rm e}^{\rm p}}{\sigma_{\rm y}} = \frac{\sqrt{3}k^2}{x^2(k^2-1)}\frac{p}{\sigma_{\rm y}} = \frac{1.805108}{x^2}$$

The load-bearing capacity is different from that based on the maximum shear stress theory, but the equivalent operation stress is the same as that based on the maximum shear stress theory.

From equations (3.29) or (3.154), the equivalent total stress

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}$$

Then, in the plastic region,

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \equiv 1$$

In the elastic region

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{2.5}{x^2}$$

The equivalent total stress is the same as that based on the maximum shear stress theory.

The autofrettage pressure

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{p}{\sigma_{\mathrm{y}}}$$

3. $k_{\mathrm{j}} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right)$

When $k = 2.5$, from equation (3.146) or table 3.1, the load-bearing capacity $\frac{p}{\sigma_{\mathrm{y}}} = 0.706831$. The load-bearing capacity is $\frac{2}{\sqrt{3}}$ times of that based on the maximum shear stress theory. When $\frac{p}{\sigma_{\mathrm{y}}} = 0.706831$, from equation (3.134), $k_{\mathrm{j}} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right) \triangleq k_{\mathrm{j}3} = 1.844363$.

The equivalent residual stress is as follows.

From equation (3.141), in the plastic region,

$$\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} = 1 - \frac{2}{x^2}$$

From equation (3.145), in the elastic region

$$\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} = \frac{k - 1 - \ln k}{k^2 - 1}\frac{k^2}{x^2} = \frac{1.401675}{x^2}$$

The equivalent residual stress is the same as that based on the maximum shear stress theory.

From equation (1.10), the equivalent operation stress is as follows.

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{\sqrt{3}k^2}{x^2(k^2 - 1)}\frac{p}{\sigma_{\mathrm{y}}} = \frac{1.805108}{x^2}$$

The load-bearing capacity is different from that based on the maximum shear stress theory, but the equivalent operation stress is the same as that based on the maximum shear stress theory.

From equations (3.29) or (3.154), the equivalent total stress

$$\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}}=\frac{\sigma_\mathrm{e}'}{\sigma_\mathrm{y}}+\frac{\sigma_\mathrm{e}^\mathrm{p}}{\sigma_\mathrm{y}}$$

Thus, in the plastic region,

$$\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}}=1-\frac{0.194892}{x^2}$$

In the elastic region

$$\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}}=\frac{3.206783}{x^2}$$

The equivalent total stress is the same as that based on the maximum shear stress theory.

The autofrettage pressure

$$\frac{p_\mathrm{a}}{\sigma_\mathrm{y}}=\frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}=0.969948$$

When $k=2.5$, the figures for the distributions of the equivalent residual stress and the equivalent total stress are the same as figures 2.43 and 2.44, respectively (see last chapter).

3.6.2 $k = 2 < k_c$

1. $k_\mathrm{j}=\frac{k+1}{2}$

$$k_\mathrm{j}=\frac{k+1}{2}\triangleq k_{\mathrm{j}1}=1.5$$

The equivalent residual stress is as follows.

From equation (3.109), in plastic region,

$$\frac{\sigma_\mathrm{e}'}{\sigma_\mathrm{y}}=1-\frac{k^2-\frac{(k+1)^2}{4}+k^2\ln\frac{(k+1)^2}{4}}{k^2-1}\frac{1}{x^2}=1-1.664574\frac{1}{x^2}$$

From equation (3.113), in elastic region

$$\frac{\sigma_\mathrm{e}'}{\sigma_\mathrm{y}}=\frac{\frac{(k+1)^2}{4}-1-\ln\frac{(k+1)^2}{4}}{k^2-1}\frac{k^2}{x^2}=\frac{0.585426}{x^2}$$

The equivalent residual stress is the same as that based on the maximum shear stress theory.

From equation (3.117), the load-bearing capacity.

$$\frac{p}{\sigma_{\mathrm{y}}}=\frac{1}{\sqrt{3}}-\frac{(k+1)^2}{4\sqrt{3}k^2}+\frac{2}{\sqrt{3}}\ln\frac{k+1}{2}=0.720781$$

The load-bearing capacity is $\frac{2}{\sqrt{3}}$ times of that based on the maximum shear stress theory.

From equation (1.10), the equivalent operation stress

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}=\frac{\sqrt{3}k^2}{x^2(k^2-1)}\frac{p}{\sigma_{\mathrm{y}}}=\frac{1.664574}{x^2}$$

The load-bearing capacity is different from that based on the maximum shear stress theory, but the equivalent operation stress is the same as that based on the maximum shear stress theory.

From equations (3.29) or (3.154), the equivalent total stress

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}}+\frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}$$

Then, in the plastic region,

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\equiv 1$$

In elastic region

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{2.25}{x^2}$$

The equivalent total stress is the same as that based on the maximum shear stress theory.

The autofrettage pressure

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{p}{\sigma_{\mathrm{y}}}$$

2. $k_{\mathrm{j}}=\sqrt{k}$

$$k_{\mathrm{j}}=\sqrt{k}\triangleq k_{\mathrm{j}2}=1.414214$$

The equivalent residual stress is as follows.

From equation (3.124), in the plastic region,

$$\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}}=1-\frac{k^2-k+k^2\ln k}{k^2-1}\frac{1}{x^2}=1-\frac{1.590863}{x^2}$$

From equation (3.128), in the elastic region

$$\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} = \frac{k - 1 - \ln k}{k^2 - 1} \frac{k^2}{x^2} = \frac{0.409137}{x^2}$$

The equivalent residual stress is the same as that based on the maximum shear stress theory.

From equation (3.131), the load-bearing capacity.

$$\frac{p}{\sigma_{\mathrm{y}}} = \frac{k - 1}{\sqrt{3}k} + \frac{1}{\sqrt{3}} \ln k = 0.688864$$

The load-bearing capacity is $\frac{2}{\sqrt{3}}$ times of that based on the maximum shear stress theory.

From equation (1.10), the equivalent operation stress

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{\sqrt{3}k^2}{x^2(k^2 - 1)} \frac{p}{\sigma_{\mathrm{y}}} = \frac{1.590863}{x^2}$$

The load-bearing capacity is different from that based on the maximum shear stress theory, but the equivalent operation stress is the same as that based on the maximum shear stress theory.

From equations (3.29) or (3.154), the equivalent total stress

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}$$

Hence, in the plastic region,

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \equiv 1$$

In the elastic region

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{2}{x^2}$$

The equivalent total stress is the same as that based on the maximum shear stress theory.

The autofrettage pressure

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{p}{\sigma_{\mathrm{y}}}$$

3. $k_{\mathrm{j}} = \exp\left(\frac{\sqrt{3}}{2} \frac{p}{\sigma_{\mathrm{y}}}\right)$ and $k_{\mathrm{j}} = k$

When $k = 2 < k_{\mathrm{c}}$, $k_{\mathrm{j}} = k$, the load-bearing capacity $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}} \ln k = 0.800377$, The load-bearing capacity is $\frac{2}{\sqrt{3}}$ times of that based on the maximum shear stress theory, and it is greater than the load-bearing capacity when $k = 2.5$ and $k_\mathrm{j} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_\mathrm{y}}\right)$. When $\frac{p}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}\ln k$, $k_\mathrm{j} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_\mathrm{y}}\right) = k$. There is no elastic region when $k_\mathrm{j} = k$.

From equation (3.80), the equivalent residual stress

$$\frac{\sigma'_\mathrm{e}}{\sigma_\mathrm{y}} = 1 - \frac{k^2 \ln k^2}{(k^2-1)x^2} = 1 - \frac{1.848392}{x^2}$$

This is the same as that based on the maximum shear stress theory.

From equation (1.10), the equivalent operation stress

$$\frac{\sigma^\mathrm{p}_\mathrm{e}}{\sigma_\mathrm{y}} = \frac{1.848392}{x^2}$$

This is the same as that based on the maximum shear stress theory.

The equivalent total stress

$$\frac{\sigma^\mathrm{T}_\mathrm{e}}{\sigma_\mathrm{y}} \equiv 1$$

This is the same as that based on the maximum shear stress theory.

The autofrettage pressure

$$\frac{p_\mathrm{a}}{\sigma_\mathrm{y}} = \frac{p}{\sigma_\mathrm{y}} = 0.800377$$

When $k = 2$, the figures for the distributions of the equivalent residual stress and the equivalent total stress are the same as figures 2.45 and 2.46, respectively (see last chapter).

3.7 Chapter Summary

The main equations and conclusion are listed in table 3.2.

TAB. 3.2 – The main equations and conclusion of this chapter.

The components of residual stresses in general forms
Within the plastic region ($1 \leq x \leq k_\mathrm{j}$)
$\frac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} = \frac{1}{\sqrt{3}}\left[\frac{k_\mathrm{j}^2}{k^2} + \ln\frac{x^2}{k_\mathrm{j}^2} - \left(1 - \frac{k_\mathrm{j}^2}{k^2} + \ln k_\mathrm{j}^2\right)\frac{1}{k^2-1}\right]$
$\frac{\sigma'_\mathrm{r}}{\sigma_\mathrm{y}} = \frac{1}{\sqrt{3}}\left[\frac{k_\mathrm{j}^2}{k^2} - 1 + \ln\frac{x^2}{k_\mathrm{j}^2} - \left(1 - \frac{k_\mathrm{j}^2}{k^2} + \ln k_\mathrm{j}^2\right)\frac{1}{k^2-1}\left(1 - \frac{k^2}{x^2}\right)\right]$
$\frac{\sigma'_\theta}{\sigma_\mathrm{y}} = \frac{1}{\sqrt{3}}\left[\frac{k_\mathrm{j}^2}{k^2} + 1 + \ln\frac{x^2}{k_\mathrm{j}^2} - \left(1 - \frac{k_\mathrm{j}^2}{k^2} + \ln k_\mathrm{j}^2\right)\frac{1}{k^2-1}\left(1 + \frac{k^2}{x^2}\right)\right]$
$\frac{\sigma'_\mathrm{e}}{\sigma_\mathrm{y}} = 1 - \frac{k^2 - k_\mathrm{j}^2 + k^2\ln k_\mathrm{j}^2}{k^2-1}\frac{1}{x^2}$

TAB. 3.2 – (continued).

Within the elastic region ($k_\mathrm{j} \leq x \leq k$) $\dfrac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} = \dfrac{k_\mathrm{j}^2 - 1 - \ln k_\mathrm{j}^2}{\sqrt{3}(k^2-1)} = \text{constant} > 0$ $\dfrac{\sigma'_\mathrm{r}}{\sigma_\mathrm{y}} = \left(1 - \dfrac{k^2}{x^2}\right)\dfrac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} = -\dfrac{k^2 - x^2}{x^2}\dfrac{k_\mathrm{j}^2 - 1 - \ln k_\mathrm{j}^2}{\sqrt{3}(k^2-1)} < 0$ $\dfrac{\sigma'_\theta}{\sigma_\mathrm{y}} = \left(1 + \dfrac{k^2}{x^2}\right)\dfrac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} = \dfrac{k^2 + x^2}{x^2}\dfrac{k_\mathrm{j}^2 - 1 - \ln k_\mathrm{j}^2}{\sqrt{3}(k^2-1)} > 0$ $\dfrac{\sigma'_\mathrm{e}}{\sigma_\mathrm{y}} = \dfrac{k_\mathrm{j}^2 - 1 - \ln k_\mathrm{j}^2}{k^2-1}\dfrac{k^2}{x^2} > 0$ Within the elastic region, when $x < x_\mathrm{a}$ $(=\sqrt{\sqrt{3}-1}k = 0.8556k)$, $\dfrac{\sigma'_\mathrm{e}}{\sigma_\mathrm{y}} > \dfrac{\sigma'_\theta}{\sigma_\mathrm{y}}$; when $x > x_\mathrm{a}$, $\dfrac{\sigma'_\mathrm{e}}{\sigma_\mathrm{y}} < \dfrac{\sigma'_\theta}{\sigma_\mathrm{y}}$
General autofrettage pressure p_a $\dfrac{p}{\sigma_\mathrm{y}} = \dfrac{2}{\sqrt{3}}\ln k_\mathrm{j} + \dfrac{k^2 - k_\mathrm{j}^2}{\sqrt{3}k^2} = \dfrac{p_a}{\sigma_\mathrm{y}}$
The abscissa of the intersection of curves of the residual stress or the abscissa at which $\sigma'_\mathrm{e} = 0$ $x_0 = \dfrac{r}{r_\mathrm{i}} = \sqrt{\dfrac{k^2 - k_\mathrm{j}^2 + k^2 \ln k_\mathrm{j}^2}{k^2-1}} < k_\mathrm{j}$
The value of residual stresses at $x = x_0$ $y_0^\mathrm{d} = \dfrac{\sigma'_\mathrm{zx0}}{\sigma_\mathrm{y}} = \dfrac{{\sigma_{\theta\mathrm{x0}}}'}{\sigma_\mathrm{y}} = \dfrac{\sigma'_\mathrm{rx0}}{\sigma_\mathrm{y}} = \dfrac{\sigma'_\mathrm{rmin}}{\sigma_\mathrm{y}} = \dfrac{2}{\sqrt{3}}\ln\dfrac{x_0}{k_\mathrm{j}} + \dfrac{\sigma'_\mathrm{zj}}{\sigma_\mathrm{y}} = \tfrac{2}{\sqrt{3}}\ln\sqrt{\dfrac{k^2 - k_\mathrm{j}^2 + k^2\ln k_\mathrm{j}^2}{k_\mathrm{j}^2(k^2-1)}} + \dfrac{k_\mathrm{j}^2 - 1 - \ln k_\mathrm{j}^2}{\sqrt{3}(k^2-1)}$
The abscissa at which $\sigma'_\mathrm{z} = 0$ $x_1 = \sqrt{\mathrm{e}^{\frac{1-k_\mathrm{j}^2 + k^2\ln k_\mathrm{j}^2}{k^2-1}}}$
The critical radius ratio $k_\mathrm{c} = 2.218\ 457\ 489\ 916\ 7\ldots$ k_c is the solution of $\dfrac{k^2\ln k}{k^2-1} = 1$, and $\dfrac{k^2\ln k}{k^2-1} = 1$ is equivalent to $2p_\mathrm{e} = p_\mathrm{y}$, *i.e* $2\dfrac{k^2-1}{\sqrt{3}k^2} = \dfrac{2}{\sqrt{3}}\ln k$
The optimum plastic depth $k_{\mathrm{j}*}$ $k^2\ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 \leq 0$ or $k^2 = \dfrac{k_\mathrm{j}^2 - 2}{\ln k_\mathrm{j}^2 - 1}$ $(\mathrm{e}^{0.5} \leq k_{\mathrm{j}*} \leq k_\mathrm{c}$ and $k \geq k_\mathrm{c})$ When $k \leq k_\mathrm{c}$, $k_\mathrm{j} = k$
The results under the case that $k_\mathrm{j} = k_{\mathrm{j}*}$ 1. Residual stresses in axial, radial, and circumferential direction $\begin{cases} \dfrac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} = \dfrac{2}{\sqrt{3}}\ln x - \dfrac{1}{\sqrt{3}} \\ \dfrac{\sigma'_\mathrm{r}}{\sigma_\mathrm{y}} = \dfrac{2}{\sqrt{3}}\ln x + \dfrac{2}{\sqrt{3}x^2} - \dfrac{2}{\sqrt{3}} \\ \dfrac{\sigma'_\theta}{\sigma_\mathrm{y}} = \dfrac{2}{\sqrt{3}}\ln x - \dfrac{2}{\sqrt{3}x^2} \end{cases} \quad (1 \leq x \leq k_{\mathrm{j}*}) \qquad \begin{cases} \dfrac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} = \dfrac{k_{\mathrm{j}*}^2 - 2}{\sqrt{3}k^2} \\ \dfrac{\sigma'_\mathrm{r}}{\sigma_\mathrm{y}} = \left(1 - \dfrac{k^2}{x^2}\right)\dfrac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} \\ \dfrac{\sigma'_\theta}{\sigma_\mathrm{y}} = \left(1 + \dfrac{k^2}{x^2}\right)\dfrac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} \end{cases} \quad (k_{\mathrm{j}*} \leq x \leq k)$

TAB. 3.2 – (continued).

2. Equivalent residual stress

$$\frac{\sigma'_e}{\sigma_y} = 1 - \frac{2}{x^2} \quad (1 \le x \le k_{j*}) \qquad \frac{\sigma'_e}{\sigma_y} = \frac{k_{j*}^2 - 2}{x^2} \ (k_{j*} \le x \le k)$$

3. The optimum load-bearing capability and autofrettage pressure

$$\frac{p}{\sigma_y} = \frac{p_a}{\sigma_y} = 2\frac{p_e}{\sigma_y} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$$

4. $x_0 \equiv \sqrt{2} = \text{constant}$

$x_1 \equiv \sqrt{e} = \text{constant}$

5. $y_0^d = \dfrac{\sigma'_z}{\sigma_y} = \dfrac{\sigma'_r}{\sigma_y} = \dfrac{\sigma'_\theta}{\sigma_y} = \dfrac{\sigma'_{rmin}}{\sigma_y} = \dfrac{\sigma'_{rx0}}{\sigma_y} \equiv -\dfrac{1-\ln 2}{\sqrt{3}} = \text{constant}$

y_0^d is not related to k and k_j

The results under the case that $k_j = k_{j*}$, $p = p_a = 2p_e$

1. The components of total stresses

$$\begin{cases} \dfrac{\sigma_z^T}{\sigma_y} = \dfrac{\ln x^2}{\sqrt{3}} - \dfrac{1}{\sqrt{3}} + \dfrac{2}{\sqrt{3}k^2} \\ \dfrac{\sigma_r^T}{\sigma_y} = \dfrac{\ln x^2}{\sqrt{3}} - \dfrac{2}{\sqrt{3}} + \dfrac{2}{\sqrt{3}k^2} \\ \dfrac{\sigma_\theta^T}{\sigma_y} = \dfrac{\ln x^2}{\sqrt{3}} + \dfrac{2}{\sqrt{3}k^2} \end{cases} (1 \le x \le k_{j*}) \qquad \begin{cases} \dfrac{\sigma_z^T}{\sigma_y} = \dfrac{k_j^2}{\sqrt{3}k^2} = \text{constant} \\ \dfrac{\sigma_r^T}{\sigma_y} = -\dfrac{k_j^2}{\sqrt{3}}\left(\dfrac{1}{x^2} - \dfrac{1}{k^2}\right) \\ \dfrac{\sigma_\theta^T}{\sigma_y} = \dfrac{k_j^2}{\sqrt{3}}\left(\dfrac{1}{x^2} + \dfrac{1}{k^2}\right) \end{cases} (k_{j*} \le x \le k)$$

2. Equivalent total stress

$$\frac{\sigma_e^T}{\sigma_y} \equiv 1 \ (1 \le x \le k_{j*}) \qquad \frac{\sigma_e^T}{\sigma_y} = \frac{k_j^2}{x^2} \ (k_{j*} \le x \le k)$$

(1) When $k \le k_r = 1 + \sqrt{3} = 2.732051$, $p \le \sigma_y$, hence $\sigma_{ri}^T > -\sigma_y$

(2) When $k \ge k_f = 2.300122\ldots$, $k_{j*} < \sqrt{\sqrt{3}-1}k (=x_a)$, thus $\sigma_{\theta j}^T < \sigma_y$;

when $k < k_f$, $k_{j*} > \sqrt{\sqrt{3}-1}k$, thus $\sigma_{\theta j}^T > \sigma_y$

(3) When $x \ge \exp\left(\dfrac{\sqrt{3}k^2 - 2}{2k^2}\right) = \exp\left(\dfrac{k^2-1}{k^2} - \dfrac{2-\sqrt{3}}{2}\right) = \exp\left(\dfrac{\sqrt{3}}{2}\dfrac{p}{\sigma_y} - \dfrac{2-\sqrt{3}}{2}\right) = x_z$, $\sigma_e^T \le \sigma_\theta^T$

(4) When $k_j = k_{j*}$, if $k^2 \ln[(\sqrt{3}-1)k^2] - \sqrt{3}k^2 + 2 = 0$ or $k = k_f$, $x_z = x_a = k_{j*} = 1.967984\ldots$

(When $k_j = \dfrac{k+1}{2} = \sqrt{\sqrt{3}-1}k$, or $k = k_d$, $x_z = x_a = k_j$; When $k_j = \sqrt{k} = \sqrt{\sqrt{3}-1}k$, or

$k = k_g$, $x_z = x_a = k_j$.)

(5) When $k_j < \sqrt{\sqrt{3}-1}k (=x_a)$, x_a exists and x_z does not exist; When $k_j > x_a$, x_a does not exists and x_z exist

The results under the case that $k_j = k$

1. The optimum load-bearing capability and autofrettage pressure

$$\frac{p}{\sigma_y} = \frac{p_a}{\sigma_y} = \frac{2}{\sqrt{3}}\ln k = \frac{p_y}{\sigma_y}$$

2. The total stresses under the case that $k_j = k$ and $p = p_y = \sigma_y \ln k$

$$\frac{\sigma_z^T}{\sigma_y} = \frac{2}{\sqrt{3}}\left(\frac{1}{2} + \ln x - \ln k\right)$$

Tab. 3.2 – (continued).

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{2}{\sqrt{3}}(\ln x-\ln k)$$

$$\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{2}{\sqrt{3}}(1+\ln x-\ln k)$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\equiv 1$$

3. $x=\exp\left(\ln k+\frac{\sqrt{3}}{2}-1\right)=x_{\mathrm{b}}=\exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}-\frac{2-\sqrt{3}}{2}\right)=x_{\mathrm{z}}$

When $x<x_{\mathrm{b}}$, $\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}<\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{y}}=1$; When $x>x_{\mathrm{b}}$, $\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}>\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{y}}=1$. When $p=p_{\mathrm{y}}$, $x_{\mathrm{b}}=x_{\mathrm{z}}$

4. The total stresses on inner surface under the case that $k_{\mathrm{j}}=k$ and $p=p_{\mathrm{y}}=\sigma_{\mathrm{y}}\ln k$

$$\frac{\sigma_{\mathrm{zi}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{2}{\sqrt{3}}\left(\frac{1}{2}-\ln k\right)$$

$$\frac{\sigma_{\mathrm{ri}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=-\frac{2}{\sqrt{3}}\ln k=-\frac{p}{\sigma_{\mathrm{y}}}$$

$$\frac{\sigma_{\theta\mathrm{i}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{2}{\sqrt{3}}(1-\ln k)$$

$$\frac{\sigma_{\mathrm{ei}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=1$$

The results under the case that $k_{\mathrm{j}}=\frac{k+1}{2}$

1. When $k\leq k_{\mathrm{a}}=2.618253\ldots$, $\left|\sigma_{\mathrm{ei}}'\right|\leq\sigma_{\mathrm{y}}$; When $k\geq k_{\mathrm{a}}$, $\left|\sigma_{\mathrm{ei}}'\right|\geq\sigma_{\mathrm{y}}$

2. When $k=k_{\mathrm{a}}$, $k_{\mathrm{j}}=\frac{k+1}{2}=1.809126\ldots$

3. The maximum load-bearing capacity and autofrettage pressure

$$\frac{p}{\sigma_{\mathrm{y}}}=\frac{1}{\sqrt{3}}-\frac{(k+1)^{2}}{4\sqrt{3}k^{2}}+\frac{2}{\sqrt{3}}\ln\frac{k+1}{2}\leq\frac{2}{\sqrt{3}}\ln k=\frac{p_{\mathrm{y}}}{\sigma_{\mathrm{y}}}$$

4. When $k\leq k_{\mathrm{a}}$, $\frac{1}{\sqrt{3}}-\frac{(k+1)^{2}}{4\sqrt{3}k^{2}}+\frac{2}{\sqrt{3}}\ln\frac{k+1}{2}<\frac{2}{\sqrt{3}}\frac{k^{2}-1}{k^{2}}$;

When $k\geq k_{\mathrm{a}}$, $\frac{1}{\sqrt{3}}-\frac{(k+1)^{2}}{4\sqrt{3}k^{2}}+\frac{2}{\sqrt{3}}\ln\frac{k+1}{2}>\frac{2}{\sqrt{3}}\frac{k^{2}-1}{k^{2}}$, but when $k\geq k_{\mathrm{a}}$, $\left|\sigma_{\mathrm{ei}}'\right|>\sigma_{\mathrm{y}}$

5. When $k\geq\frac{1}{\sqrt{\sqrt{3}-1}-1}=1.406076=k_{\mathrm{d}}$, $k_{\mathrm{j}}\leq\sqrt{\sqrt{3}-1}k=x_{\mathrm{a}}$, and $\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\geq\frac{\sigma_{\theta\mathrm{j}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$

6. When $k\leq k_{\mathrm{z}}=2.654911\ldots$, $p\leq\sigma_{\mathrm{y}}$ and $\sigma_{\mathrm{ri}}^{\mathrm{T}}\geq-\sigma_{\mathrm{y}}$; When $k\geq k_{\mathrm{z}}$, $p\geq\sigma_{\mathrm{y}}$ and $\sigma_{\mathrm{ri}}^{\mathrm{T}}\leq-\sigma_{\mathrm{y}}$;
When $k=k_{\mathrm{z}}$, $p=\sigma_{\mathrm{y}}$ and $\sigma_{\mathrm{ri}}^{\mathrm{T}}=-\sigma_{\mathrm{y}}$
When $k\leq k_{\mathrm{a}}$, there must be $k\leq k_{\mathrm{z}}$

7. When $k\geq k_{\mathrm{a}}$, $\sigma_{\mathrm{e}}^{\mathrm{T}}>\sigma_{\mathrm{y}}$

8. When $k_{\mathrm{j}}=\frac{k+1}{2}=\sqrt{\sqrt{3}-1}k$, or $k=k_{\mathrm{d}}$, $x_{\mathrm{z}}=x_{\mathrm{a}}=k_{\mathrm{j}}$

(When $k_{\mathrm{j}}=\sqrt{k}=\sqrt{\sqrt{3}-1}k$, or $k=k_{\mathrm{g}}$, $x_{\mathrm{z}}=x_{\mathrm{a}}=k_{\mathrm{j}}$; When $k=k_{\mathrm{f}}$, $x_{\mathrm{z}}=x_{\mathrm{a}}=k_{\mathrm{j}^{*}}$.)

9. Autofrettage pressure $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{1}{\sqrt{3}}-\frac{(k+1)^{2}}{4\sqrt{3}k^{2}}+\frac{2}{\sqrt{3}}\ln\frac{k+1}{2}=\frac{p}{\sigma_{\mathrm{y}}}$

10. $\frac{p}{\sigma_{\mathrm{y}}}=\frac{1}{\sqrt{3}}-\frac{(k+1)^{2}}{4\sqrt{3}k^{2}}+\frac{2}{\sqrt{3}}\ln\frac{k+1}{2}\left(k_{\mathrm{j}}=\frac{k+1}{2}\right)>\frac{p}{\sigma_{\mathrm{y}}}=\frac{k-1}{\sqrt{3}k}+\frac{1}{\sqrt{3}}\ln k\ (k_{\mathrm{j}}=\sqrt{k})$

Tab. 3.2 – (continued).

The results under the case that $k_{\mathrm{j}} = \sqrt{k}$

1. When $k \leq k_{\mathrm{b}} = 3.042297\ldots$, $\left|\sigma'_{\mathrm{ei}}\right| \leq \sigma_{\mathrm{y}}$; When $k \geq k_{\mathrm{b}}$, $\left|\sigma'_{\mathrm{ei}}\right| \geq \sigma_{\mathrm{y}}$
2. When $k = k_{\mathrm{b}}$, $k_{\mathrm{j}} = \sqrt{k} = 1.744218\ldots$
3. The maximum load-bearing capacity and autofrettage pressure

$$\frac{p}{\sigma_{\mathrm{y}}} = \frac{k-1}{\sqrt{3}k} + \frac{1}{\sqrt{3}}\ln k$$

4. $\frac{p}{\sigma_{\mathrm{y}}} = \frac{k-1}{\sqrt{3}k} + \frac{1}{\sqrt{3}}\ln k \ (k_{\mathrm{j}} = \sqrt{k}) < \frac{p}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}} - \frac{(k+1)^2}{4\sqrt{3}k^2} + \frac{2}{\sqrt{3}}\ln\frac{k+1}{2}\left(k_{\mathrm{j}} = \frac{k+1}{2}\right)$
5. When $k \leq k_{\mathrm{b}}$, $\frac{k-1}{\sqrt{3}k} + \ln\sqrt{k} < \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$; when $k \geq k_{\mathrm{b}}$, $\frac{k-1}{\sqrt{3}k} + \ln\sqrt{k} > \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$, but when $k \geq k_{\mathrm{b}}$, $\left|\sigma'_{\mathrm{ei}}\right| > \sigma_{\mathrm{y}}$
6. When $k \geq k_{\mathrm{g}}$, $k_{\mathrm{j}} \leq \sqrt{\sqrt{3}-1}k$, accordingly $\frac{\sigma^{\mathrm{T}}_{\mathrm{ej}}}{\sigma_{\mathrm{y}}} \geq \frac{\sigma^{\mathrm{T}}_{\theta\mathrm{j}}}{\sigma_{\mathrm{y}}}$
7. When $k_{\mathrm{j}} = \sqrt{k} = \sqrt{\sqrt{3}-1}k$, or $k = k_{\mathrm{g}}$, $x_{\mathrm{z}} = x_{\mathrm{a}} = k_{\mathrm{j}}$

(When $k_{\mathrm{j}} = \frac{k+1}{2} = \sqrt{\sqrt{3}-1}k$, or $k = k_{\mathrm{d}}$, $x_{\mathrm{z}} = x_{\mathrm{a}} = k_{\mathrm{j}}$; When $k = k_{\mathrm{f}}$, $x_{\mathrm{z}} = x_{\mathrm{a}} = k_{\mathrm{j}^*}$.)

The results under the case that $k_{\mathrm{j}} = \frac{r_{\mathrm{j}}}{r_{\mathrm{i}}}\exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right)$

1. Overstrain $\varepsilon = \frac{\exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right)-1}{k-1}$
2. When $k_{\mathrm{j}} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right)$, the equivalent stress σ_{ej} is the minimum, that is

$$\frac{\sigma^{\mathrm{T}}_{\mathrm{ej}}}{\sigma_{\mathrm{y}}} = \frac{k^2}{k^2-1}\frac{\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right)-1}{\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right)} = \frac{k^2}{k^2-1}\frac{k_{\mathrm{j}}^2-1}{k_{\mathrm{j}}^2}$$

3. The allowable load for $\sigma'_{\mathrm{ei}}/\sigma_{\mathrm{y}} \geq -1$ when $k \geq k_{\mathrm{c}}$

$$\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right) + k^2 - \sqrt{3}k^2\frac{p}{\sigma_{\mathrm{y}}} - 2 \geq 0 \quad \text{or} \quad k^2 \leq \frac{\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right)-2}{\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}-1}$$

4. The relationship between autofrettage pressure and the allowable load

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{p}{\sigma_{\mathrm{y}}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}k^2}\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right)$$

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} \geq \frac{p}{\sigma_{\mathrm{y}}}$$

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Chapter 4

Mechanical Autofrettage Technology by Limiting Circumferential Residual Stress Based on Mises Yield Criterion

When the maximum distortion strain energy theory (Mises's yield criterion) is applied to study the autofrettage for pressure vessels, there are some problems needed to be studied further. Based on our previous work[1–3], this chapter is intended to do relevant research with the help of the mathematical analysis of ref.[4].

4.1 The Optimum Plastic Depth When Circumferential Residual Stress on the Inside Surface Controlled

The residual stresses based on the maximum distortion strain energy theory (Mises's criterion) are given as equations (3.2)–(3.9). The depth of plastic region (k_j) affects the residual stresses, the greater k_j is, the greater the equivalent residual stress σ'_e is, so it is inadvisable to simply raise k_j, otherwise, compressive yield may occur after removing the autofrettage pressure, p_a. However, raising k_j can raise the load-bearing capacity of a pressure vessel, thereupon, in our previous studies, we put forward an equation for $k_{\mathrm{j}*}$, the maximum plastic depth (k_j) for a certain k to ensure $\sigma'_\mathrm{ei} \geq -\sigma_\mathrm{y}$:

$$k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 = 0 \tag{4.1}$$

where $\mathrm{e}^{0.5} \leq k_{\mathrm{j}*} \leq k_\mathrm{c} = 2.2184574899167\ldots$ and $k \geq k_\mathrm{c}$. When $k \leq k_\mathrm{c}$, $\sigma'_\mathrm{ei} \leq -\sigma_\mathrm{y}$ never occurs irrespective of k_j. Equation (4.1) is equation (2.34). $k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 = 0$ is applicable for both the maximum shear stress theory (Tresca yield criterion) and the maximum distortion strain energy theory (Mises's yield criterion).

When k_j is determined by equation (4.1)—we write $k_\mathrm{j} = k_{\mathrm{j}*}$, and according to the maximum distortion strain energy theory, the equations for residual stresses are found in equations (3.43)–(3.50).

DOI: 10.1051/978-2-7598-3111-1.c004
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By taking $k = 3$, $k_{\mathrm{j}} = 1.5 < k_{\mathrm{j}*}$, $k_{\mathrm{j}} = 1.748442 = k_{\mathrm{j}*}$ and $k_{\mathrm{j}} = 2 > k_{\mathrm{j}*}$, the residual stresses based on the maximum distortion strain energy theory are illustrated as shown in figure 4.1a–c, respectively, to show the effect of k_{j} on residual stresses.

In figure 4.1, points a, b, and c are intersections of the three curves for the residual stresses, their abscissas are calculated by $x_0 = \frac{r}{r_{\mathrm{i}}} = \sqrt{\frac{k^2 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2}{k^2 - 1}} < k_{\mathrm{j}}$ (equation (2.22) or equation (3.22)), their coordinates are a (1.325159, −0.11142), b ($x = \sqrt{2}$, −0.17717) and c (1.478033, −0.23276).

When $k = 3$, $k_{\mathrm{j}} = 1.5 < k_{\mathrm{j}*}$, both $|\sigma'_{\mathrm{ei}}|$ and $|\sigma'_{\theta\mathrm{i}}|$ are below σ_{y}. When $k = 3$, $k_{\mathrm{j}} = 1.748442 = k_{\mathrm{j}*}$, $|\sigma'_{\mathrm{ei}}|$ just reach σ_{y}, but $|\sigma'_{\theta\mathrm{i}}|$ exceeds σ_{y}. When $k = 3$, $k_{\mathrm{j}} = 2 > k_{\mathrm{j}*}$, both $|\sigma'_{\mathrm{ei}}|$ and $|\sigma'_{\theta\mathrm{i}}|$ exceed σ_{y}.

Small k_{j} causes small residual stresses and the effect of autofrettage is not obvious; Great k_{j} causes great residual stresses and the effect of autofrettage is obvious, but there will be reversed yielding after releasing autofrettage pressure p_{a} if the plastic depth is too deep. For this reason, we presented a formula to control k_{j} so that $|\sigma'_{\mathrm{ei}}| \leq \sigma_{\mathrm{y}}$, that is equation (4.1). Nevertheless, we have found that when $k_{\mathrm{j}} = k_{\mathrm{j}*}$, at the inside surface, $|\sigma'_{\mathrm{ei}}|$ and $|\sigma'_{\theta\mathrm{i}}|$ just simultaneously reach σ_{y} according to the maximum shear stress theory (Tresca yield criterion), and $|\sigma'_{\theta\mathrm{i}}|$ exceeds σ_{y} when $|\sigma'_{\mathrm{ei}}|$ just reaches σ_{y} according to the maximum distortion strain energy theory (Mises's yield criterion) although $k_{\mathrm{j}} = k_{\mathrm{j}*}$. The reasons are as follows.

When based on the maximum distortion strain energy theory, with the help of ref.[5], we obtain the equivalent stress $\sigma_{\mathrm{e}}^{\mathrm{d}}$ as follows,

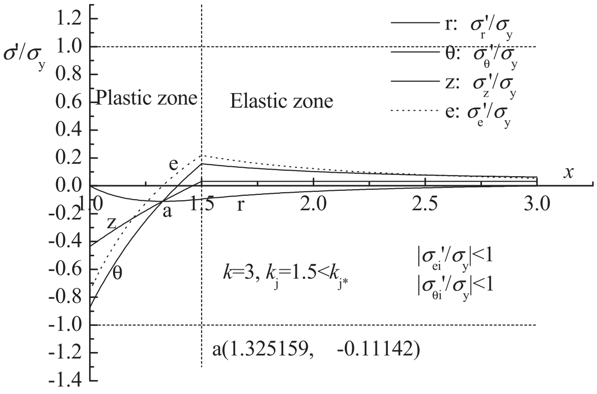
$$\sigma_{\mathrm{e}}^{\mathrm{d}} = \frac{\sqrt{3}}{2}(\sigma_\theta - \sigma_{\mathrm{r}}) \quad \text{or} \quad \sigma_\theta - \sigma_{\mathrm{r}} = \frac{2}{\sqrt{3}}\sigma_{\mathrm{e}}^{\mathrm{d}} \tag{4.2}$$

When $\sigma_{\mathrm{e}}^{\mathrm{d}}$ reaches the strength σ_{y}, the material becomes yielded. At the inside surface ($r = r_{\mathrm{i}}$), the radial residual stress $\sigma'_{\mathrm{ri}} = 0$, then, circumferential residual stress $\sigma'_{\theta\mathrm{i}} = \frac{2}{\sqrt{3}}\sigma_{\mathrm{y}}$, which exceeds the strength σ_{y}. This may lower the safety of a pressure vessel. So, it is necessary to limit $|\sigma'_{\theta\mathrm{i}}|$ in order to find the maximum plastic depth k_{j} for $|\sigma'_{\theta\mathrm{i}}|$ to be below σ_{y}, such k_{j} is written as $k_{\mathrm{j}\theta}$. Therefore, letting $\frac{\sigma'_{\theta\mathrm{i}}}{\sigma_{\mathrm{y}}} = -\frac{2}{\sqrt{3}(k^2-1)}(1 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2) \geq -1$ obtains:

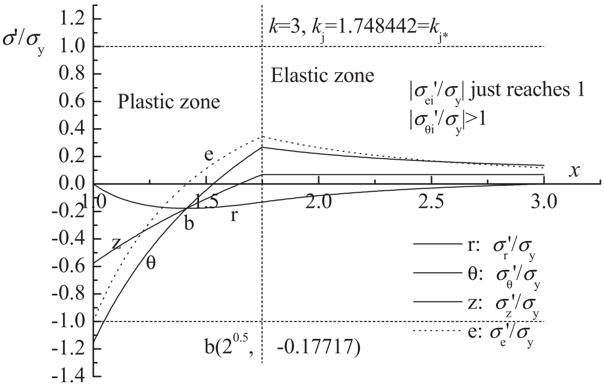
$$\begin{gathered} 2k^2 \ln k_{\mathrm{j}\theta}^2 - 2k_{\mathrm{j}\theta}^2 - \sqrt{3}k^2 + (2+\sqrt{3}) \leq 0 \quad \text{or} \\ k = \sqrt{\frac{2\left(k_{\mathrm{j}\theta}^2 - 1\right) - \sqrt{3}}{2\ln k_{\mathrm{j}\theta}^2 - \sqrt{3}}} \; (\mathrm{e}^{\sqrt{3}/4} \leq k_{\mathrm{j}\theta} \leq k_{\mathrm{c}\theta},\ k \geq k_{\mathrm{c}\theta}) \end{gathered} \tag{4.3}$$

Replacing $\sqrt{3}$ with 2 just leads to $k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 = 0$. This is the result of letting $\sigma'_{\mathrm{ei}} = -\sigma_{\mathrm{y}}$.

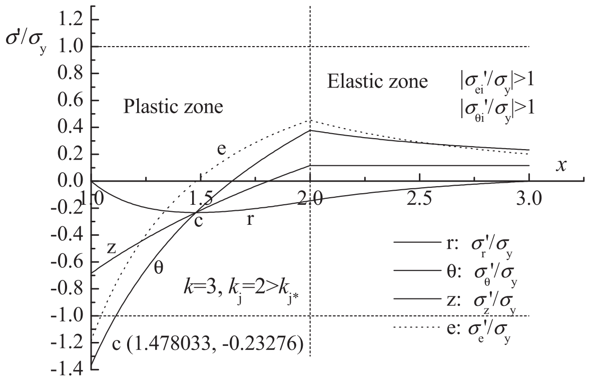
Limiting circumferential stress is essentially the maximum principal stress theory.



(a) k=3, k_j=1.5<k_{j*}



(b) k=3, k_j=1.748442=k_{j*}



(c) k=3, k_j=2>k_{j*}

FIG. 4.1 – Distributions of the residual stresses based on the maximum distortion strain energy theory.

From equation (4.3),

$$k \to \infty, k_{j\theta} = e^{\sqrt{3}/4} = 1.541896\ldots = k_{j\theta\infty} \tag{4.4}$$

Letting $k_{j\theta} = k$ in equation (4.3) obtains:

$$\frac{k^2 \ln k^2}{k^2 - 1} = \frac{2+\sqrt{3}}{2} \quad \text{or} \quad \frac{k^2 \ln k}{k^2 - 1} = \frac{2+\sqrt{3}}{4} \tag{4.5}$$

The solution of equation (4.5) is

$$k = 2.024678965\ldots = k_{c\theta} < k_c = 2.2184574899167 \tag{4.6}$$

Therefore, when $k \le k_{c\theta}$, no matter how deep the depth of the plastic region is, even if the whole cylinder is yielded, or $k_j = k$, $|\sigma'_{\theta i}|$ can not exceed σ_y after removing p_a from the cylinder; when $k > k_{c\theta}$, if k_j is not greater than the value determined by equation (4.3), or $k_j \le k_{j\theta}$, $|\sigma'_{\theta i}|$ will not exceed σ_y after removing p_a, otherwise, $|\sigma'_{\theta i}|$ will exceed σ_y after removing p_a. Clearly, when $k = k_{c\theta}$, $k_{j\theta} = k = k_{c\theta}$.

From equation (4.3), the domain of k is $k_{j\theta} \ge e^{\sqrt{3}/4} = k_{j\theta\infty}$. and $k_{j\theta} \le \sqrt{\frac{2+\sqrt{3}}{2}} = 1.366025\ldots$.

The relation between $k_{j\theta}$ and k or equation (4.3) is shown in figure 4.2. The graph of equation (4.3) has two branches: curve od and curve abc. Curve od corresponds to $k_{j\theta} \le \sqrt{\frac{2+\sqrt{3}}{2}} = 1.366025$; Curve abc corresponds to $k_{j\theta} \ge e^{\sqrt{3}/4} = 1.541896\ldots$. For curve od, $k < 1$. This is unreasonable and, thus, it is of no significance in practice. For section ab of curve abc, $k_j > k$, therefore, this section is of no significance in practice. The meaningful control curve is section bc of curve abc. At point b, $k_j = k = k_{c\theta} = 2.024678965\ldots$.

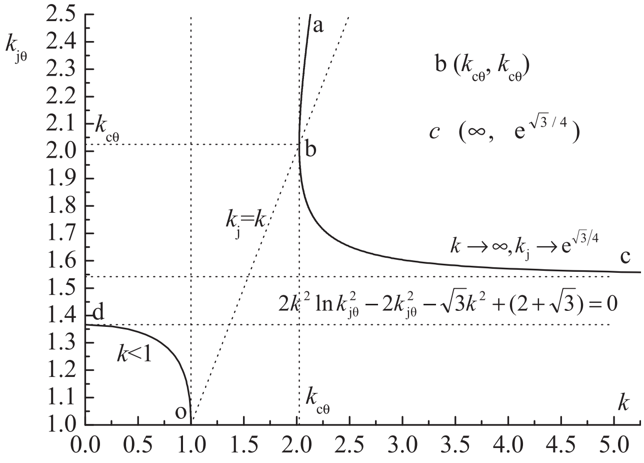


FIG. 4.2 – The graph of relation between $k_{j\theta}$ and k.

Limiting $\sigma'_{\mathrm{ei}} \geq -\sigma_{\mathrm{y}}$, we get $k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 = 0$ ($\mathrm{e}^{0.5} \leq k_{\mathrm{j}*} \leq k_{\mathrm{c}}$ and $k \geq k_{\mathrm{c}}$). For comparison, the meaningful part of $2k^2 \ln k_{\mathrm{j}\theta}^2 - 2k_{\mathrm{j}\theta}^2 - \sqrt{3}k^2 + (2+\sqrt{3}) = 0$ is plotted along with the meaningful part of $k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 = 0$ in figure 4.3.

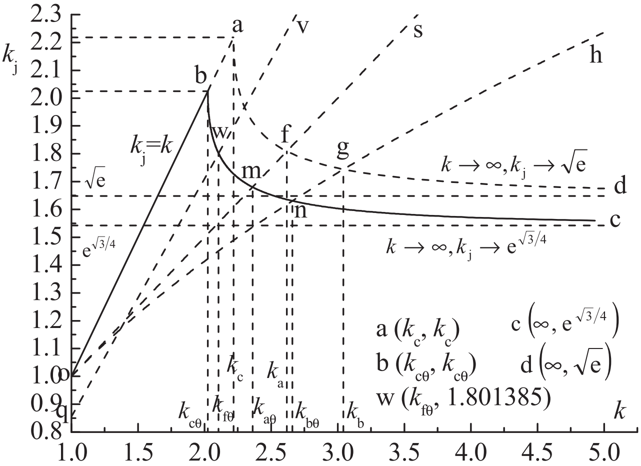


FIG. 4.3 – The plastic depth.

From figure 4.3, it is seen that for the same k, $k_{\mathrm{j}*} > k_{\mathrm{j}\theta}$. To avoid $\sigma'_{\theta\mathrm{i}} \leq -\sigma_{\mathrm{y}}$, $k_{\mathrm{j}\theta}$ should be smaller than $k_{\mathrm{j}*}$.

In figure 4.3, $k_{\mathrm{j}} = \frac{k+1}{2}$ and $k_{\mathrm{j}} = \sqrt{k}$ are plotted along with $2k^2 \ln k_{\mathrm{j}\theta}^2 - 2k_{\mathrm{j}\theta}^2 - \sqrt{3}k^2 + (2+\sqrt{3}) = 0$ and $k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 = 0$, where curve obwmnc is for $2k^2 \ln k_{\mathrm{j}\theta}^2 - 2k_{\mathrm{j}\theta}^2 - \sqrt{3}k^2 + (2+\sqrt{3}) = 0$, curve obafgd is for $k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 = 0$, curve omfs is for $k_{\mathrm{j}} = \frac{k+1}{2}$ and curve ongh is for $k_{\mathrm{j}} = \sqrt{k}$.

Points c and d are at infinity, their coordinates are c $\left(\infty, \mathrm{e}^{\sqrt{3}/4}\right)$ and d $(\infty, \sqrt{\mathrm{e}})$.

Point a is the intersection point of straight line oba ($k_{\mathrm{j}} = k$) and curve afgd ($k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 = 0$), its coordinate is a ($k_{\mathrm{c}}$, k_{c}).

Point b is the intersection point of straight line oba ($k_{\mathrm{j}} = k$) and curve bwmnc ($2k^2 \ln k_{\mathrm{j}\theta}^2 - 2k_{\mathrm{j}\theta}^2 - \sqrt{3}k^2 + (2+\sqrt{3}) = 0$), its coordinate is b ($k_{\mathrm{c}\theta}$, $k_{\mathrm{c}\theta}$).

Point f is the intersection point of straight line omfs ($k_{\mathrm{j}} = \frac{k+1}{2}$) and curve afgd ($k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 = 0$), its coordinate is f ($k_{\mathrm{a}}$, 1.80913) = f (2.618253, 1.80913).

Point g is the intersection point of curve ongh ($k_{\mathrm{j}} = \sqrt{k}$) and curve afgd ($k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 = 0$), its coordinate is g ($k_{\mathrm{b}}$, 1.74422) = g (3.042297, 1.74422).

When based on the maximum distortion strain energy theory and $k_{\mathrm{j}} = \frac{k+1}{2}$, from equation (3.108),

$$\frac{\sigma'_{\theta\mathrm{i}}}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}} \left[\frac{k+3}{2(k+1)} + \frac{4k^2}{k^2-1} \ln \frac{2}{k+1} \right] \tag{4.7}$$

It is easy to prove that when $k \rightarrow 1$, $\frac{\sigma'_{\theta i}}{\sigma_y} \rightarrow 0$. Equation (4.7) is plotted in figure 4.4.

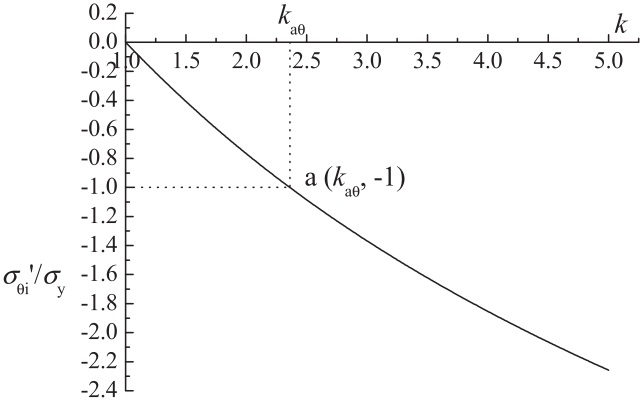


FIG. 4.4 – The relationship between $\frac{\sigma'_{\theta i}}{\sigma_y}$ and k when $k_j = \frac{k+1}{2}$.

Letting $\frac{\sigma'_{\theta i}}{\sigma_y} = -1$ obtains

$$8k^2 \ln \frac{k+1}{2} = (2\sqrt{3}+1)k^2 + 2k - 3 - 2\sqrt{3} \tag{4.8}$$

The solution of equation (4.8) is

$$k \leq 2.360344\ldots = k_{a\theta} < k_a \tag{4.9}$$

When based on the maximum distortion strain energy theory and $k_j = \frac{k+1}{2}$, for $\sigma'_{\theta i} \geq -\sigma_y$, $k \leq k_{a\theta}$ or $k \leq 2.36035\ldots$.

On the other hand, substituting $k_j = \frac{k+1}{2}$ into $2k^2 \ln k_{j\theta}^2 - 2k_{j\theta}^2 - \sqrt{3}k^2 + (2+\sqrt{3}) = 0$ also results in equation (4.8). When $k = k_{a\theta}$, $k_j = \frac{k+1}{2} = 1.680175\ldots$. Therefore, in figure 4.3, point m is the intersection point of straight line omfs ($k_j = \frac{k+1}{2}$) and curve bwmnc ($2k^2 \ln k_{j\theta}^2 - 2k_{j\theta}^2 - \sqrt{3}k^2 + (2+\sqrt{3}) = 0$), its coordinate is m ($k_{a\theta}$, 1.680175) = m (2.360344, 1.680175).

When based on the maximum distortion strain energy theory and $k_j = \sqrt{k}$, from equation (3.103),

$$\frac{\sigma'_{\theta i}}{\sigma_y} = \frac{2}{\sqrt{3}} \frac{k - 1 - k^2 \ln k}{k^2 - 1} \tag{4.10}$$

It is easy to prove that when $k \to 1$, $\frac{\sigma'_{\theta i}}{\sigma_y} \to 0$. Equation (4.10) is plotted in figure 4.5.

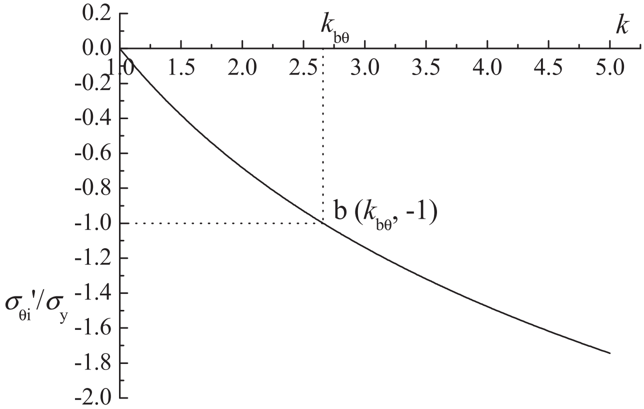


FIG. 4.5 – The relationship between $\frac{\sigma'_{\theta i}}{\sigma_y}$ and k when $k_j = \sqrt{k}$.

Letting $\frac{\sigma'_{\theta i}}{\sigma_y} = -1$ obtains

$$2k^2 \ln k - 2k - \sqrt{3}k^2 + 2 + \sqrt{3} = 0 \tag{4.11}$$

The solution of equation (4.11) is

$$k = 2.65972\ldots = k_{b\theta} < k_b \tag{4.12}$$

When based on the maximum distortion strain energy theory and $k_j = \sqrt{k}$, for $\sigma'_{\theta i} \geq -\sigma_y$, $k \leq k_{b\theta}$ or $k \leq 2.65972\ldots$.

On the other hand, substituting $k_j = \sqrt{k}$ into $2k^2 \ln k_{j\theta}^2 - 2k_{j\theta}^2 - \sqrt{3}k^2 + (2+\sqrt{3}) = 0$ also results in equation (4.11). When $k = k_{b\theta}$, $k_j = \sqrt{k} = 1.630865\ldots$. Therefore, in figure 4.3, point n is the intersection point of curve ongh ($k_j = \sqrt{k}$) and curve bwmnc ($2k^2 \ln k_{j\theta}^2 - 2k_{j\theta}^2 - \sqrt{3}k^2 + (2+\sqrt{3}) = 0$), and its coordinate is n ($k_{b\theta}$, 1.630865) = n (2.65972, 1.630865).

Converting the relation of $k_j \sim k$ into the relation of $\frac{k_j}{k} \sim k$, we obtain figure 4.6. In figure 4.6, curve abg is for $\frac{k_j}{k}$ obtained from $2k^2 \ln k_{j\theta}^2 - 2k_{j\theta}^2 - \sqrt{3}k^2 + (2+\sqrt{3}) = 0$; curve abcf is for $\frac{k_j}{k}$ obtained from $k^2 \ln k_{j*}^2 - k^2 - k_{j*}^2 + 2 = 0$; curve ad is for $\frac{k_j}{k}$ obtained from $k_j = \frac{k+1}{2}$ and curve ae is for $\frac{k_j}{k}$ obtained from $k_j = \sqrt{k}$.

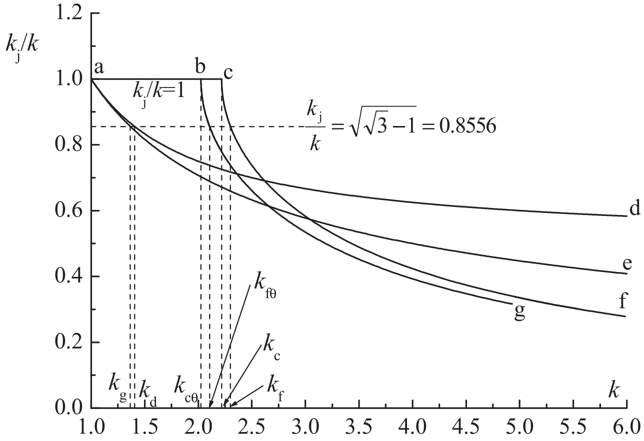


FIG. 4.6 – The relation of $\frac{k_j}{k} \sim k$.

Substituting $k_j = \sqrt{\sqrt{3}-1}k$ into $2k^2 \ln k_{j\theta}^2 - 2k_{j\theta}^2 - \sqrt{3}k^2 + (2+\sqrt{3}) = 0$ obtains

$$2k^2 \ln[(\sqrt{3}-1)k^2] - (3\sqrt{3}-2)k^2 + 2 + \sqrt{3} = 0 \tag{4.13}$$

Solving equation (4.13) results in

$$k = 2.105406\ldots = k_{f\theta} < k_f \tag{4.14}$$

$k_{f\theta}$ is marked in figures 4.3 and 4.6. That is to say when k_j is restrained by $2k^2 \ln k_{j\theta}^2 - 2k_{j\theta}^2 - \sqrt{3}k^2 + (2+\sqrt{3}) = 0$, if $k \geq k_{f\theta}$, $k_j \leq \sqrt{\sqrt{3}-1}k$, accordingly $\frac{\sigma_{ej}^T}{\sigma_y} \geq \frac{\sigma_{\theta j}^T}{\sigma_y}$.

In order to facilitate the engineering application, some data calculated by equation (4.3) are listed in table 4.1 along with some data calculated by $k^2 \ln k_{j*}^2 - k^2 - k_{j*}^2 + 2 = 0$ for comparison.

4.2 The Distribution of Residual Stresses When Circumferential Residual Stress on the Inside Surface Controlled

4.2.1 General Discussion

When $k_j = k_{j\theta}$, the residual stresses within the plastic region are as follows.

$$\frac{\sigma_z'}{\sigma_y} = \frac{1}{\sqrt{3}} \ln\left(\frac{r}{r_i}\right)^2 - \frac{1}{2} = \frac{1}{\sqrt{3}} \ln x^2 - \frac{1}{2} \tag{4.15}$$

Tab. 4.1 – Some data of $k_{j\theta}$ and k_{j*}.

k	$k_{j\theta}$	$k_{j\theta}/k$	k_{j*}	k_{j*}/k	$k_{j\theta}/k_{j*}$
$2.024678965 = k_{c\theta}$	$2.024678965 = k_{c\theta}$	1	–	–	–
2.05	1.882849	0.918463	–	–	–
2.075	1.837525	0.885554	–	–	–
2.1	1.806908	0.860432	–	–	–
2.15	1.764911	0.820889	–	–	–
2.2	1.735954	0.78907	–	–	–
$2.2184574899167 = k_c$	1.72725	0.778582	$2.2184574899167 = k_c$	1	0.778582
2.25	1.71415	0.761844	2.046308	0.90947	0.837679
2.3	1.696894	0.73778	1.968122	0.855705	0.862189
2.35	1.682783	0.716078	1.92208	0.817906	0.875501
2.4	1.670971	0.696238	1.889454	0.787273	0.884367
2.45	1.660907	0.677921	1.864437	0.760995	0.890836
2.5	1.652212	0.660885	1.844363	0.737745	0.895817
2.55	1.644614	0.644947	1.827762	0.716769	0.899797
2.6	1.637911	0.629966	1.813731	0.697589	0.903062
2.65	1.63195	0.61583	1.801676	0.679878	0.905795
2.7	1.626613	0.602449	1.791183	0.663401	0.908122
2.75	1.621806	0.589748	1.781952	0.647983	0.910129
2.8	1.617453	0.577662	1.773758	0.633485	0.911879
2.85	1.613492	0.566138	1.766432	0.619801	0.913419
2.9	1.609874	0.555129	1.759838	0.606841	0.914785
2.95	1.606556	0.544595	1.75387	0.594532	0.916006
3	1.603502	0.534501	1.748442	0.582814	0.917103
3.25	1.591296	0.48963	1.7273	0.531477	0.921262
3.5	1.582618	0.452177	1.712755	0.489359	0.924019
3.75	1.576167	0.420311	1.702177	0.453914	0.925971
4	1.571211	0.392803	1.694172	0.423543	0.927421

TAB. 4.1 – (continued).

k	$k_{j\theta}$	$k_{j\theta}/k$	k_{j*}	k_{j*}/k	$k_{j\theta}/k_{j*}$
4.5	1.564158	0.347591	1.682956	0.37399	0.929411
5	1.559444	0.311889	1.675565	0.335113	0.930697
5.5	1.556118	0.282931	1.670397	0.303709	0.931586
5.9	1.554108	0.263408	1.667294	0.282592	0.932114
6	1.553673	0.258946	1.666625	0.277771	0.932227
∞	$\exp\left(\frac{\sqrt{3}}{4}\right)$	0	$\sqrt{e}$	0	$\sqrt{e}/\exp\left(\frac{\sqrt{3}}{4}\right) = 0.935207$

$$\frac{\sigma_{\mathrm{r}}'}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}\left(\ln x^2 + \frac{2+\sqrt{3}}{2x^2} - \frac{2+\sqrt{3}}{2}\right) \tag{4.16}$$

$$\frac{\sigma_{\theta}'}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}\left(\ln x^2 - \frac{2+\sqrt{3}}{2x^2} + \frac{2-\sqrt{3}}{2}\right) \tag{4.17}$$

$$\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} = 1 - \frac{\sqrt{3}+2}{2(r/r_{\mathrm{i}})^2} = 1 - \frac{\sqrt{3}+2}{2x^2} \tag{4.18}$$

It is seen that equations (4.15)–(4.18) are unconcerned with k_{j} and k, this means that when $\sigma_{\theta\mathrm{i}}'$ is controlled, or $k_{\mathrm{j}} = k_{\mathrm{j}\theta}$, the residual stresses, and their equivalent stress are determined only by radial relative location ($r/r_{\mathrm{i}} = x$) and independent of k_{j} and k, or at a certain x, the residual stresses and their equivalent stress are identical for any k_{j} and k.

Equation (4.18) shows the following conclusion.

When $x \leq \sqrt{\frac{\sqrt{3}+2}{2}} = 1.366025$, $\sigma_{\mathrm{e}}' \leq 0$, at the inside surface where $x = 1$, $\frac{\sigma_{\mathrm{ei}}'}{\sigma_{\mathrm{y}}} = -\frac{\sqrt{3}}{2}$, that is the minimum within the whole wall because $\frac{\mathrm{d}\sigma_{\mathrm{e}}'}{\mathrm{d}x} > 0$; when $x \geq \sqrt{\frac{\sqrt{3}+2}{2}}$, $\sigma_{\mathrm{e}}' \geq 0$; within the whole wall of a pressure vessel, $|\sigma_{\mathrm{e}}'| < \sigma_{\mathrm{y}}$; at the elastic-plastic juncture where $x = k_{\mathrm{j}}$, $0 < \frac{\sigma_{\mathrm{ej}}'}{\sigma_{\mathrm{y}}} = 1 - \frac{\sqrt{3}+2}{2k_{\mathrm{j}}^2} < 1$, since $k_{\mathrm{j}}^2 \geq \exp\left(\frac{\sqrt{3}}{2}\right) > 1 + \sqrt{3}/2$, this means that σ_{ej}' is tension and can not reach σ_{y}.

At $x = \frac{r}{r_{\mathrm{i}}} = \sqrt{\frac{\sqrt{3}+2}{2}}$, $\sigma_{\mathrm{e}}' = 0$, and this is just the location where $\sigma_{\theta}' = \sigma_{\mathrm{r}}' = \sigma_{\mathrm{z}}'$. That is to say, when $k_{\mathrm{j}} = k_{\mathrm{j}\theta}$, no matter how great k is, the three curves for the residual stresses collect at the same point within the plastic region and the intersection is: $\left(\sqrt{\frac{\sqrt{3}+2}{2}},\ \frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}+2}{2} - \frac{1}{2}\right) = (1.366025\ldots, -0.13984\ldots)$.

We once presented the location (x_0) of the intersection of the three residual stress curves or the abscissa at which $\sigma_{\mathrm{e}}' = 0$ in general form, that is as follows

$$x_0 = \frac{r}{r_{\mathrm{i}}} = \sqrt{\frac{k^2 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2}{k^2 - 1}} < k_{\mathrm{j}} \tag{4.19}$$

Equation (4.19) is equation (2.10). When k and k_{j} meet the relation expressed by equation (4.3), x_0 just becomes

$$x_{0\theta} = \sqrt{\frac{\sqrt{3}+2}{2}} \tag{4.20}$$

Setting $\sigma'_\mathrm{z} = 0$ obtains

$$x_{1\theta} = \mathrm{e}^{\frac{\sqrt{3}}{4}} = k_{\mathrm{j}\theta\infty} \tag{4.21}$$

When $k_\mathrm{j} = k_{\mathrm{j}\theta}$, the residual stresses within the elastic region are as follows,

$$\frac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} = \frac{k_\mathrm{j}^2 - \frac{2+\sqrt{3}}{2}}{\sqrt{3}k^2} \tag{4.22}$$

$$\frac{\sigma'_\mathrm{r}}{\sigma_\mathrm{y}} = \left(1 - \frac{k^2}{(r/r_\mathrm{i})^2}\right)\frac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} = \left(1 - \frac{k^2}{x^2}\right)\frac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} \tag{4.23}$$

$$\frac{\sigma'_\theta}{\sigma_\mathrm{y}} = \left(1 + \frac{k^2}{(r/r_\mathrm{i})^2}\right)\frac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} = \left(1 + \frac{k^2}{x^2}\right)\frac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} \tag{4.24}$$

$$\frac{\sigma'_\mathrm{e}}{\sigma_\mathrm{y}} = \frac{k_\mathrm{j}^2 - (\sqrt{3}+2)/2}{x^2} \tag{4.25}$$

The distribution of equivalent residual stress within the whole wall is demonstrated in figure 4.7, the related parameters are shown in the figure.

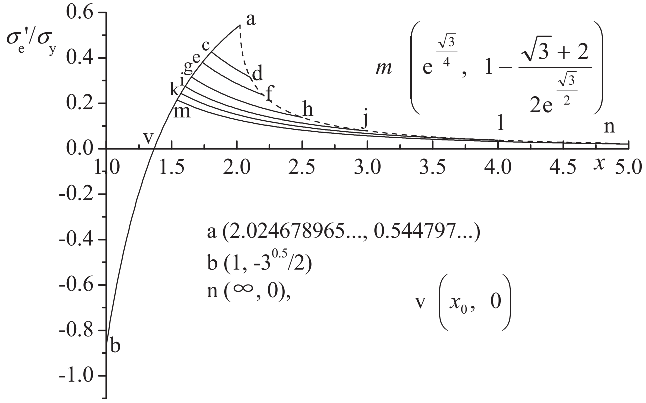


FIG. 4.7 – The distribution of equivalent residual stress within the whole wall for various $k_{\mathrm{j}\theta}$.

For figure 4.7, we make the following explanation.

(1) Curve baa: This is the curve of the equivalent residual stress when $k = k_{\mathrm{j}\theta} = k_{\mathrm{c}\theta} = 2.024678965\ldots$ In this case, the cylinder is entirely yielded, or there is no elastic zone in the wall, x varies from 1 to $k_{\mathrm{j}\theta}$ ($=k_{\mathrm{c}\theta}$) within the plastic region (from point b to point a). When the plastic depth is controlled by

equation (4.3), or $k_{\mathrm{j}} = k_{\mathrm{j}\theta}$, all curves of the equivalent residual stresses in the plastic zone for any k and $k_{\mathrm{j}\theta}$ are located on curve ba, starting from point b and ending at some point before point a.

(2) Curve *bcd*: $k = 2.101833\ldots$, $k_{\mathrm{j}\theta} = 1.805\ldots$. The curve of the equivalent residual stress within the plastic region is curve bc and the curve of the equivalent residual stress within the elastic zone is curve cd. x varies from 1 to $k_{\mathrm{j}\theta} = 1.805\ldots$ within the plastic region (point b to c), and from $k_{\mathrm{j}\theta}$ to k (point c to d) within the elastic zone.

(3) Curve *bef*: $k = 2.2$, $k_{\mathrm{j}\theta} = 1.7359545\ldots$. The curve of the equivalent residual stress within the plastic region is curve be and the curve of the equivalent residual stress within the elastic zone is curve ef. x varies from 1 to $k_{\mathrm{j}\theta} = 1.7359545\ldots$ within the plastic zone (point b to e), and from $k_{\mathrm{j}\theta}$ to k (point e to f) within the elastic zone.

(4) Curve *bgh*: $k = 2.5$, $k_{\mathrm{j}\theta} = 1.652212\ldots$. The curve of the equivalent residual stress within the plastic zone is curve bg and the curve of the equivalent residual stress within the elastic zone is curve gh. x varies from 1 to $k_{\mathrm{j}\theta} = 1.652212\ldots$ within the plastic zone (point b to g), and from $k_{\mathrm{j}\theta}$ to k (point g to h) within the elastic zone.

(5) Curve *bij*: $k = 3$, $k_{\mathrm{j}\theta} = 1.603502\ldots$. The curve of the equivalent residual stress within the plastic zone is curve bi and the curve of the equivalent residual stress within the elastic zone is curve ij. x varies from 1 to $k_{\mathrm{j}\theta} = 1.603502\ldots$ within the plastic zone (point b to i), and from $k_{\mathrm{j}\theta}$ to k (point i to j) within the elastic zone.

(6) Curve *bkl*: $k = 4$, $k_{\mathrm{j}\theta} = 1.571211\ldots$. The curve of the equivalent residual stress within the plastic zone is curve bk and the curve of the equivalent residual stress within the elastic zone is curve kl. x varies from 1 to $k_{\mathrm{j}\theta} = 1.571211\ldots$ within the plastic zone (point b to k), and from $k_{\mathrm{j}\theta}$ to k (point k to l) within the elastic zone.

(7) Curve *bmn*: $k = \infty$, $k_{\mathrm{j}\theta} = \mathrm{e}^{\sqrt{3}/4} = 1.541896\ldots$. The curve of the equivalent residual stress within the plastic zone is curve bm and the curve of the equivalent residual stress within the elastic zone is curve mn. x varies from 1 to $k_{\mathrm{j}\theta} = \mathrm{e}^{\sqrt{3}/4} = 1.541896\ldots$ within the plastic zone (point b to m), and from $k_{\mathrm{j}\theta}$ to ∞ (point m to n) within the elastic zone.

The coordinates of some key points in figure 4.7 are a $(k_{\mathrm{c}\theta},\ 0.544797\ldots)$, $b\left(1, -\frac{\sqrt{3}}{2}\right)$, $m\left(\mathrm{e}^{\frac{\sqrt{3}}{4}}, 1 - \frac{\sqrt{3}+2}{2\mathrm{e}^{\frac{\sqrt{3}}{2}}}\right) = m\ (1.541896,\ 0.215112\ldots)$, $n\ (\infty,\ 0)$, $\mathrm{v}\,(x_0, 0) = \mathrm{v}\left(\sqrt{\frac{\sqrt{3}+2}{2}}, 0\right)$.

From figure 4.7 and equation (4.18), it is known that all curves of the equivalent residual stresses for any k and $k_{\mathrm{j}\theta}$ within the plastic region are located on the identical curve ba and pass through the same point $v\left(\sqrt{\frac{\sqrt{3}+2}{2}}, 0\right)$, except that the different curve for different k and $k_{\mathrm{j}\theta}$ is located on a different section of curve ba. Saying, the above curves for the plastic zone, ba $(k = k_{\mathrm{j}\theta} = k_{\mathrm{c}\theta})$, bc $(k = 2.101833\ldots$, $k_{\mathrm{j}\theta} = 1.805\ldots)$, be $(k = 2.2,\ k_{\mathrm{j}\theta} = 1.7359545\ldots)$, bg $(k = 2.5,\ k_{\mathrm{j}\theta} = 1.652212\ldots)$, bi $(k = 3,\ k_{\mathrm{j}\theta} = 1.603502\ldots)$, bk $(k = 4,\ k_{\mathrm{j}\theta} = 1.571211\ldots)$ and bm $(k = \infty,\ k_{\mathrm{j}\theta} = \mathrm{e}^{\sqrt{3}/4})$ are all on curve ba, or they coincide with each other.

Corresponding to figures 4.7–4.10 are axial residual stress, radial residual stress, and circumferential residual stress respectively.

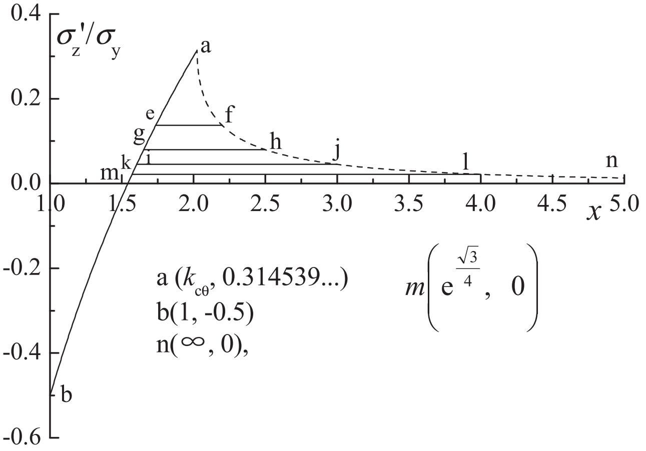


FIG. 4.8 – The distribution of axial residual stress within the whole wall for various $k_{j\theta}$.

When $k = \infty$, in the elastic region, $\frac{\sigma_z'}{\sigma_y} = 0$. The coordinates of some key points in figure 4.8 are a $(k_{c\theta}, 0.314539\ldots)$, b $(1, -0.5)$, $m\left(e^{\frac{\sqrt{3}}{4}}, 0\right) = m\,(1.541896, 0)$, $n\,(\infty, 0)$.

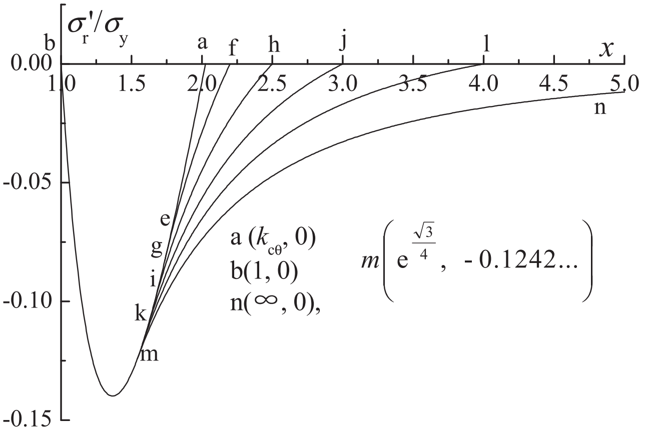


FIG. 4.9 – The distribution of radial residual stress within the whole wall for various $k_{j\theta}$.

When $k = \infty$, $k_j = \exp\left(\frac{\sqrt{3}}{4}\right)$, in the elastic region, $\frac{\sigma_r'}{\sigma_y} = -\frac{k_j^2-(\sqrt{3}+2)/2}{\sqrt{3}x^2} = -\frac{\exp\left(\frac{\sqrt{3}}{2}\right)-(\sqrt{3}+2)/2}{\sqrt{3}x^2}$. The coordinates of some key points in figure 4.9 are a $(k_{c\theta}, 0)$, b $(1, 0)$, $m\left(e^{\frac{\sqrt{3}}{4}}, -0.124195\ldots\right) = m\,(1.541896, -0.124195)$, $n\,(\infty, 0)$.

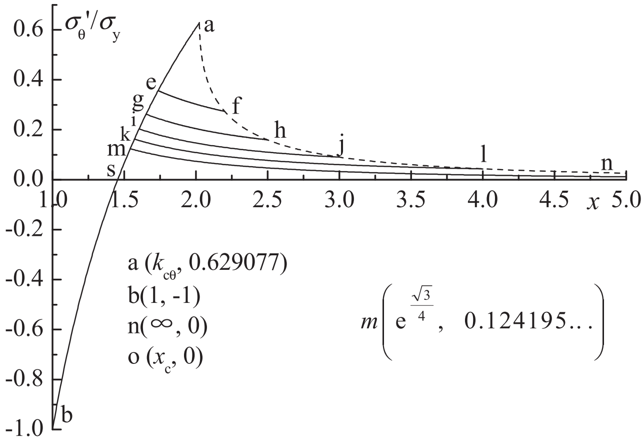


FIG. 4.10 – The distribution of circumferential residual stress within the whole wall for various $k_{j\theta}$.

When $k = \infty$, $k_j = \exp\left(\frac{\sqrt{3}}{4}\right)$, in the elastic region,

$$\frac{\sigma_\theta'}{\sigma_y} = \frac{k_j^2 - (\sqrt{3}+2)/2}{\sqrt{3}x^2} = \frac{\exp\left(\frac{\sqrt{3}}{2}\right) - (\sqrt{3}+2)/2}{\sqrt{3}x^2}$$

The coordinates of some key points in figure 4.10 are a ($k_{c\theta}$, 0.629077), b (1, −1), $m\left(e^{\frac{\sqrt{3}}{4}}, 0.124195...\right) = m$ (1.541896, 0.124195), n (∞, 0) and s (x_c, 0) = s (1.4540151, 0). x_c is the solution of equation (4.26) that is obtained by letting $\frac{\sigma_\theta'}{\sigma_y} = 0$ for equation (4.24).

$$2x^2 \ln x^2 + \left(2 - \sqrt{3}\right)x^2 + 2 + \sqrt{3} = 0 \tag{4.26}$$

In figures 4.8–4.10, the curves with the same labeled letters represent the residual stresses under the same k and $k_{j\theta}$, saying curves bij in figures 4.8–4.10 represent the residual stresses under $k = 3$ and $k_{j\theta} = 1.603502$, *i.e.* curve bij in figure 4.8 is the axial residual stress when $k = 3$ and $k_{j\theta} = 1.603502$; curve bij in figure 4.9 is the radial residual stress when $k = 3$ and $k_{j\theta} = 1.603502$; curve bij in figure 4.10 is the circumferential residual stress when $k = 3$ and $k_{j\theta} = 1.603502$, and so on.

About the terminal connection curve (dotted curve) in figures 4.8–4.10: For a certain $k_{j\theta}$, from equation (4.3) we get k, letting $x = k$, we get $\frac{\sigma_{zo}'}{\sigma_y}$, $\frac{\sigma_{\theta o}'}{\sigma_y}$ and $\frac{\sigma_{eo}'}{\sigma_y}$, connecting all $\frac{\sigma_{zo}'}{\sigma_y}$ (or $\frac{\sigma_{\theta o}'}{\sigma_y}$, $\frac{\sigma_{eo}'}{\sigma_y}$) we get the terminal connection curve, $\frac{\sigma_{ro}'}{\sigma_y} \equiv 0$ for any k and $k_{j\theta}$.

Figure 4.11 shows the axial residual stress together with radial residual stress and circumferential residual stress for $k = 3$ and $k_{j\theta} = 1.603502$.

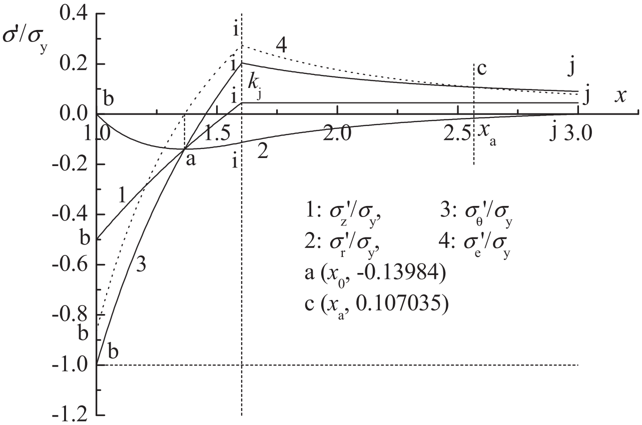


FIG. 4.11 – The axial, radial, and circumferential residual stress for $k = 3$ and $k_{j\theta} = 1.603502$.

The coordinate of point a in figure 4.11 is a $(x_0, -0.13984) = a\left(\sqrt{\frac{\sqrt{3}+2}{2}}, -0.13984\right) =$ a $(1.366025, -0.13984)$, the coordinate of point c in figure 4.11 is c $(x_{\mathrm{a}}, 0.107035) =$ c $(2.566799, 0.107035)$, where x_{a} is the abscissa of the intersection point of curves $\frac{\sigma'_\theta}{\sigma_y}$ and $\frac{\sigma'_e}{\sigma_y}$, $x_a = \sqrt{\sqrt{3}-1}k$.

However, if $k_j \neq k_{j\theta}$, or the correlation between k_j and k does not meet equation (4.3), the above features do not exist, or even $\left|\frac{\sigma'_{\theta i}}{\sigma_y}\right| > 1$ or $\left|\frac{\sigma'_{ei}}{\sigma_y}\right| > 1$. This case is illustrated in figure 4.12, where curves 1 and 2 coincide with each other in the plastic region and both pass through the point o $(x_{\mathrm{c}}, 0)$ because $k_j = k_{j\theta}$, but curves 3 and 4 do not coincide with each other in the plastic region and do not pass through the point o $(x_{\mathrm{c}}, 0)$, and they do not coincide with curve 1 and 2 either because $k_j \neq k_{j\theta}$. In figure 4.12, curve 1 is curve bij in figure 4.10; curve 2 is curve bkl in figure 4.10; curve 3 is for $k = 4$ and $k_j = 1.7 > k_{j\theta}$; curve 4 is for $k = 4$ and $k_j = 1.5 < k_{j\theta}$. Curves 3 and 4 are plotted from equations (3.4) and (3.8). When $k = 4$, $k_j = 1.7 > k_{j\theta}$, $|\sigma'_{\theta i}| > \sigma_y$; When $k = 4$, $k_j = 1.5 < k_{j\theta}$, $|\sigma'_{\theta i}| < \sigma_y$, but the load-bearing capacity will be smaller than that under the case of $k_j = k_{j\theta}$.

When $k_j = k_{j\theta}$, within the plastic region, all curves of the residual stresses for any k and $k_{j\theta}$ are located on the same curve, the greater $k_{j\theta}$ is, the longer the curve of the residual stress for this $k_{j\theta}$ is. Because the deepest plastic depth is $k_{j\theta} = k_{c\theta}$, the longest curve of the residual stress is the one with $k_{j\theta} = k_{c\theta}$, see curve *ba* in figures 4.7–4.10, where x varies from 1 to $k_{j\theta} = k_{c\theta} = 2.024678965$. The distributions of the equivalent residual stress, axial residual stress, radial residual stress, and

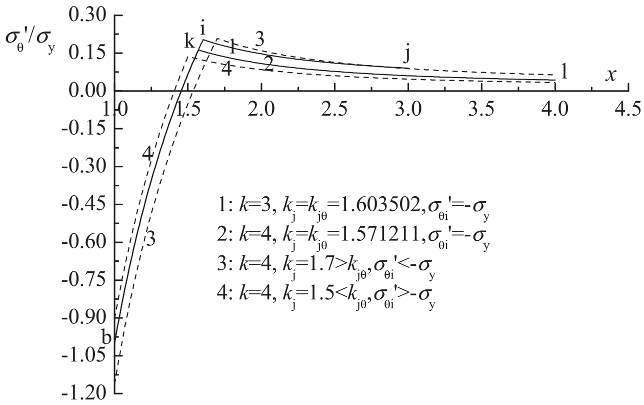


FIG. 4.12 – Comparison between the circumferential residual stresses under different k and k_j.

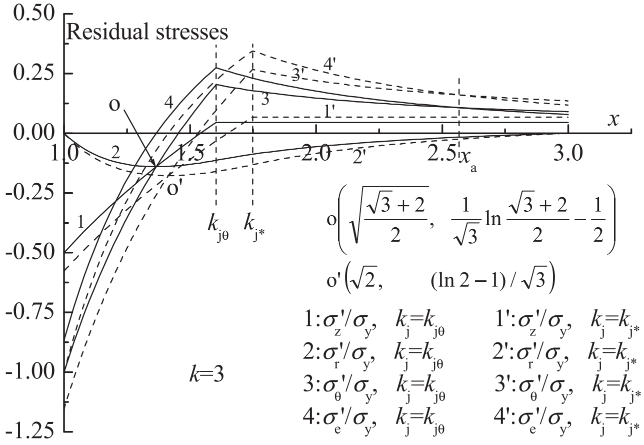
circumferential residual stress within the whole wall for different k and $k_{j\theta}$ are illustrated in figures 4.7–4.10, respectively.

For comparison, when $k = 3$, $k_{j*} = 1.748442\ldots$ and $k = 3$, $k_{j\theta} = 1.603502\ldots$, the residual stresses are plotted in figure 4.13a; when $k = 4$, $k_{j*} = 1.694172\ldots$ and $k = 4$, $k_{j\theta} = 1.571211\ldots$, the residual stresses are plotted in figure 4.13b. In figure 4.13, solid curves, which ensures $|\sigma'_{\theta i}| < \sigma_y$, are for $k_j = k_{j\theta}$; and dash curves, which ensures $|\sigma'_{ei}| \leq \sigma_y$, are for $k_j = k_{j*}$.

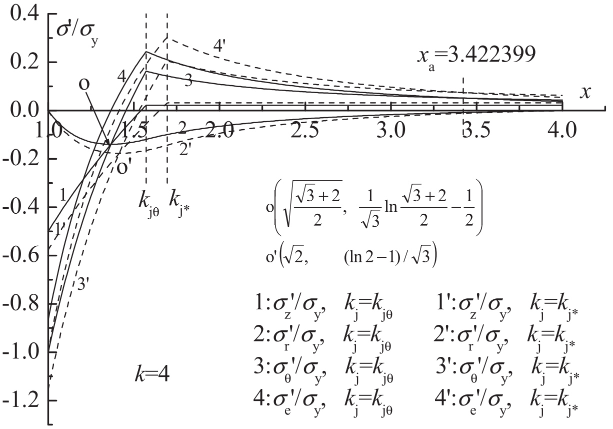
When $k_j = k_{j\theta}$, $x_0 = \sqrt{\frac{\sqrt{3}+2}{2}}$, $\frac{\sigma'_{zx0}}{\sigma_y} = \frac{\sigma'_{\theta x0}}{\sigma_y} = \frac{\sigma'_{rx0}}{\sigma_y} = \frac{\sigma'_{rmin}}{\sigma_y} = -\left(\frac{1}{2} - \frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}+2}{2}\right)$, as a result, irrespective of k, the three curves for residual stress collect at the fixed point within the plastic region: $o\left(\sqrt{\frac{\sqrt{3}+2}{2}},\ \frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}+2}{2} - \frac{1}{2}\right)$. When $k_j = k_{j*}$, $x_0 = \sqrt{2}$, $\frac{\sigma'_{zx0}}{\sigma_y} = \frac{\sigma'_{\theta x0}}{\sigma_y} = \frac{\sigma'_{rx0}}{\sigma_y} = \frac{\sigma'_{rmin}}{\sigma_y} = -\frac{1-\ln 2}{\sqrt{3}}$, as a result, irrespective of k, the three curves for residual stress collect at the fixed point within the plastic region: $o'\left(\sqrt{2},\ -\frac{1-\ln 2}{\sqrt{3}}\right)$.

Since $k_{j\theta} < k_{j*}$ for the same k, the residual stresses under the case $k_j = k_{j\theta}$ are smaller than those under the case $k_j = k_{j*}$.

When $k_j = k_{j*}$, $\frac{\sigma'_{\theta i}}{\sigma_y} = -\frac{2}{\sqrt{3}} < -1$. To make $\frac{\sigma'_{\theta i}}{\sigma_y} = -1$, just replace $\sqrt{3}$ with 2. In fact, on many occasions, replacing $\sqrt{3}$ in the results from the case $k_j = k_{j\theta}$ with 2 can obtain the results in the case $k_j = k_{j*}$. For examples, replacing $\sqrt{3}$ in $2k^2\ln k_{j\theta}^2 - 2k_{j\theta}^2 - \sqrt{3}k^2 + (2+\sqrt{3}) = 0$ (equation (4.3)) with 2 just obtains $k^2\ln k_{j*}^2 - k^2 - k_{j*}^2 + 2 = 0$ (equation (2.34)); replacing $\sqrt{3}$ in $x_{0\theta} = \sqrt{\frac{\sqrt{3}+2}{2}}$ (equation (4.20)) with 2 just obtains $x_0 = \sqrt{2}$; replacing $\sqrt{3}$ in $x_{1\theta} = e^{\frac{\sqrt{3}}{4}}$ (equation (4.21)) with 2 just obtains $x_0 = \sqrt{e}$.



(a) k=3, k_{j*}=1.748442… and k=3, $k_{j\theta}$=1.603502…



(b) k=4, k_{j*}=1.694172… and k=3, $k_{j\theta}$=1.571211…

FIG. 4.13 – Comparison between residual stresses when $k_j = k_{j\theta}$ and $k_j = k_{j*}$.

4.2.2 The Residual Stresses for Entire Yield

Under the entire yield state, or if $k_j = k$, when $\sigma_{ei}' = -\sigma_y$ (in this case $k_j = k = k_c$), $\sigma_{\theta i}' < -\sigma_y$ (see figure 3.23). To control $\sigma_{\theta i}' \geq -\sigma_y$, setting $\frac{\sigma_{\theta i}'}{\sigma_y} = \frac{1}{\sqrt{3}}\left(2 + \ln\frac{1}{k^2} - \frac{\ln k^2}{k^2-1} - \frac{k^2\ln k^2}{k^2-1}\right) \geq -1$ (see equation (3.59)) results in equation (4.5). Thereupon, when $k_j = k$, to control $\sigma_{\theta i}' \geq -\sigma_y$, we only need to limit $k \leq k_{c\theta}$, if

$k \leq k_{c\theta}$, $\sigma'_{\theta i} \geq -\sigma_y$ when $k_j = k$. Figures 4.14 and 4.15 show the distributions of the residual stresses for $k = k_{c\theta} = 2.024678965$ and $k = 1.5 < k_{c\theta}$, respectively.

Compared with figure 3.20, $|\sigma'_{\theta i}|$ has been restrained within σ_y.

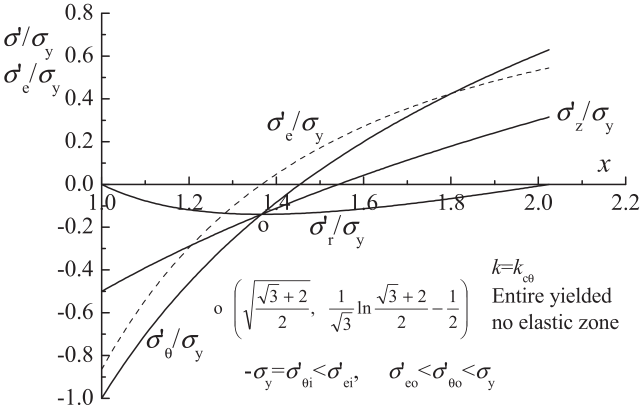


FIG. 4.14 – The distributions of the residual stresses for $k_j = k$ and $k = k_{c\theta}$.

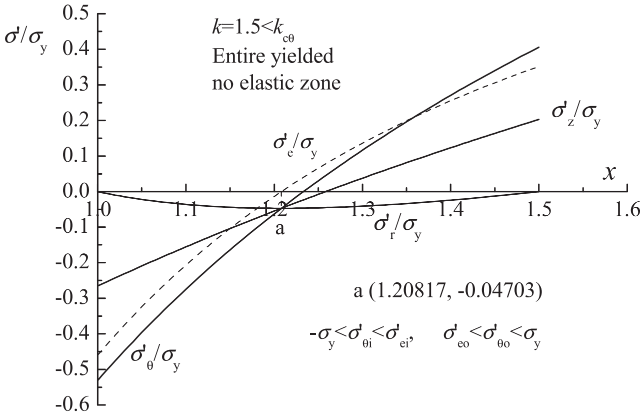


FIG. 4.15 – The distributions of the residual stresses for $k_j = k$ and $k = 1.5 < k_{c\theta}$.

4.2.3 The Residual Stresses with Radius of Elastic-Plastic Juncture Being Arithmetic Mean Radius of Inside Radius and Outside Radius

When $k_j = \frac{k+1}{2}$ and $k = k_a = 2.618253$, $\sigma'_{ei} = -\sigma_y$, but $\sigma'_{\theta i} < -\sigma_y$ (see figure 3.32). From equation (3.88), $\frac{\sigma'_{\theta i}}{\sigma_y} = \frac{1}{\sqrt{3}}\left[\frac{(k+1)^2}{4k^2} + 1 + 2\ln\frac{2}{k+1} - \left(1 - \frac{(k+1)^2}{4k^2} + \ln\frac{(k+1)^2}{4}\right)\frac{k^2+1}{k^2-1}\right]$, which can be arranged as $\frac{\sigma'_{\theta i}}{\sigma_y} = \frac{1}{\sqrt{3}}\left[\frac{k+3}{2(k+1)} + \frac{4k^2}{k^2-1}\ln\frac{2}{k+1}\right]$, that is just equation (4.7), setting $\frac{\sigma'_{\theta i}}{\sigma_y} \geq -1$ must result in $k \leq k_{a\theta}$.

Therefore, if $k_\mathrm{j} = \frac{k+1}{2}$, when $k \le k_{\mathrm{a}\theta}$, $\sigma'_{\theta\mathrm{i}} \ge -\sigma_\mathrm{y}$. Figure 4.16 shows the distributions of the residual stresses for $k = k_{\mathrm{a}\theta} = 2.360344$, $k_\mathrm{j} = (k+1)/2 = 1.680172$.

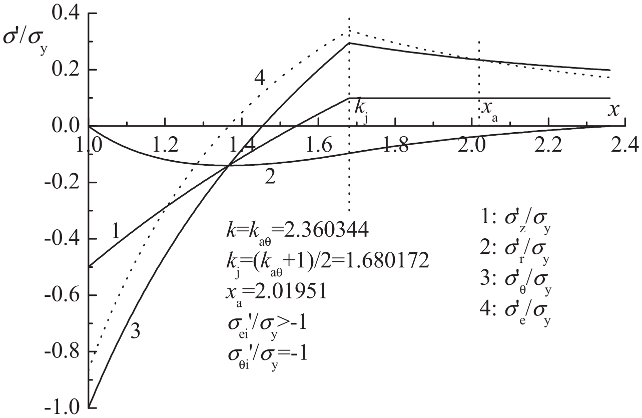


FIG. 4.16 – The distributions of the residual stresses for $k = k_{\mathrm{a}\theta}$ and $k_\mathrm{j} = (k+1)/2$.

Compared with figure 3.32, $|\sigma'_{\theta\mathrm{i}}|$ has been restrained within σ_y.

Figure 4.17 shows the distributions of the residual stresses for $k = 2.2 < k_{\mathrm{a}\theta}$ and $k_\mathrm{j} = (k+1)/2 = 1.6$.

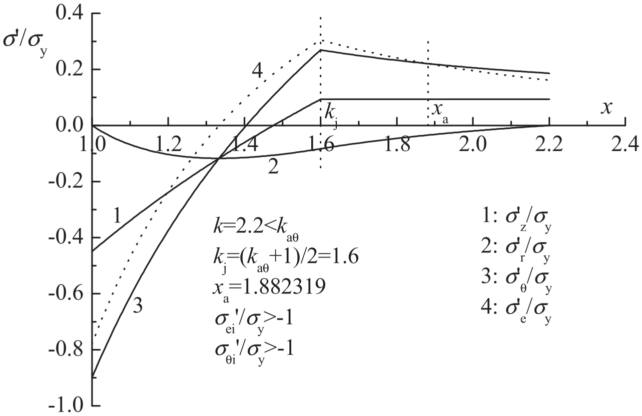


FIG. 4.17 – The distributions of the residual stresses for $k < k_{\mathrm{a}\theta}$ and $k_\mathrm{j} = (k+1)/2$.

In the case of $k = 2.2 < k_{\mathrm{a}\theta}$, $\frac{\sigma'_{\theta\mathrm{i}}}{\sigma_\mathrm{y}} = -0.89899 > -1$.

Figure 4.18 shows the distributions of the residual stresses for $k = 2.5 < k_{\mathrm{a}\theta}$ and $k_\mathrm{j} = (k+1)/2 = 1.75$.

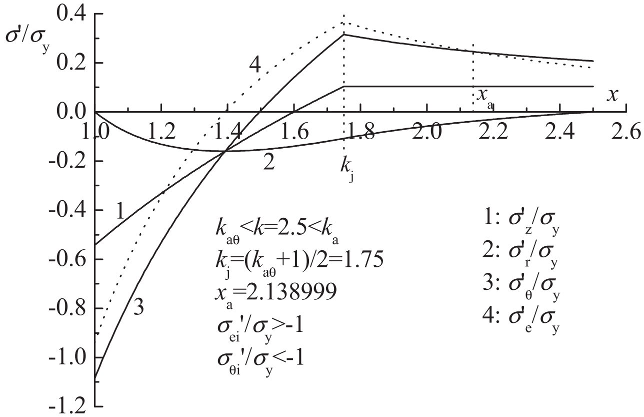


FIG. 4.18 – The distributions of the residual stresses for $k > k_{a\theta}$ and $k_j = (k+1)/2$.

In the case of $k = 2.5 > k_{a\theta}$, $\frac{\sigma'_{\theta i}}{\sigma_y} = -1.08491 < -1$.

4.2.4 The Residual Stresses with Radius of Elastic-Plastic Juncture Being Geometrical Mean Radius of Inside Radius and Outside Radius

When $k_j = \sqrt{k}$ and $k = k_b = 3.042297$, $\sigma'_{ei} = -\sigma_y$, but $\sigma'_{\theta i} < -\sigma_y$ (see figure 3.42). To control $\sigma'_{\theta i} \geq -\sigma_y$, setting $\frac{\sigma'_{\theta i}}{\sigma_y} = \frac{2}{\sqrt{3}} \frac{k - k^2 \ln k - 1}{k^2 - 1} \geq -1$ obtains equation (4.10).

Therefore, if $k_j = \sqrt{k}$, when $k \leq k_{b\theta}$, $\sigma'_{\theta i} \geq -\sigma_y$. Figure 4.19 shows the distributions of the residual stresses for $k = k_{b\theta} = 2.659725$, $k_j = \sqrt{k} = 1.630866$.

Compared with figure 3.42, $|\sigma'_{\theta i}|$ has been restrained within σ_y.

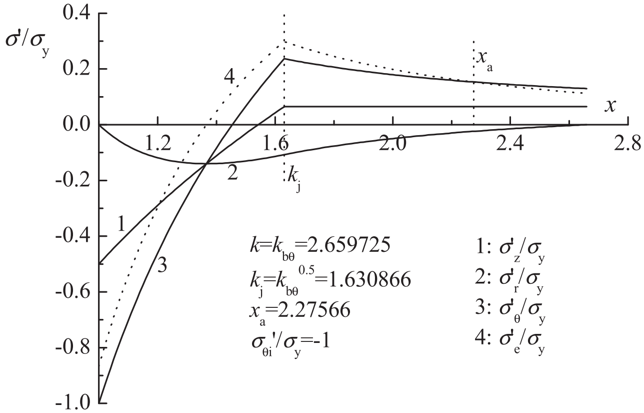


FIG. 4.19 – The distributions of the residual stresses for $k = k_{b\theta}$ and $k_j = \sqrt{k}$.

4.2.5 The Residual Stresses When Equivalent Total Stress on Elastic-Plastic Juncture is the Minimum

When $k_{\mathrm{j}} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right)$, the circumferential residual stress is equation (3.120), at the inner surface

$$\frac{\sigma'_{\theta\mathrm{i}}}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}\left[\frac{\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right)}{k^2} + 1 - \sqrt{3}\frac{p}{\sigma_{\mathrm{y}}} - \left(1 - \frac{\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right)}{k^2} + \sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right)\frac{k^2+1}{k^2-1}\right] \tag{4.27}$$

Setting $\frac{\sigma'_{\theta\mathrm{i}}}{\sigma_{\mathrm{y}}} \geq -1$ obtains

$$\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right) + \frac{\sqrt{3}}{2}k^2 - \sqrt{3}k^2\frac{p}{\sigma_{\mathrm{y}}} - \frac{2+\sqrt{3}}{2} \geq 0 \quad \text{or} \quad k^2 \leq \frac{\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right) - \frac{2+\sqrt{3}}{2}}{\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}} - \frac{\sqrt{3}}{2}} \tag{4.28}$$

Equation (4.28) is the load-bearing capacity $(\frac{p}{\sigma_{\mathrm{y}}})$ of a cylinder with a certain k greater than $k_{\mathrm{c}\theta}$ when $k_{\mathrm{j}} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right)$ and $|\sigma'_{\theta\mathrm{i}}| = \sigma_{\mathrm{y}}$.

For equation (4.28), the domain of k^2 is $\frac{p}{\sigma_{\mathrm{y}}} \geq \frac{1}{2} = p_1$ and $\frac{p}{\sigma_{\mathrm{y}}} \leq \frac{1}{\sqrt{3}}\ln\frac{2+\sqrt{3}}{2} = p_2$. The function $k^2 \leq \frac{\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right) - \frac{2+\sqrt{3}}{2}}{\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}} - \frac{\sqrt{3}}{2}}$ is plotted in figure 4.20. There are two branches for the graphs of the function: curve og and curve bafh. In addition, the function $k_{\mathrm{j}} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right)$ is plotted in figure 4.20 as the solid curve oadi and the function $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$ is plotted in figure 4.20 as the dash curve ofdc. In figure 4.20, curve og corresponds to the domain of $\frac{p}{\sigma_{\mathrm{y}}} \leq \frac{1}{\sqrt{3}}\ln\frac{2+\sqrt{3}}{2}$, and $k < 1$ on curve og, thus it is of no practical significance; curve bafh corresponds to the domain of $\frac{p}{\sigma_{\mathrm{y}}} \geq \frac{1}{2}$. For $k_{\mathrm{j}} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right)$, letting $k_{\mathrm{j}} = k$ and substituting $k = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right)$ into equation (4.28) gives $\frac{k^2\ln k^2}{k^2-1} = \frac{2+\sqrt{3}}{2}$, that is just equation (4.3). Thus, the ordinate of the intersection (point a) of curve bah and curve oadi is $k_{\mathrm{c}\theta} = 2.024678965$. When $k = k_{\mathrm{c}\theta}$, from equation (4.28), the corresponding load-bearing capacity $\frac{p}{\sigma_{\mathrm{y}}} = 0.814539\ldots$. Therefore, the coordinate of point a is a $(0.814539,\ k_{\mathrm{c}\theta})$. Substituting $k = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right)$ into $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$ leads to $\frac{2}{\sqrt{3}}\frac{k^2-1}{k^2} = \frac{2}{\sqrt{3}}\ln k$ or $\frac{k^2\ln k}{k^2-1} = 1$, that is just equation (2.28). Therefore, the coordinate of the intersection (point d) of curve ofdc and curve oadi is d $(0.920081,\ k_{\mathrm{c}})$. Substituting $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$ into equation (4.28) obtains $\exp\left(2\frac{k^2-1}{k^2}\right) - \left(2 - \frac{\sqrt{3}}{2}\right)k^2 + 1 - \frac{\sqrt{3}}{2} = 0$, the solution of which is $k = 2.0336493\ldots$. The corresponding load-bearing capacity $\frac{p}{\sigma_{\mathrm{y}}} = 0.8754994\ldots$. Therefore, the coordinate of the intersection (point f) of curve ofdc and curve bah is f $(0.8754994,\ 2.0336493)$.

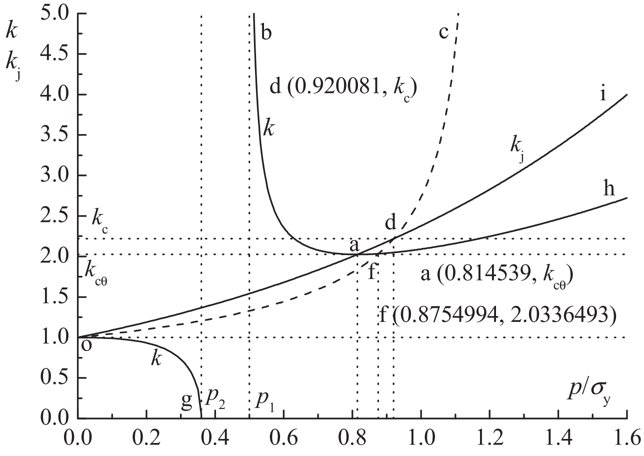


FIG. 4.20 – The relationship between k (k_j) and $\frac{p}{\sigma_\mathrm{y}}$.

For the function $k^2 \le \frac{\exp\left(\sqrt{3}\frac{p}{\sigma_\mathrm{y}}\right)-\frac{2+\sqrt{3}}{2}}{\sqrt{3}\frac{p}{\sigma_\mathrm{y}}-\frac{\sqrt{3}}{2}}$, setting $\frac{\mathrm{d}k^2}{\mathrm{d}\left(\sqrt{3}\frac{p}{\sigma_\mathrm{y}}\right)}=0$ gives

$$\sqrt{3}\,\frac{p}{\sigma_\mathrm{y}}\exp\left(\sqrt{3}\,\frac{p}{\sigma_\mathrm{y}}\right)-\frac{2+\sqrt{3}}{2}\exp\left(\sqrt{3}\,\frac{p}{\sigma_\mathrm{y}}\right)+\frac{2+\sqrt{3}}{2}=0 \tag{4.29}$$

Solving equation (4.29) obtains the horizontal ordinate of the minimum value point a

$$\frac{p}{\sigma_\mathrm{y}}=0.814539\ldots \tag{4.30}$$

Substituting $\frac{p}{\sigma_\mathrm{y}}=0.814539\ldots$ into $k^2=\frac{\exp\left(\sqrt{3}\frac{p}{\sigma_\mathrm{y}}\right)-\frac{2+\sqrt{3}}{2}}{\sqrt{3}\frac{p}{\sigma_\mathrm{y}}-\frac{\sqrt{3}}{2}}$ leads to $k=k_{\mathrm{c}\theta}$. It is thus evident that point a is the minimum value point. For curve bah, on section ba, $k>k_\mathrm{j}$; on section ah, $k<k_\mathrm{j}$, where k_j is determined by $k_\mathrm{j}=\exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_\mathrm{y}}\right)$.

Letting $k^2=\frac{\exp\left(\sqrt{3}\frac{p}{\sigma_\mathrm{y}}\right)-\frac{2+\sqrt{3}}{2}}{\sqrt{3}\frac{p}{\sigma_\mathrm{y}}-\frac{\sqrt{3}}{2}}=k_\mathrm{j}^2=\exp\left(\sqrt{3}\,\frac{p}{\sigma_\mathrm{y}}\right)$ also obtains equation (4.29), and when $k^2=k_\mathrm{j}^2=\exp\left(\sqrt{3}\,\frac{p}{\sigma_\mathrm{y}}\right)$, equation (4.29) is just equation (4.3). When $k=k_\mathrm{j}$, substituting $k^2=k_\mathrm{j}^2=\exp\left(\sqrt{3}\,\frac{p}{\sigma_\mathrm{y}}\right)$ into $\frac{p}{\sigma_\mathrm{y}}=\frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$ obtains $\frac{k^2\ln k}{k^2-1}=1$. From figure 4.20, when $\frac{p}{\sigma_\mathrm{y}}<0.814539$ (corresponding to section ba), $k>k_{\mathrm{c}\theta}$, and k is larger than k_j determined by $k_\mathrm{j}=\exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_\mathrm{y}}\right)$, and k is larger than the value determined by $\frac{p}{\sigma_\mathrm{y}}=\frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$. This is reasonable and makes sense; when $\frac{p}{\sigma_\mathrm{y}}>0.814539$ (corresponding to section ha), $k > k_{c\theta}$, and k is smaller than k_j determined by $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right)$, this is unreasonable and makes no sense. Curve odac can ensure $k > k_j$, but can not ensure $\sigma'_{\theta i} \geq -\sigma_y$. When $k > k_{c\theta}$, the meaningful control curve is curve ba. Besides, when $\frac{p}{\sigma_y} \to \frac{1}{2}$, $k \to \infty$. Curve ba is re-plotted in figure 4.21.

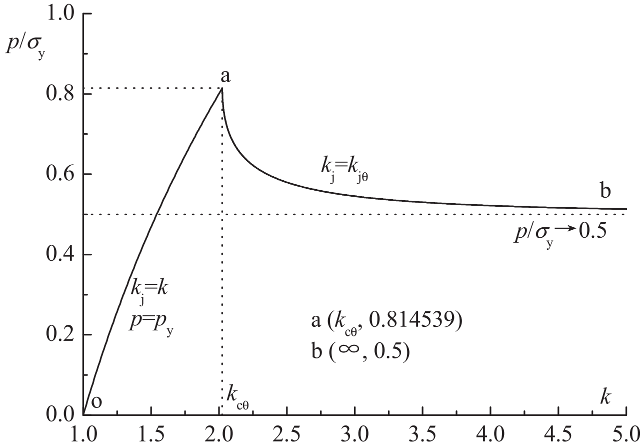


FIG. 4.21 – The reasonable relationship between k and $\frac{p}{\sigma_y}$.

Since $k_{j\theta}$ obtained by ensuring $\sigma'_{\theta i} \geq -\sigma_y$ is smaller than k_{j*} obtained by ensuring $\sigma'_{ei} \geq -\sigma_y$, the load-bearing capacity $\frac{p}{\sigma_y}$ in the case of $k_j = k_{j\theta}$ is certainly smaller than that in the of $k_j = k_{j*}$. When $k < k_{c\theta}$, there is $k_j = k$, thus $\frac{p}{\sigma_y} = \frac{2}{\sqrt{3}}\ln k$, that is expressed by curve oa in figure 4.21.

If the design conditions are controlled by curve ba when $k > k_{c\theta}$, there are both $k_j < k$ and $\sigma'_{\theta i} \geq -\sigma_y$. For example, when $k = 3$, from equation (4.28), $\frac{p}{\sigma_y} = 0.5452382\ldots$. When $k = 3$, $\frac{p}{\sigma_y} = 0.5452382$, $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right) = 1.603502\ldots = k_{j\theta}$. The residual stresses (expressed by equations (3.118)–(3.125)) within the whole wall are plotted in figure 4.22. The meaning of each curve and the main parameters are marked in the figure.

Compared with figure 3.48, the residual stresses are reduced, because when $\sigma'_{\theta i} \geq -\sigma_y$, $k_j = k_{j\theta}$, and when $\sigma'_{ei} \geq -\sigma_y$, $k_j = k_{j*}$, and $k_{j\theta}$ is smaller than k_{j*}. However, the load-bearing capacity is reduced.

In fact, when $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right)$, equations (4.29) and (4.28) are equivalent. When $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right)$ (but $k > k_j$), equation (4.28) becomes equation (4.3).

Therefore, curve bc in figure 4.2 or figure 4.3 is also the relation between k and k_j when $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right)$ and $\sigma'_{\theta i} = -\sigma_y$. However, when the load-bearing capacity $\frac{p}{\sigma_y} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$, $\sigma_{ej} \equiv \sigma_y$ within the whole plastic region; when the load-bearing capacity

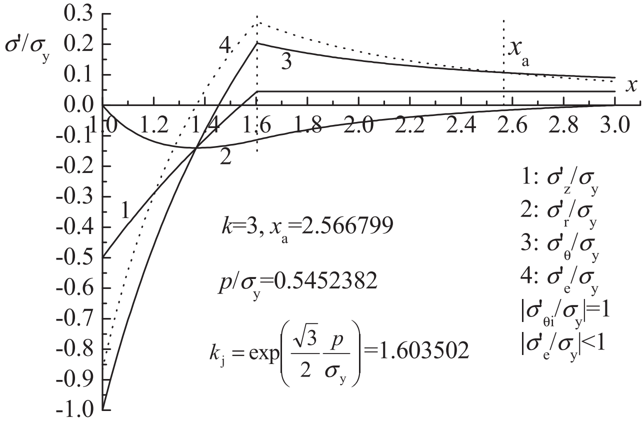


FIG. 4.22 – The residual stresses when $k = 3$, $\frac{p}{\sigma_y} = 0.5452382$.

$\frac{p}{\sigma_y} = \frac{2}{\sqrt{3}} \ln k_j$ (equivalent to $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right)$), where k_j is restrained by $k^2 \ln k_j^2 - k^2 - k_j^2 + 2 \leq 0$, $\sigma_{ej} < \sigma_y$ at the elastic-plastic juncture.

Equation (4.28) or equation (4.29) can ensure $k_j < k$ and $\sigma'_{\theta i} \geq -\sigma_y$, but under the premise of ensuring $k_j < k$ and $\sigma'_{\theta i} \geq -\sigma_y$ as well as σ_{ej} being the minimum, the load-bearing capacity $\frac{p}{\sigma_y}$ calculated by $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right)$ may not the most satisfactory. From equation (4.28) or equation (4.29), the greater k is, the smaller k_j is, or the smaller $\frac{p}{\sigma_y}$ is. When $k = \infty$, $k_j = \exp\left(\frac{\sqrt{3}}{4}\right)$, hence $\frac{p}{\sigma_y} = \frac{1}{2}$, that is merely the maximum elastic load with $k = \infty$ based on the maximum shear stress theory (Tresca yield criterion). Besides, when $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right)$, from figure 4.21, the load-bearing capacity under the case of $k > k_{c\theta}$ is unexpectedly smaller than that under the case of $k < k_{c\theta}$. So, if it is not necessary, $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right)$ is not a satisfactory solution, the higher load-bearing capacity is needed to be further explored.

When $k_j = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_y}\right)$, autofrettage pressure

$$\frac{p_a}{\sigma_y} = \frac{p}{\sigma_y} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}k^2}\exp\left(\sqrt{3}\frac{p}{\sigma_y}\right) \tag{4.31}$$

When $\frac{p}{\sigma_y} \leq \frac{2}{\sqrt{3}} \ln k$, $\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}k^2}\exp\left(\sqrt{3}\frac{p}{\sigma_y}\right) \geq 0$. Therefore, $\frac{p_a}{\sigma_y} \geq \frac{p}{\sigma_y}$.

When $k_{j\theta} = k = k_{c\theta}$, $k_j^2 = \exp\left(\sqrt{3}\frac{p}{\sigma_y}\right) = k^2$, then, from equation (4.31), $\frac{p_a}{\sigma_y} = \frac{p}{\sigma_y} = \frac{2}{\sqrt{3}} \ln k_{c\theta} = 0.814539$.

When $\frac{p}{\sigma_y} = \frac{2}{\sqrt{3}} \ln k$, $\frac{p_a}{\sigma_y}$ reaches the maximum value, which is $\frac{p_a}{\sigma_y} = \frac{p}{\sigma_y} = \frac{2}{\sqrt{3}} \ln k$. It is known that only when $k \leq k_{c\theta}$, $\frac{p}{\sigma_y} = \frac{2}{\sqrt{3}} \ln k$. For $k \geq k_{c\theta}$, from equation (4.28), we obtain $\frac{p}{\sigma_{\mathrm{y}}}-\frac{1}{\sqrt{3}k^2}\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right)=\frac{1}{2}-\frac{2+\sqrt{3}}{2\sqrt{3}k^2}$. Substituting this equation into equation (4.31) gives

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{2+\sqrt{3}}{2\sqrt{3}}\frac{k^2-1}{k^2} \tag{4.32}$$

Equation (3.18) indicates that in general when the load-bearing capacity $\frac{p}{\sigma_{\mathrm{y}}}$ equals autofrettage pressure, or $\frac{p}{\sigma_{\mathrm{y}}}=\frac{p_a}{\sigma_{\mathrm{y}}}$, it is the optimum, in this case $\sigma_{\mathrm{e}}^{\mathrm{T}}\equiv\sigma_{\mathrm{y}}$ in the whole plastic region. So, when $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{2+\sqrt{3}}{2\sqrt{3}}\frac{k^2-1}{k^2}$ $(k\geq k_{\mathrm{c}\theta})$, the load-bearing capacity is also $\frac{p}{\sigma_{\mathrm{y}}}=\frac{2+\sqrt{3}}{2\sqrt{3}}\frac{k^2-1}{k^2}$. In the case that $k_{\mathrm{j}}=\exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right)$, if $k\geq k_{\mathrm{c}\theta}$, when $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{2+\sqrt{3}}{2\sqrt{3}}\frac{k^2-1}{k^2}$, $\frac{p}{\sigma_{\mathrm{y}}}$ is controlled by equation (4.28), $\frac{p}{\sigma_{\mathrm{y}}}<\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$, $\sigma_{\mathrm{ej}}^{\mathrm{T}}<\sigma_{\mathrm{y}}$ and $\sigma'_{\theta\mathrm{i}}=-\sigma_{\mathrm{y}}$. For example, when $k=3$, $k_{\mathrm{j}\theta}=1.603502$, $\frac{p}{\sigma_{\mathrm{y}}}=0.5452382$ and $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=0.957645\ldots$ (from equation (4.31)), $\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=0.687464$ (equation (3.115)).

When $k\leq k_{\mathrm{c}\theta}$, the optimum load-bearing capacity is $\frac{p}{\sigma_{\mathrm{y}}}=\frac{p_a}{\sigma_{\mathrm{y}}}=\frac{2}{\sqrt{3}}\ln k$, and in this case $\sigma_{\mathrm{e}}^{\mathrm{T}}\equiv\sigma_{\mathrm{y}}$ in the whole plastic region.

When $k=\infty$, from equation (4.31), $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{2+\sqrt{3}}{2\sqrt{3}}$; from equation (4.28), $\frac{p}{\sigma_{\mathrm{y}}}=\frac{1}{2}$, when $\frac{p}{\sigma_{\mathrm{y}}}=\frac{1}{2}$, from equation (4.31), we also have $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{2+\sqrt{3}}{2\sqrt{3}}$. When $k=\infty$, equation (4.31) become

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{p}{\sigma_{\mathrm{y}}}+\frac{1}{\sqrt{3}} \tag{4.33}$$

For equation (4.31), when $k\leq k_{\mathrm{c}\theta}$, $\frac{p}{\sigma_{\mathrm{y}}}=0$ to $\frac{2}{\sqrt{3}}\ln k_{\mathrm{c}\theta}$; when $k\geq k_{\mathrm{c}\theta}$, $\frac{p}{\sigma_{\mathrm{y}}}=\frac{2}{\sqrt{3}}\ln k_{\mathrm{c}\theta}$ to the value determined by equation (4.28). For a pressure vessel with a certain radius ratio $k\geq k_{\mathrm{c}\theta}$, the maximum load for σ_{ej} to reach the minimum is the value determined by equation (4.28). When $\frac{p}{\sigma_{\mathrm{y}}}$ equals the value determined by equation (4.28), $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{2+\sqrt{3}}{2\sqrt{3}}\frac{k^2-1}{k^2}=\frac{2+\sqrt{3}}{2}\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$. For the convenience of engineering applications, some data calculated by equation (4.28) are listed in table 4.2.

Equation (4.31) is plotted in figure 4.23. Equation (4.31) is just equation (3.120). Therefore, when $k\leq k_{\mathrm{c}\theta}$, the curves for equation (4.31) is just those in figure 3.51. When $k\geq k_{\mathrm{c}\theta}$, the curves for equation (4.31) is those in figure 3.34 except that the ends are different from each other. The connection of the ends of the curves in figure 4.23 is curve cd, which is lower than ab in figure 3.34 of the last chapter because the maximum load-bearing capacity $\frac{p}{\sigma_{\mathrm{y}}}$ and the maximum autofrettage pressure $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$ are lower than those in chapter 3.

In figure 4.23, the curve cd is the connection of the ends of the curves described by equation (4.31), which shows the maximum load-bearing capacity $\frac{p}{\sigma_{\mathrm{y}}}$ and the maximum autofrettage pressure $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$ for $k\geq k_{\mathrm{c}\theta}$. Substituting k^2 in equation (4.28) into

TAB. 4.2 – Numerical value from $\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right)+\frac{\sqrt{3}}{2}k^2-\sqrt{3}k^2\frac{p}{\sigma_{\mathrm{y}}}-\frac{2+\sqrt{3}}{2}=0$.

k	$\frac{p}{\sigma_{\mathrm{y}}}$	$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$	k	$\frac{p}{\sigma_{\mathrm{y}}}$	$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$
$k_{\mathrm{c}\theta}$	0.814539	0.814539	3.2	0.537939	0.97214
2.05	0.730678	0.820991	3.3	0.534988	0.97842
2.1	0.683141	0.833053	3.4	0.532395	0.984154
2.2	0.636884	0.854757	3.5	0.5301	0.989403
2.3	0.610605	0.873692	3.6	0.528057	0.994221
2.4	0.592829	0.89031	3.7	0.526228	0.998654
2.5	0.579793	0.904974	3.8	0.524583	1.002742
2.6	0.569754	0.917979	3.9	0.523096	1.006519
2.7	0.561762	0.929566	4	0.521747	1.010016
2.8	0.555241	0.939933	4.5	0.516553	1.024148
2.9	0.549817	0.949247	5	0.513068	1.034256
3	0.5452382	0.957645	10	0.503025	1.066577
3.1	0.541323	0.965243	∞	0.5	1.07735

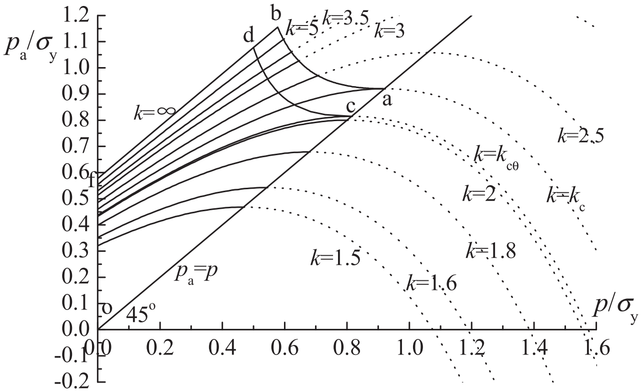


FIG. 4.23 – The graph of $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{p}{\sigma_{\mathrm{y}}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}k^2}\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right)$.

equation (4.32) obtains $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$, as a result, the equation of curve cd in figure 4.23 is generated, that is

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{2}{\sqrt{3}}\frac{\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right)-\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}-1}{\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right)-\frac{2+\sqrt{3}}{2}}\frac{2+\sqrt{3}}{4} \tag{4.34}$$

When $k=k_{\mathrm{c}\theta}$, $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{p}{\sigma_{\mathrm{y}}}=\frac{2}{\sqrt{3}}\ln k=\frac{2}{\sqrt{3}}\ln k_{\mathrm{c}\theta}$ and

$$\frac{k_{\mathrm{c}\theta}^2\ln k_{\mathrm{c}\theta}}{k_{\mathrm{c}\theta}^2-1}=\frac{2+\sqrt{3}}{4}\quad\text{or}\quad k_{\mathrm{c}\theta}^2-1=\frac{4}{2+\sqrt{3}}k_{\mathrm{c}\theta}^2\ln k_{\mathrm{c}\theta}\quad\text{or}\quad\ln k_{\mathrm{c}\theta}^2=\frac{2+\sqrt{3}}{2}\frac{k_{\mathrm{c}\theta}^2-1}{k_{\mathrm{c}\theta}^2} \tag{4.35}$$

Equation (4.34) is equivalent to $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{2}{\sqrt{3}}\frac{k_{\mathrm{c}\theta}^2-\ln k_{\mathrm{c}\theta}^2-1}{k_{\mathrm{c}\theta}^2-\frac{2+\sqrt{3}}{2}}\frac{k_{\mathrm{c}\theta}^2\ln k_{\mathrm{c}\theta}}{k_{\mathrm{c}\theta}^2-1}$. Applying equation (4.35), we obtain $\frac{k_{\mathrm{c}\theta}^2-\ln k_{\mathrm{c}\theta}^2-1}{k_{\mathrm{c}\theta}^2-\frac{2+\sqrt{3}}{2}}=\frac{k_{\mathrm{c}\theta}^2-1}{k_{\mathrm{c}\theta}^2}$. Then, $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{p}{\sigma_{\mathrm{y}}}=\frac{2}{\sqrt{3}}\ln k=\frac{2}{\sqrt{3}}\ln k_{\mathrm{c}\theta}$. This confirms the above conclusion.

The coordinate of point c is c $(\frac{2}{\sqrt{3}}\ln k_{\mathrm{c}\theta},\frac{2}{\sqrt{3}}\ln k_{\mathrm{c}\theta})$. As mentioned above, the meaningful domain of k^2 is $\frac{p}{\sigma_{\mathrm{y}}}\geq\frac{1}{2}$, and when $\frac{p}{\sigma_{\mathrm{y}}}=\frac{1}{2}$, $k^2=\infty$. Consequently, from equation (4.31) or equation (4.32) or equation (4.33), $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{2+\sqrt{3}}{2\sqrt{3}}$, or the coordinate of point d is d $(\frac{1}{2},\ \frac{2+\sqrt{3}}{2\sqrt{3}})$. The coordinate of point c is f $(0,\ \frac{1}{\sqrt{3}})$. The practically significant curves for a relation between $\frac{p}{\sigma_{\mathrm{y}}}$ and $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$ are located in a quadrilateral cdfo, where the straight line bdf is parallel to the straight line aco.

For $k\leq k_{\mathrm{c}}$, the ends of the curves described by equation (4.31) are on the straight-line segment oc where $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{p}{\sigma_{\mathrm{y}}}$. Thereupon the included angle between the straight line oa and the abscissa axis or vertical axis is 45°.

It is similar to the previous chapter, in figure 4.23, the dash curves have no practical significance, they are merely an extension of solid curves or only the elongation of the mathematical relation expressed by equation (4.31).

4.3 The Total Stresses and the Load-Bearing Capacity When Circumferential Residual Stress on the Inside Surface Controlled

The equivalent total stress

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}}+\frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}\tag{4.36}$$

The equivalent stress of the stresses caused by the internal pressure p is:

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}=\frac{\sqrt{3}}{2}\left(\frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}-\frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}\right)=\frac{\sqrt{3}k^2}{k^2-1}\frac{p}{\sigma_{\mathrm{y}}}\frac{1}{x^2}\tag{4.37}$$

In plastic region

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=1-\frac{k^2-k_{\mathrm{j}}^2+k^2\ln k_{\mathrm{j}}^2}{k^2-1}\frac{1}{x^2}+\frac{\sqrt{3}k^2}{x^2(k^2-1)}\frac{p}{\sigma_{\mathrm{y}}}\tag{4.38}$$

In elastic region

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{k_{\mathrm{j}}^2-1-\ln k_{\mathrm{j}}^2}{k^2-1}\frac{k^2}{x^2}+\frac{\sqrt{3}k^2}{x^2(k^2-1)}\frac{p}{\sigma_{\mathrm{y}}}\tag{4.39}$$

At the elastic-plastic juncture ($x = r/r_\mathrm{i} = k_\mathrm{j}$), equations (4.38) and (4.39) both become:

$$\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} = \frac{k^2(k_\mathrm{j}^2 - 1 - \ln k_\mathrm{j}^2)}{(k^2-1)k_\mathrm{j}^2} + \frac{\sqrt{3}k^2 p/\sigma_\mathrm{y}}{(k^2-1)k_\mathrm{j}^2} \tag{4.40}$$

When

$$\frac{p}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}\ln k_\mathrm{j} + \frac{k^2 - k_\mathrm{j}^2}{\sqrt{3}k^2} = \frac{p_\mathrm{a}}{\sigma_\mathrm{y}} \tag{4.41}$$

$\frac{\sigma_\mathrm{e}^\mathrm{T}}{\sigma_\mathrm{y}} \equiv 1$ within the whole plastic region. In this case, the load-bearing capacity $\frac{p}{\sigma_\mathrm{y}}$ is the most ideal.

When k_j is determined by equation (4.3), or $k_\mathrm{j} = k_{\mathrm{j}\theta}$, equation (4.41) becomes

$$\frac{p}{\sigma_\mathrm{y}} = \frac{2+\sqrt{3}}{2\sqrt{3}}\frac{k^2-1}{k^2} = \frac{2+\sqrt{3}}{2}\frac{p_\mathrm{e}}{\sigma_\mathrm{y}} = \frac{p_\theta}{\sigma_\mathrm{y}} \tag{4.42}$$

Equation (4.42) is the ultimate allowable loading of a pressure vessel with $k_\mathrm{j} = k_{\mathrm{j}\theta}$ under the condition $\sigma'_{\theta\mathrm{i}} = -\sigma_\mathrm{y}$ ($|\sigma'_\mathrm{ei}| < \sigma_\mathrm{y}$) and based on the maximum distortion strain energy theory (Mises's yield criterion), denoted by p_θ. When $k \geq k_{\mathrm{c}\theta}$ and under the condition $\sigma'_{\theta\mathrm{i}} = -\sigma_\mathrm{y}$ ($|\sigma'_\mathrm{ei}| < \sigma_\mathrm{y}$), the load-bearing capacity of an autofrettaged pressure vessel should be determined by equation (4.42). In fact, equation (4.42) is just equation (4.32).

The value determined by equation (4.42) is smaller than that determined by another equation proposed in chapter 3 based on the maximum distortion strain energy theory and $k_\mathrm{j} = k_{\mathrm{j}}*$: $\frac{p_\mathrm{m}}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$, but greater than that proposed in chapter 2 based on the maximum shear stress theory and $k_\mathrm{j} = k_{\mathrm{j}}*$: $\frac{p_\mathrm{t}}{\sigma_\mathrm{y}} = \frac{k^2-1}{k^2}$, and it is greater than the maximum elastic load-bearing capability (initial yield pressure) of an unautofrettaged pressure vessel: $\frac{p_\mathrm{e}}{\sigma_\mathrm{y}} = \frac{k^2-1}{\sqrt{3}k^2}$, or it is $1 + \frac{\sqrt{3}}{2}$ (=1.866025) times of the initial yield pressure of an unautofrettaged pressure vessel. It is seen that $p_\theta = \frac{p_\mathrm{m}+p_\mathrm{t}}{2}$, this is wonderful. Another interesting thing is that letting equation (4.42) equal the entire yield pressure $\frac{p_\mathrm{y}}{\sigma_\mathrm{y}}$, *i.e.* $\frac{p}{\sigma_\mathrm{y}} = \frac{\sqrt{3}+2}{2\sqrt{3}}\frac{k^2-1}{k^2} = \frac{p_\mathrm{y}}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}\ln k$ obtains $\frac{k^2 \ln k^2}{k^2-1} = \frac{2+\sqrt{3}}{2}$, or equation (4.5). Setting $\frac{p_\theta}{\sigma_\mathrm{y}} \leq 1$ results in

$$k \leq k_{\mathrm{r}\theta} = 2 + \sqrt{3} = 3.732051\ldots \tag{4.43}$$

When $k \leq k_{\mathrm{r}\theta}$, $p \leq \sigma_\mathrm{y}$, hence $\sigma_\mathrm{ri}^\mathrm{T} > -\sigma_\mathrm{y}$.

$\frac{p_\theta}{\sigma_\mathrm{y}}$, $\frac{p_\mathrm{t}}{\sigma_\mathrm{y}}$, $\frac{p_\mathrm{m}}{\sigma_\mathrm{y}}$, $\frac{p_\mathrm{e}}{\sigma_\mathrm{y}}$ and $\frac{p_\mathrm{y}}{\sigma_\mathrm{y}}$ are plotted in figure 4.24, where curve ad is for $\frac{p_\theta}{\sigma_\mathrm{y}} = \frac{2+\sqrt{3}}{2\sqrt{3}}\frac{k^2-1}{k^2}$; curve oab is for $\frac{p_\mathrm{y}}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}\ln k$ (based on Mises yield criterion); curve oc is for $\frac{p_\mathrm{y}}{\sigma_\mathrm{y}} = \ln k$ (based on Tresca yield criterion); curve cg is for $\frac{p_\mathrm{t}}{\sigma_\mathrm{y}} = \frac{k^2-1}{k^2}$ (based on yield Tresca criterion); curve bf is for $\frac{p_\mathrm{m}}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$ (based on Mises yield criterion); curve on is for

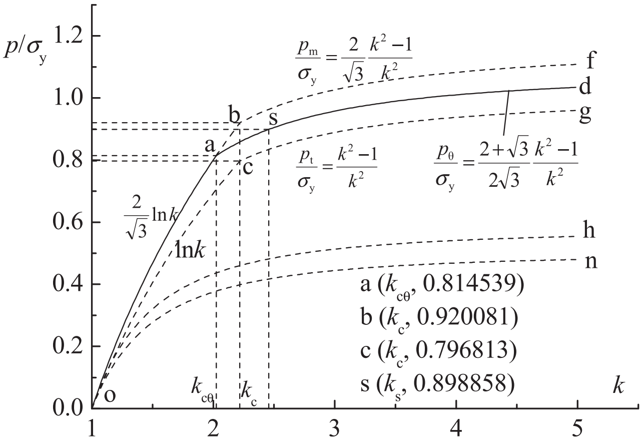


FIG. 4.24 – Comparison between load-bearing capacity.

$\frac{p_e}{\sigma_y} = \frac{k^2-1}{2k^2}$ (based on Tresca yield criterion); curve oh is for $\frac{p_e}{\sigma_y} = \frac{k^2-1}{\sqrt{3}k^2}$ (based on Mises yield criterion).

The application conditions for curve ad are: (1) $k_j = k_{j\theta}$; (2) $\sigma'_{\theta i} = -\sigma_y$ ($|\sigma'_{ei}| < \sigma_y$); (3) based on the maximum distortion strain energy theory (Mises's yield criterion); (4) $k \geq k_{c\theta}$; (5) $\frac{p}{\sigma_y} = \frac{p_\theta}{\sigma_y} = \frac{2+\sqrt{3}}{2\sqrt{3}}\frac{k^2-1}{k^2}$.

The application conditions for curve oa are: (1) $k_j = k$; (2) $|\sigma'_{\theta i}| \leq \sigma_y$, $|\sigma'_{ei}| < \sigma_y$; (3) based on the maximum distortion strain energy theory (Mises's yield criterion); (4) $k \leq k_{c\theta}$; (5) $\frac{p}{\sigma_y} = \frac{p_y}{\sigma_y} = \frac{2}{\sqrt{3}}\ln k$.

The application conditions for curve oab are: (1) $k_j = k$; (2) $|\sigma'_{ei}| \leq \sigma_y$; (3) based on the maximum distortion strain energy theory (Mises's yield criterion); (4) $k \leq k_c$; (5) $\frac{p}{\sigma_y} = \frac{p_y}{\sigma_y} = \frac{2}{\sqrt{3}}\ln k$.

The application conditions for curve oc are: (1) $k_j = k$; (2) $|\sigma'_{ei}| \leq \sigma_y$; (3) based on the maximum shear stress theory (Tresca yield criterion); (4) $k \leq k_c$; (5) $\frac{p}{\sigma_y} = \frac{p_y}{\sigma_y} = \ln k$.

The application conditions for curve cg are: (1) $k_j = k_{j*}$; (2) $\sigma'_{\theta i} = \sigma'_{ei} = -\sigma_y$; (3) based on the maximum shear stress theory (Tresca yield criterion); (4) $k \geq k_c$; (5) $\frac{p}{\sigma_y} = \frac{p_t}{\sigma_y} = \frac{k^2-1}{k^2}$.

The application conditions for curve bf are: (1) $k_j = k_{j*}$; (2) $\sigma'_{\theta i} = -\sigma_y$ and $-\sigma_y < \sigma'_{ei} < 0$; (3) based on the maximum distortion strain energy theory (Mises's yield criterion); (4) $k \geq k_c$; (5) $\frac{p}{\sigma_y} = \frac{p_m}{\sigma_y} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$.

Point a is the intersection of curve ad and curve oa. Setting $\frac{p_\theta}{\sigma_y} = \frac{2+\sqrt{3}}{2\sqrt{3}}\frac{k^2-1}{k^2} = \frac{p_y}{\sigma_y} = \frac{2}{\sqrt{3}}\ln k$ just results in $\frac{k^2 \ln k^2}{k^2-1} = \frac{2+\sqrt{3}}{2}$, that is equation (4.5). Therefore, the horizontal ordinate of point a is $k_{c\theta}$. When $k \leq k_{c\theta}$, curve oa works; When $k \geq k_{c\theta}$, curve ad works.

Point b is the intersection of curve bf and curve oab. Setting $\frac{p_{\mathrm{m}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}} \frac{k^2-1}{k^2} = \frac{p_{\mathrm{y}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}} \ln k$ just results in $\frac{k^2 \ln k^2}{k^2-1} = 1$, that is equation (2.28). Therefore, the horizontal ordinate of point b is k_{c}. When $k \leq k_{\mathrm{c}}$, curve oab works; When $k \geq k_{\mathrm{c}}$, curve bf works.

Point c is the intersection of curve cg and curve oc. Setting $\frac{p_{\mathrm{t}}}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{k^2} = \frac{p_{\mathrm{y}}}{\sigma_{\mathrm{y}}} = \ln k$ just results in the same equation $\frac{k^2 \ln k^2}{k^2-1} = 1$. Therefore, the horizontal ordinate of point c is also k_{c}. When $k \leq k_{\mathrm{c}}$, curve oc works; When $k \geq k_{\mathrm{c}}$, curve cg works.

Point s is the intersection of curve ad and the extension of curve oc. Setting $\frac{p_{\theta}}{\sigma_{\mathrm{y}}} = \frac{2+\sqrt{3}}{2\sqrt{3}} \frac{k^2-1}{k^2} = \frac{p_{\mathrm{y}}}{\sigma_{\mathrm{y}}} = \ln k$ obtains

$$\frac{k^2 \ln k^2}{k^2-1} = \frac{2+\sqrt{3}}{\sqrt{3}} \tag{4.44}$$

The solution of equation (4.44) is

$$k = 2.456797\ldots = k_{\mathrm{s}} \tag{4.45}$$

When $k = k_{\mathrm{s}}$, $\frac{p_{\theta}}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{y}}}{\sigma_{\mathrm{y}}} = 0.898858\ldots$. Therefore, the coordinate of point s is s (k_{s}, 0.898858).

Further, if $k_{\mathrm{j}} = k_{\mathrm{j}\theta}$, equations (4.38) and (4.39) become respectively:

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1 - \frac{\sqrt{3}+2}{2x^2} + \frac{\sqrt{3}k^2}{k^2-1} \frac{p/\sigma_{\mathrm{y}}}{x^2} \tag{4.46}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}}^2 - (\sqrt{3}+2)/2}{x^2} + \frac{\sqrt{3}k^2}{k^2-1} \frac{p/\sigma_{\mathrm{y}}}{x^2} \tag{4.47}$$

From equation (4.46), it is known that: (1) Provided $\frac{p}{\sigma_{\mathrm{y}}} \geq -\frac{4x^2-(\sqrt{3}+2)}{2\sqrt{3}} \frac{k^2-1}{k^2}$(negative), $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \geq -1$, this is definitely feasible for $p > 0$ in engineering; (2) As long as $\frac{p}{\sigma_{\mathrm{y}}} \geq \frac{(\sqrt{3}+2)-2x^2}{2\sqrt{3}} \frac{k^2-1}{k^2}$, $\sigma_{\mathrm{e}}^{\mathrm{T}} > 0$, while $\frac{(\sqrt{3}+2)-2x^2}{2\sqrt{3}} \frac{k^2-1}{k^2} < \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$, so when $p > p_{\mathrm{e}}$, $\sigma_{\mathrm{e}}^{\mathrm{T}} > 0$; (3) So long as $\frac{p}{\sigma_{\mathrm{y}}} \leq \frac{\sqrt{3}+2}{2\sqrt{3}} \frac{k^2-1}{k^2} = \frac{\sqrt{3}+2}{2} \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$, $\sigma_{\mathrm{e}}^{\mathrm{T}} \leq \sigma_{\mathrm{y}}$. When $x = 1$, if $\frac{p}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{2k^2} = \frac{\sqrt{3}}{2} \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$, then $\sigma_{\mathrm{ei}}^{\mathrm{T}} = 0$.

From equation (4.47), it is known that: (1) Provided $\frac{p}{\sigma_{\mathrm{y}}} > -\frac{k_{\mathrm{j}}^2-(\sqrt{3}+2)/2}{\sqrt{3}} \frac{k^2-1}{k^2}$(negative), $\sigma_{\mathrm{e}}^{\mathrm{T}} > 0$, this is definitely feasible for $p > 0$ in engineering, so the equivalent residual stress within the elastic zone is always tension; (2) As long as $\frac{p}{\sigma_{\mathrm{y}}} < \frac{x^2-k_{\mathrm{j}}^2+(\sqrt{3}+2)/2}{\sqrt{3}} \frac{k^2-1}{k^2}$, $\sigma_{\mathrm{e}}^{\mathrm{T}} < \sigma_{\mathrm{y}}$, so when $\frac{p}{\sigma_y} \leq \frac{\sqrt{3}+2}{2\sqrt{3}} \frac{k^2-1}{k^2} = \frac{\sqrt{3}+2}{2} \frac{p_{\mathrm{e}}}{\sigma_y}$, $\sigma_{\mathrm{e}}^{\mathrm{T}} \leq \sigma_{\mathrm{y}}$.

At inside surface, $x = r/r_{\mathrm{i}} = 1$, then, from equation (4.46):

$$\frac{\sigma_{\mathrm{ei}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sqrt{3}k^2}{k^2-1} \frac{p}{\sigma_{\mathrm{y}}} - \frac{\sqrt{3}}{2} \tag{4.48}$$

Unless $\frac{p}{\sigma_{\mathrm{y}}} < -\frac{2-\sqrt{3}}{2\sqrt{3}}\frac{k^2-1}{k^2}$(negative), $\sigma_{\mathrm{ei}}^{\mathrm{T}}$ cannot be lower than $-\sigma_{\mathrm{y}}$; Unless $p > 2p_{\mathrm{e}}$, $\sigma_{\mathrm{ei}}^{\mathrm{T}}$ cannot be higher than σ_{y}. So, when $0 \le \frac{p}{\sigma_{\mathrm{y}}} \le \frac{\sqrt{3}+2}{2}\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$, $-\sigma_{\mathrm{y}} \le \sigma_{\mathrm{ei}}^{\mathrm{T}} \le \sigma_{\mathrm{y}}$. Especially, when $\frac{p}{\sigma_{\mathrm{y}}} = \frac{\sqrt{3}+2}{2}\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$, $\sigma_{\mathrm{e}}^{\mathrm{T}} \equiv \sigma_{\mathrm{y}}$ within the whole plastic zone.

At the elastic-plastic juncture, $x = r/r_{\mathrm{i}} = k_{\mathrm{j}}$, from (4.46) or (4.47):

$$\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}}^2 - (\sqrt{3}+2)/2}{k_{\mathrm{j}}^2} + \frac{\sqrt{3}k^2}{k^2-1}\frac{p/\sigma_{\mathrm{y}}}{k_{\mathrm{j}}^2} \tag{4.49}$$

Clearly, $\sigma_{\mathrm{ej}}^{\mathrm{T}} > 0$ within the whole elastic zone. If $\frac{p}{\sigma_{\mathrm{y}}} \le \frac{\sqrt{3}+2}{2}\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$, $\sigma_{\mathrm{e}}^{\mathrm{T}}$ can not be higher than σ_{y}. So, when $0 \le \frac{p}{\sigma_{\mathrm{y}}} \le \frac{\sqrt{3}+2}{2}\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$, $0 < \sigma_{\mathrm{ej}}^{\mathrm{T}} \le \sigma_{\mathrm{y}}$. Especially, when $\frac{p}{\sigma_{\mathrm{y}}} = \frac{\sqrt{3}+2}{2}\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$, $\sigma_{\mathrm{ej}}^{\mathrm{T}} = \sigma_{\mathrm{y}}$ at the elastic-plastic juncture and $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}}^2}{x^2}$ at a general location within the elastic zone.

When $\frac{p}{\sigma_{\mathrm{y}}} = \frac{\sqrt{3}+2}{2}\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$, or the load-bearing capacity is determined by equations (4.42), (4.46) and (4.47) become respectively:

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \equiv 1 \tag{4.50}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}}^2}{x^2} \tag{4.51}$$

Figure 4.25 shows the distribution of the equivalent stress of total stress.

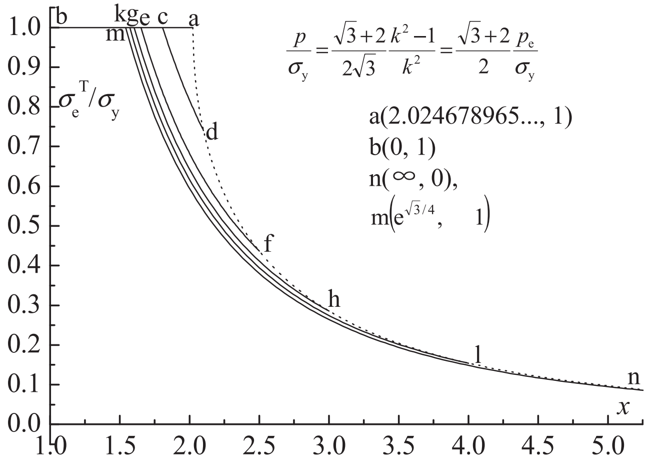


FIG. 4.25 – The distribution of the equivalent stress of total stress for $\frac{p}{\sigma_{\mathrm{y}}} = \frac{\sqrt{3}+2}{2}\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$ and $k_{\mathrm{j}} = k_{\mathrm{j}\theta}$.

The additional remarks for figure 4.25 are as follows.

(1) Horizontal line baa: $k = 2.024678965... = k_{\mathrm{c\theta}}$, $k_{\mathrm{j}} = k_{\mathrm{j\theta}} = k = k_{\mathrm{c\theta}}$. Within the plastic zone, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ is a horizontal line: $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1$, x varies from 1 to $k_{\mathrm{j\theta}}$ ($=k_{\mathrm{c\theta}} = k$) or from point b to point a, and then from 2.024678965... ($k_{\mathrm{j\theta}}$) to 2.024678965... (k) or from point a to point a within the elastic zone (no elastic zone, the "curve" of the equivalent total stress is actually a point within the elastic zone).

(2) Curve bcd: $k = 2.1...$, $k_{\mathrm{j\theta}} = 1.806908...$. Within the plastic zone, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ is a horizontal line: $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1$, x varies from 1 to $k_{\mathrm{j\theta}}$, or from point b to point c, and then from 1.806908... to 2.1..., or from point c to point d within the elastic zone.

(3) Curve bef: $k = 2.5...$, $k_{\mathrm{j\theta}} = 1.6522121...$. Within the plastic zone, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ is a horizontal line: $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1$, x varies from 1 to $k_{\mathrm{j\theta}}$, or from point b to point e, and then from 1.6522121... to 2.5..., or from point e to point f within the elastic zone.

(4) Curve bgh: $k = 3$, $k_{\mathrm{j\theta}} = 1.60350225...$. Within the plastic zone, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ is a horizontal line: $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1$, x varies from 1 to $k_{\mathrm{j\theta}}$, or from point b to point g, and then from 1.844363... to 3, or from point g to point h within the elastic zone.

(5) Curve bkl: $k = 4$, $k_{\mathrm{j\theta}} = 1.57121054...$. Within the plastic zone, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ is a horizontal line: $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1$, x varies from 1 to $k_{\mathrm{j\theta}}$, or from point b to point k, and then from 1.57121054... to 4, or from point k to point l within the elastic zone.

(6) Curve bmn: $k = \infty$, $k_{\mathrm{j\theta}} = \mathrm{e}^{\sqrt{3}/4}$. Within the plastic zone, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ is a horizontal line: $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1$, x varies from 1 to $k_{\mathrm{j\theta}}$, or from point b to point m, and then from $k_{\mathrm{j\theta}}$ to ∞, or from point m to point n within the elastic zone.

The prerequisite to the above arguments is that k and k_{j} meet equation (4.3) or $k_{\mathrm{j}} = k_{\mathrm{j\theta}}$ and $\frac{p}{\sigma_{\mathrm{y}}} = \frac{\sqrt{3}+2}{2\sqrt{3}}\frac{k^2-1}{k^2} = \frac{\sqrt{3}+2}{2}\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$. Grasping these laws is helpful to the design of high and ultrahigh pressure vessels. If $k_{\mathrm{j}} \neq k_{\mathrm{j\theta}}$ or $\frac{p}{\sigma_{\mathrm{y}}} \neq \frac{\sqrt{3}+2}{2\sqrt{3}}\frac{k^2-1}{k^2}$, the above facts are untenable, and $|\sigma_{\mathrm{e}}^{\mathrm{T}}|$ may exceed σ_{y}.

When $\frac{p}{\sigma_{\mathrm{y}}} = \frac{\sqrt{3}+2}{2}\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$, the stresses caused by operation pressure are as follows:

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{\sqrt{3}+2}{2\sqrt{3}k^2} \tag{4.52}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}\left(1 - \frac{k^2}{x^2}\right) = \frac{\sqrt{3}+2}{2\sqrt{3}k^2} - \frac{\sqrt{3}+2}{2\sqrt{3}x^2} \tag{4.53}$$

$$\frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}\left(1 + \frac{k^2}{x^2}\right) = \frac{\sqrt{3}+2}{2\sqrt{3}k^2} + \frac{\sqrt{3}+2}{2\sqrt{3}x^2} \tag{4.54}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}=\frac{\sqrt{3}}{2}\left(\frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}-\frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}\right)=\frac{\sqrt{3}+2}{2x^2} \tag{4.55}$$

When $\frac{p}{\sigma_{\mathrm{y}}}=\frac{\sqrt{3}+2}{2}\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$ and $k_{\mathrm{j}}=k_{\mathrm{j}\theta}$, the components of the total stresses are:
Within the plastic zone:

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\sigma_{\mathrm{z}}'}{\sigma_{\mathrm{y}}}+\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}=\frac{\ln x^2}{\sqrt{3}}-\frac{1}{2}+\frac{\sqrt{3}+2}{2\sqrt{3}k^2} \tag{4.56}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\ln x^2}{\sqrt{3}}+\frac{\sqrt{3}+2}{2\sqrt{3}k^2}-\frac{\sqrt{3}+2}{2\sqrt{3}} \tag{4.57}$$

$$\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\ln x^2}{\sqrt{3}}+\frac{\sqrt{3}+2}{2\sqrt{3}k^2}+\frac{2-\sqrt{3}}{2\sqrt{3}} \tag{4.58}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\equiv 1 \tag{4.59}$$

Letting $\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\ln x^2}{\sqrt{3}}+\frac{\sqrt{3}+2}{2\sqrt{3}k^2}+\frac{2-\sqrt{3}}{2\sqrt{3}}\geq 1$ gives

$$x\geq\sqrt{\exp\left(\sqrt{3}-\frac{2+\sqrt{3}}{2k^2}-\frac{2-\sqrt{3}}{2}\right)}=x_{\mathrm{z}\theta} \tag{4.60}$$

From equations (4.57) and (4.58), $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\sqrt{3}}{2}\left(\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}-\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\right)\equiv 1$, the same as equation (4.50).

Within the elastic zone:

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\sigma_{\mathrm{z}}'}{\sigma_{\mathrm{y}}}+\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}=\frac{k_{\mathrm{j}\theta}^2}{\sqrt{3}k^2} \tag{4.61}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\sigma_{\mathrm{r}}'}{\sigma_{\mathrm{y}}}+\frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}=\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\left(1-\frac{k^2}{x^2}\right)=-\frac{k_{\mathrm{j}\theta}^2}{\sqrt{3}}\left(\frac{1}{x^2}-\frac{1}{k^2}\right) \tag{4.62}$$

$$\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\sigma_{\theta}'}{\sigma_{\mathrm{y}}}+\frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}=\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\left(1-\frac{k^2}{x^2}\right)=\frac{k_{\mathrm{j}\theta}^2}{\sqrt{3}}\left(\frac{1}{x^2}+\frac{1}{k^2}\right) \tag{4.63}$$

From equations (4.62) and (4.63), $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{k_{\mathrm{j}}^2}{x^2}$, the same as equation (4.51).

Figure 4.26 is a comparison between the equivalent stresses of total stresses under different internal pressure and $k_{\mathrm{j}}=k_{\mathrm{j}\theta}$, from which it is known that only when

$\frac{p}{\sigma_y} = \frac{\sqrt{3}+2}{2\sqrt{3}}\frac{k^2-1}{k^2} = \frac{\sqrt{3}+2}{2}\frac{p_e}{\sigma_y}$ and $k_j = k_{j\theta}$, the operation state is optimum. Otherwise, if $\frac{p}{\sigma_y} \neq \frac{\sqrt{3}+2}{2}\frac{p_e}{\sigma_y}$ and/or $k_j \neq k_{j\theta}$, either $\sigma_e^T > \sigma_y$ or the load-bearing capacity is lowered. In figure 4.26, curve 1 is just curve bgh in figure 4.25, in this case, $\frac{p}{\sigma_y} = \frac{\sqrt{3}+2}{2}\frac{p_e}{\sigma_y}$, $\sigma_e^T \equiv \sigma_y$ within the whole plastic zone; Curve 2: $\frac{p}{\sigma_y} = \frac{\sqrt{3}+2.1}{2}\frac{p_e}{\sigma_y}$, $\sigma_{ei}^T > \sigma_y$; Curve 3: $\frac{p}{\sigma_y} = \frac{\sqrt{3}+1.9}{2}\frac{p_e}{\sigma_y}$, $0 < \sigma_{ei}^T < \sigma_y$; Curve 4: $p = p_e$, $0 < \sigma_{ei}^T \ll \sigma_y$; Curve 5: $\frac{p}{\sigma_y} = \frac{k^2-1}{2k^2} = \frac{\sqrt{3}}{2}\frac{p_e}{\sigma_y}$, $\sigma_{ei}^T/\sigma_y = 0$; Curve 6: $p = 0.8p_e$, $\sigma_{ei}^T/\sigma_y < 0$.

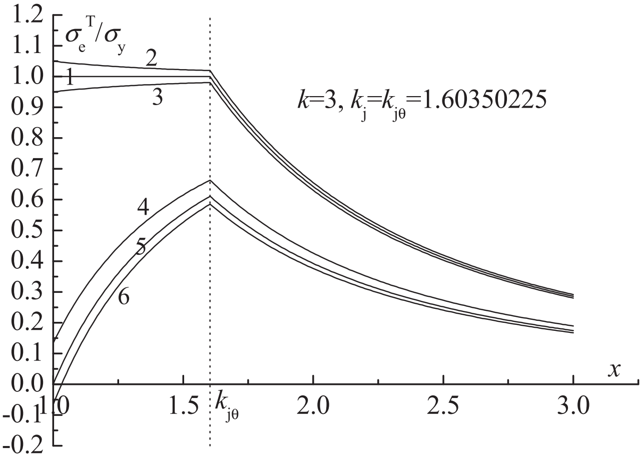


FIG. 4.26 – Comparison between the equivalent total stresses under different p and $k_j = k_{j\theta}$.

Besides, for certain k, when $k_j < k_{j\theta}$, though the residual stresses are smaller than those when $k_j = k_{j\theta}$, the load-bearing capacity is dropped. For example, for $k = 3$, if $k_j = k_{j\theta}$ (=1.60350225…), from equation (4.41) or (4.42), $\frac{p}{\sigma_y} = 0.9576…$; while if $k_j = 1.5$, from equation (4.41), $\frac{p}{\sigma_y} = 0.9012… <0.9576…$.

According to equations (4.58) and (4.63), at the elastic-plastic juncture, $\frac{\sigma_\theta^T}{\sigma_y}$ is maximum.

$$\frac{\sigma_{\theta j}^T}{\sigma_y} = \frac{k^2 + k_j^2}{\sqrt{3}k^2} \tag{4.64}$$

Letting $\frac{\sigma_{\theta j}^T}{\sigma_y} \leq 1$ gives $k_j \leq \sqrt{\sqrt{3}-1}k = x_a$.

As is stated above, substituting $k_j = \sqrt{\sqrt{3}-1}k$ into $2k^2 \ln k_{j\theta}^2 - 2k_{j\theta}^2 - \sqrt{3}k^2 + (2+\sqrt{3}) = 0$ results in $k = 2.105406… = k_h$. Therefore, when $k_j = k_{j\theta}$, if $k \geq k_h$, $\frac{\sigma_{\theta j}^T}{\sigma_y} \leq 1$; if $k \leq k_h$, $\frac{\sigma_{\theta j}^T}{\sigma_y} \geq 1$. It can be seen that $\frac{\sigma_{\theta j}^T}{\sigma_y} \leq 1$ is equivalent to $\frac{\sigma_{ej}^T}{\sigma_y} \geq \frac{\sigma_{\theta j}^T}{\sigma_y}$. $k_j = \sqrt{\sqrt{3}-1}k$ is plotted in figure 4.3 as the dash straight line qwv, where point w is the intersection of curve bwmnc and the straight line qwv, the coordinate of point w is w $(k_{\mathrm{h}}, k_{\mathrm{j\theta}}) = \mathrm{w}\ (k_{\mathrm{h}}, \sqrt{\sqrt{3}-1}k) = \mathrm{w}\ (k_{\mathrm{h}}, 1.801385)$.

Figure 4.27 shows the total stresses in three directions for $k = 3$ and $k_{\mathrm{j\theta}} = 1.603502$.

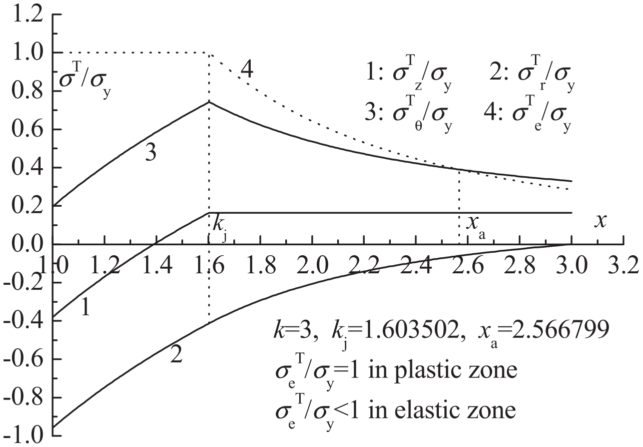


FIG. 4.27 – The total stresses for $k = 3$, $k_{\mathrm{j\theta}} = 1.603502$ and $p = p_\theta$.

Figure 4.28 shows the total stresses in three directions for $k = 2.05$ and $k_{\mathrm{j\theta}} = 1.882849$.

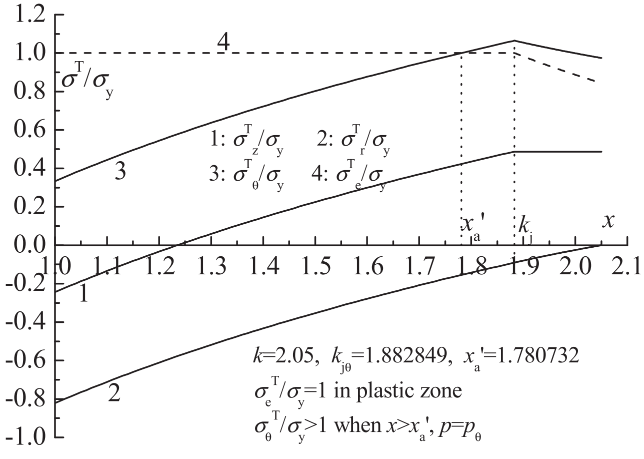


FIG. 4.28 – The total stresses for $k = 2.05$, $k_{\mathrm{j\theta}} = 1.882849$ and $p = p_\theta$.

Figure 4.29 shows the total stresses in three directions for $k = k_{r\theta} = 2 + \sqrt{3} = 3.732051$ and $k_{j\theta} = 1.576574$. In this case, $p = p_\theta = \sigma_y$ and $\sigma_r^T = -\sigma_y$.

Figure 4.30 shows the total stresses in three directions for $k = 4 > k_{r\theta}$ and $k_{j\theta} = 1.571211$. In this case, $p = p_\theta > \sigma_y$ and $\sigma_r^T < -\sigma_y$. In figure 4.27, $k = 3 < k_{r\theta}$, so $p = p_\theta < \sigma_y$ and $\sigma_r^T > -\sigma_y$.

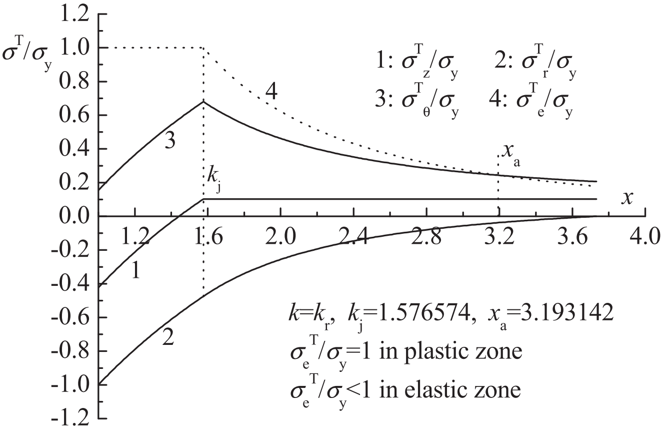


FIG. 4.29 – The total stresses for $k = k_{r\theta}$, $k_{j\theta} = 1.576574$ and $p = p_\theta = \sigma_y$.

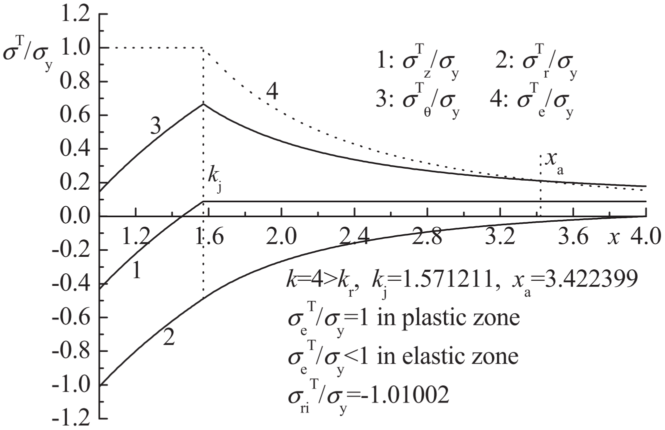


FIG. 4.30 – The total stresses for $k = 4 > k_{r\theta}$, $k_{j\theta} = 1.571211$ and $p = p_\theta > \sigma_y$.

Figure 4.31 shows the total stresses in three directions for $k = k_{f\theta} = 2.105406$ and $k_{j\theta} = 1.801385$. In this case, $k_{j\theta} = \sqrt{\sqrt{3} - 1}k = x_a$ and $\frac{\sigma_{ej}^T}{\sigma_y} = \frac{\sigma_{\theta j}^T}{\sigma_y}$; When $x < x_a$, $\frac{\sigma_e^T}{\sigma_y} > \frac{\sigma_\theta^T}{\sigma_y}$; When $x > x_a$, $\frac{\sigma_e^T}{\sigma_y} < \frac{\sigma_\theta^T}{\sigma_y}$; When $x = x_a = k_{j\theta}$, $\frac{\sigma_{ej}^T}{\sigma_y} = \frac{\sigma_{\theta j}^T}{\sigma_y}$.

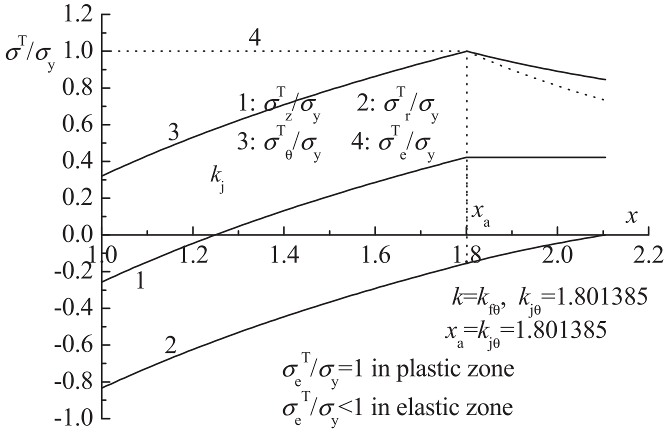


FIG. 4.31 – The total stresses for $k = k_{\mathrm{f\theta}}$, $k_{\mathrm{j\theta}} = 1.801385$ and $p = p_{\theta}$.

4.4 Control Circumferential Total Stress Directly

Circumferential total stress $\sigma_{\theta}^{\mathrm{T}}$ is expressed as

$$\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\theta}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}\left(\ln x^2 - \frac{2+\sqrt{3}}{2x^2} + \frac{2-\sqrt{3}}{2}\right) + \left(1 + \frac{k^2}{x^2}\right)\frac{1}{k^2-1}\frac{p}{\sigma_{\mathrm{y}}} \tag{4.65}$$

At the elastic-plastic juncture, $\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$ is maximum.

$$\frac{\sigma_{\theta\mathrm{j}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}\left(\ln k_{\mathrm{j}}^2 - \frac{2+\sqrt{3}}{2k_{\mathrm{j}}^2} + \frac{2-\sqrt{3}}{2}\right) + \left(1 + \frac{k^2}{k_{\mathrm{j}}^2}\right)\frac{1}{k^2-1}\frac{p}{\sigma_{\mathrm{y}}} \tag{4.66}$$

Setting $\frac{\sigma_{\theta\mathrm{j}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \leq 1$ results in

$$\frac{p}{\sigma_{\mathrm{y}}} \leq \frac{k_{\mathrm{j}}^2(k^2-1)}{k^2+k_{\mathrm{j}}^2} - \frac{2k_{\mathrm{j}}^2 \ln k_{\mathrm{j}}^2 - (2+\sqrt{3}) + (2-\sqrt{3})k_{\mathrm{j}}^2}{2\sqrt{3}}\frac{k^2-1}{k^2+k_{\mathrm{j}}^2} = \frac{p_{\theta}'}{\sigma_{\mathrm{y}}} \tag{4.67}$$

In equation (4.67), k_{j} and k are related by equation (4.3), or $k_{\mathrm{j}} = k_{\mathrm{j\theta}}$.

For the above $k = 2.05$ and $k_{\mathrm{j\theta}} = 1.882849$, $\frac{p_{\theta}'}{\sigma_{\mathrm{y}}} = 0.726638\ldots$, $\frac{p_{\theta}}{\sigma_{\mathrm{y}}} = \frac{2+\sqrt{3}}{2\sqrt{3}}\frac{k^2-1}{k^2} = 0.820991\ldots > \frac{p_{\theta}'}{\sigma_{\mathrm{y}}}$.

For the above $k = 2.05$ and $k_{\mathrm{j\theta}} = 1.882849$ and $\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\theta}'}{\sigma_{\mathrm{y}}} = 0.726638$, according to $\frac{\sigma^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'}{\sigma_{\mathrm{y}}} + \frac{\sigma^{\mathrm{p}}}{\sigma_{\mathrm{y}}}$, the total stresses are shown in figure 4.32, from which it is seen that $\sigma_{\theta}^{\mathrm{T}}$ is reduced below σ_{y} within the whole wall and $\sigma_{\theta\mathrm{j}}^{\mathrm{T}} = \sigma_{\mathrm{y}}$ at $k_{\mathrm{j\theta}}$, $\sigma_{\mathrm{e}}^{\mathrm{T}} < \sigma_{\mathrm{y}}$ within the whole wall.

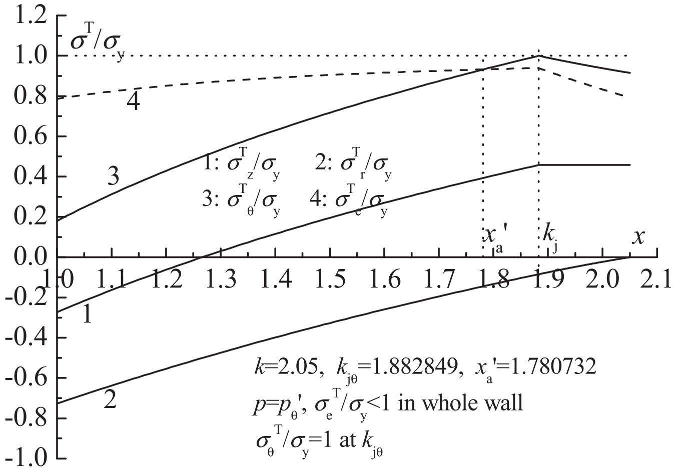


FIG. 4.32 – The total stresses for $k = 2.05$, $k_{j\theta} = 1.882849$ and $p = p'_\theta$.

In general, using equation (4.3), $\frac{p'_\theta}{\sigma_y}$ is plotted in figure 4.33 along with $\frac{p_\theta}{\sigma_y} = \frac{2+\sqrt{3}}{2\sqrt{3}}\frac{k^2-1}{k^2}$.

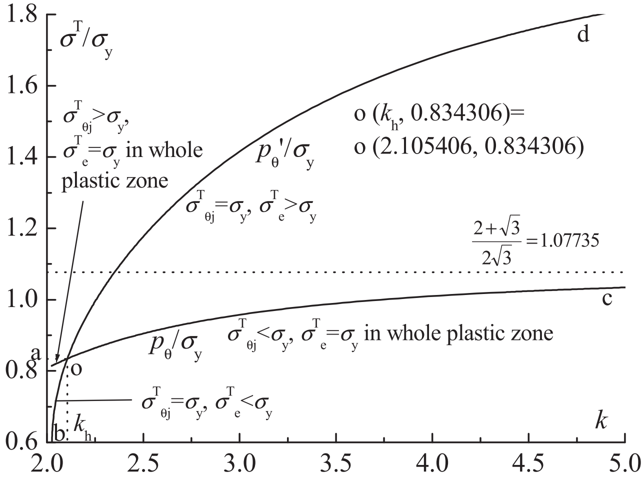


FIG. 4.33 – $\frac{p'_\theta}{\sigma_y}$ and $\frac{p_\theta}{\sigma_y}$.

In figure 4.33, curve aoc is for $\frac{p_\theta}{\sigma_y} = \frac{2+\sqrt{3}}{2\sqrt{3}}\frac{k^2-1}{k^2}$, the limit of $\frac{p_\theta}{\sigma_y}$ is $\frac{2+\sqrt{3}}{2\sqrt{3}} = 1.07735$; curve bod is for $\frac{p'_\theta}{\sigma_y}$; curve aoc ensures $\sigma^T_e \equiv \sigma_y$ within the whole wall, curve bod ensures $\sigma^T_{\theta j} = \sigma_y$; point o is the intersection of curve aoc and curve bod, the coordinate of point o is o $(k_{\rm h}, 0.834306) = $ o $(2.105406, 0.834306)$; when $k \leq k_{\rm h}$, $p'_\theta \leq p_\theta$, when $k \geq k_{\rm h}$, $p'_\theta \geq p_\theta$; on section oa, $\sigma_{\rm e}^{\rm T} \equiv \sigma_{\rm y}$ and $\sigma_{\theta{\rm j}}^{\rm T} > \sigma_{\rm y}$; on section oc, $\sigma_{\rm e}^{\rm T} \equiv \sigma_{\rm y}$ and $\sigma_{\theta{\rm j}}^{\rm T} < \sigma_{\rm y}$; on section ob, $\sigma_{\rm e}^{\rm T} < \sigma_{\rm y}$ and $\sigma_{\theta{\rm j}}^{\rm T} = \sigma_{\rm y}$; on section od, $\sigma_{\rm e}^{\rm T} \gg \sigma_{\rm y}$ because p'_θ is very great and $\sigma_{\theta{\rm j}}^{\rm T} = \sigma_{\rm y}$; sections oa and ob are quite short.

For $k = 3$ $(k > k_{\rm h})$ and $k_{{\rm j}\theta} = 1.603502$, $\frac{p'_\theta}{\sigma_{\rm y}} = 1.41576$, the total stresses are shown in figure 4.34. The meaning of each curve and the main parameters are marked in the figure.

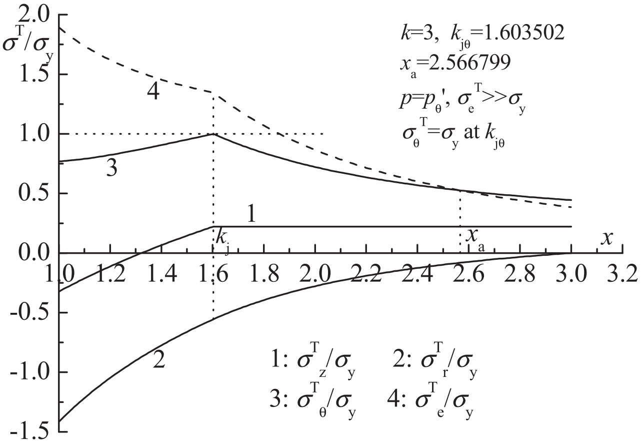


FIG. 4.34 – The total stresses for $k = 3$ $k > k_{\rm h}$, $k_{{\rm j}\theta} = 1.603502$ and $\frac{p'_\theta}{\sigma_{\rm y}} = 1.41576$.

When $k > k_{\rm h}$ and $\frac{\sigma_{\theta{\rm j}}^{\rm T}}{\sigma_{\rm y}} = 1$, the equivalent stress is quite great and the distribution of the total equivalent is unfavorable. Therefore, it is reasonable to limit equivalent stress.

4.5 Chapter Summary

(1) For $|\sigma'_{\rm ei}| \leq \sigma_{\rm y}$, the depth of plastic zone is $k^2 \ln k_{{\rm j}*}^2 - k^2 - k_{{\rm j}*}^2 + 2 = 0$ where ${\rm e}^{0.5} \leq k_{{\rm j}*} \leq k_{\rm c} = 2.2184574899167\ldots$, when $k \geq 2.2184574899167\ldots$; load-bearing capacity is $\frac{p_4}{\sigma_{\rm y}} = \frac{2}{\sqrt{3}} \frac{k^2-1}{k^2} = 2\frac{p_{\rm e}}{\sigma_{\rm y}}$.

(2) For $|\sigma'_{\theta{\rm i}}| \leq \sigma_{\rm y}$, the depth of plastic zone is $2k^2 \ln k_{{\rm j}\theta}^2 - 2k_{{\rm j}\theta}^2 - \sqrt{3}k^2 + (2+\sqrt{3}) = 0$ where ${\rm e}^{\sqrt{3}/4} \leq k_{{\rm j}\theta} \leq k_{{\rm c}\theta} = 2.024678965\ldots$, when $k \geq 2.024678965\ldots$; load-bearing capacity is $\frac{p}{\sigma_{\rm y}} = \frac{\sqrt{3}+2}{2\sqrt{3}} \frac{k^2-1}{k^2} = \frac{\sqrt{3}+2}{2} \frac{p_{\rm e}}{\sigma_{\rm y}}$. The data 2.024678965... is the solution of $\frac{k^2 \ln k^2}{k^2-1} = \frac{2+\sqrt{3}}{2}$.

(3) For an autofrettaged cylinder, $k_{\rm j} = k_{{\rm j}\theta}$ with $\frac{p}{\sigma_{\rm y}} = \frac{\sqrt{3}+2}{2\sqrt{3}} \frac{k^2-1}{k^2} = \frac{\sqrt{3}+2}{2} \frac{p_{\rm e}}{\sigma_{\rm y}}$ is the optimum operation state—not only safe but also highly capable, under the

state, $\sigma_{\mathrm{e}}^{\mathrm{T}} \equiv \sigma_{\mathrm{y}}$ within the plastic zone and $\sigma_{\mathrm{e}}^{\mathrm{T}}/\sigma_{\mathrm{y}} = k_{\mathrm{j}}^2/x^2 < 1$ within the elastic zone.

(4) For the same k, $k_{\mathrm{j}\theta} < k_{\mathrm{j}*}$, this means the depth of plastic zone to ensure $|\sigma'_{\theta\mathrm{i}}| \leq \sigma_{\mathrm{y}}$ is less than that to ensure $|\sigma'_{\mathrm{ei}}| \leq \sigma_{\mathrm{y}}$.

(5) Irrespective of k, the three curves for residual stress at a general radial location collect at the same point within the plastic zone and the intersection is $\left(\sqrt{\frac{\sqrt{3}+2}{2}},\ \frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}+2}{2}-\frac{1}{2}\right) = (1.931852\ldots,\ \ -0.13984\ldots)$ when $k_{\mathrm{j}} = k_{\mathrm{j}\theta}$, or $\left(\sqrt{2}, (\ln 2 - 1)/\sqrt{3}\right) = (1.414214\ldots, -0.17716\ldots)$ when $k_{\mathrm{j}} = k_{\mathrm{j}*}$.

(6) When $\sigma'_{\theta\mathrm{i}}$ controlled, or $k_{\mathrm{j}} = k_{\mathrm{j}\theta}$, the residual stresses, and their equivalent stress are determined only by the relative location (r/r_{i}) within the plastic zone and independent of k_{j} and k, or once the location, r/r_{i} is determined, the residual stresses and their equivalent stress within the plastic zone are identical for any k_{j} and k.

The main equations and conclusion are listed in table 4.3.

Tab. 4.3 – The main equations and conclusion of this chapter.

The critical radius ratio
When the plastic depth covers the whole wall, for $\lvert\sigma'_{\mathrm{ei}}\rvert \leq \sigma_{\mathrm{y}}$, $k \leq k_{\mathrm{c}\theta}$
$k_{\mathrm{c}\theta} = 2.024678965\ldots < k_{\mathrm{c}} = 2.218\,457\,489\,916\,7\ldots$
$k_{\mathrm{c}\theta}$ is the solution of $\dfrac{k^2\ln k}{k^2-1} = \dfrac{2+\sqrt{3}}{4}$,
and $\dfrac{k^2\ln k}{k^2-1} = \dfrac{2+\sqrt{3}}{4}$ is equivalent to $\dfrac{2}{\sqrt{3}}\ln k = \dfrac{2+\sqrt{3}}{2}\dfrac{k^2-1}{\sqrt{3}k^2}$, *i.e.*
or $\dfrac{p}{\sigma_{\mathrm{y}}} = \dfrac{p_\theta}{\sigma_{\mathrm{y}}} = \dfrac{\sqrt{3}+2}{2\sqrt{3}}\dfrac{k^2-1}{k^2} = \dfrac{p_{\mathrm{y}}}{\sigma_{\mathrm{y}}} = \dfrac{2}{\sqrt{3}}\ln k$ or $p_{\mathrm{y}} = \dfrac{2+\sqrt{3}}{2}p_{\mathrm{e}}$
If $\sqrt{3}$ in the molecule replaced with 2, then
$\dfrac{k^2\ln k}{k^2-1} = \dfrac{2+\sqrt{3}}{4}$ becomes $\dfrac{k^2\ln k}{k^2-1} = 1$;
$\dfrac{2}{\sqrt{3}}\ln k = \dfrac{2+\sqrt{3}}{2}\dfrac{k^2-1}{\sqrt{3}k^2}$ becomes $\dfrac{2}{\sqrt{3}}\ln k = 2\dfrac{k^2-1}{\sqrt{3}k^2}$;
$\dfrac{p}{\sigma_{\mathrm{y}}} = \dfrac{p_\theta}{\sigma_{\mathrm{y}}} = \dfrac{\sqrt{3}+2}{2\sqrt{3}}\dfrac{k^2-1}{k^2} = \dfrac{p_{\mathrm{y}}}{\sigma_{\mathrm{y}}} = \dfrac{2}{\sqrt{3}}\ln k$ becomes $\dfrac{p}{\sigma_{\mathrm{y}}} = \dfrac{2}{\sqrt{3}}\dfrac{k^2-1}{k^2} = \dfrac{p_{\mathrm{y}}}{\sigma_{\mathrm{y}}} = \dfrac{2}{\sqrt{3}}\ln k$;
$p_{\mathrm{y}} = \dfrac{2+\sqrt{3}}{2}p_{\mathrm{e}}$ becomes $p_{\mathrm{y}} = 2p_{\mathrm{e}}$
The optimum plastic depth $k_{\mathrm{j}\theta}$
$2k^2\ln k_{\mathrm{j}\theta}^2 - 2k_{\mathrm{j}\theta}^2 - \sqrt{3}k^2 + (2+\sqrt{3}) \leq 0$ $(\mathrm{e}^{\sqrt{3}/4} \leq k_{\mathrm{j}\theta} \leq k_{\mathrm{c}\theta},\ k \geq k_{\mathrm{c}\theta})$
Replacing $\sqrt{3}$ with 2 results in $k^2\ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 = 0$ ($\mathrm{e}^{0.5} \leq k_{\mathrm{j}*} \leq k_{\mathrm{c}}$ and $k \geq k_{\mathrm{c}}$)
For the same k, $k_{\mathrm{j}*} > k_{\mathrm{j}\theta}$, or to avoid $\sigma'_{\theta\mathrm{i}} \leq -\sigma_{\mathrm{y}}$, $k_{\mathrm{j}\theta}$ should be smaller than $k_{\mathrm{j}*}$
When $k \leq k_{\mathrm{c}}$, $k_{\mathrm{j}} = k$

TAB. 4.3 – (continued).

The results when $k_\mathrm{j} = k_{\mathrm{j}\theta}$

1. Residual stresses in axial, radial, and circumferential direction

$$\begin{cases} \dfrac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} = \dfrac{1}{\sqrt{3}}\ln x^2 - \dfrac{1}{2} = \dfrac{1}{\sqrt{3}}\left(\ln x^2 - \dfrac{\sqrt{3}}{2}\right) \\ \dfrac{\sigma'_\mathrm{r}}{\sigma_\mathrm{y}} = \dfrac{1}{\sqrt{3}}\left(\ln x^2 + \dfrac{2+\sqrt{3}}{2x^2} - \dfrac{2+\sqrt{3}}{2}\right) \\ \dfrac{\sigma'_\theta}{\sigma_\mathrm{y}} = \dfrac{1}{\sqrt{3}}\left(\ln x^2 - \dfrac{2+\sqrt{3}}{2x^2} + \dfrac{2-\sqrt{3}}{2}\right) \end{cases} (1 \le x \le k_{\mathrm{j}\theta}) \quad \begin{cases} \dfrac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} = \dfrac{k_\mathrm{j}^2 - \frac{2+\sqrt{3}}{2}}{\sqrt{3}k^2} \\ \dfrac{\sigma'_\mathrm{r}}{\sigma_\mathrm{y}} = \left(1 - \dfrac{k^2}{x^2}\right)\dfrac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} \\ \dfrac{\sigma'_\theta}{\sigma_\mathrm{y}} = \left(1 + \dfrac{k^2}{x^2}\right)\dfrac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} \end{cases} (k_{\mathrm{j}\theta} \le x \le k)$$

Replacing $\sqrt{3}$ in the molecule with 2 obtains the results when $k_\mathrm{j} = k_{\mathrm{j}*}$

2. Equivalent residual stress

$$\frac{\sigma'_\mathrm{e}}{\sigma_\mathrm{y}} = 1 - \frac{\sqrt{3}+2}{2(r/r_\mathrm{i})^2} = 1 - \frac{\sqrt{3}+2}{2x^2}\ (1 \le x \le k_{\mathrm{j}\theta}) \quad \frac{\sigma'_\mathrm{e}}{\sigma_\mathrm{y}} = \frac{k_\mathrm{j}^2 - (\sqrt{3}+2)/2}{x^2}\ (k_{\mathrm{j}\theta} \le x \le k)$$

Replacing $\sqrt{3}$ with 2 obtains the results when $k_\mathrm{j} = k_{\mathrm{j}*}$

3. The optimum load-bearing capability and autofrettage pressure

$$\frac{p}{\sigma_\mathrm{y}} = \frac{p_\theta}{\sigma_\mathrm{y}} = \frac{p_\mathrm{a}}{\sigma_\mathrm{y}} = \frac{2+\sqrt{3}}{2\sqrt{3}}\frac{k^2-1}{k^2} = \frac{2+\sqrt{3}}{2}\frac{p_\mathrm{e}}{\sigma_\mathrm{y}}$$

Replacing $\sqrt{3}$ in the molecule with 2 obtains $\dfrac{p}{\sigma_\mathrm{y}} = \dfrac{p_\mathrm{a}}{\sigma_\mathrm{y}} = 2\dfrac{p_\mathrm{e}}{\sigma_\mathrm{y}} = \dfrac{2}{\sqrt{3}}\dfrac{k^2-1}{k^2}$

When $\sqrt{3}$ in the molecule replaced with 2, then,

$$\frac{p_\theta}{\sigma_\mathrm{y}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2} = \frac{p_\mathrm{m}}{\sigma_\mathrm{y}}$$

which is the optimum load-bearing capacity based on the maximum distortion strain energy theory and $k_\mathrm{j} = k_{\mathrm{j}*}$

When based on the maximum shear stress theory and $k_\mathrm{j} = k_{\mathrm{j}*}$, the optimum load-bearing capacity is

$$\frac{p_\mathrm{t}}{\sigma_\mathrm{y}} = \frac{k^2-1}{k^2}$$

$$p_\theta = \frac{p_\mathrm{m} + p_\mathrm{t}}{2}$$

When $\dfrac{k^2\ln k^2}{k^2-1} = \dfrac{2+\sqrt{3}}{2}$, or $k = k_{\mathrm{c}\theta}$, $p_\theta = p_\mathrm{y}$

4. $x_{0\theta} \equiv \sqrt{\dfrac{\sqrt{3}+2}{2}}$ = a constant

Replacing $\sqrt{3}$ in the molecule with 2 results in $x_0 \equiv \sqrt{2}$

5. $x_{1\theta} = \mathrm{e}^{\frac{\sqrt{3}}{4}} = k_{\mathrm{j}\theta\infty}$ = constant

Replacing $\sqrt{3}$ with 2 results in $x_1 \equiv \sqrt{\mathrm{e}}$

6. $y_0^\mathrm{d} = \dfrac{\sigma'_\mathrm{z}}{\sigma_\mathrm{y}} = \dfrac{\sigma'_\mathrm{r}}{\sigma_\mathrm{y}} = \dfrac{\sigma'_\theta}{\sigma_\mathrm{y}} = \dfrac{\sigma'_\mathrm{rmin}}{\sigma_\mathrm{y}} = \dfrac{\sigma'_\mathrm{rx0}}{\sigma_\mathrm{y}} \equiv \dfrac{1}{\sqrt{3}}\ln\dfrac{\sqrt{3}+2}{2} - \dfrac{1}{2} = \dfrac{1}{\sqrt{3}}\left(\ln\dfrac{\sqrt{3}+2}{2} - \dfrac{\sqrt{3}}{2}\right)$ = a constant

Replacing $\sqrt{3}$ in the molecule with 2 obtains $y_0^\mathrm{d} = -\dfrac{1-\ln 2}{\sqrt{3}}$, the result when $k_\mathrm{j} = k_{\mathrm{j}*}$

Tab. 4.3 – (continued).

The results when $k_{\mathrm{j}} = k_{\mathrm{j}\theta}$, $p = p_\theta$

1. The components of total stresses

$$\left.\begin{aligned}
\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} &= \frac{\ln x^2}{\sqrt{3}} - \frac{1}{2} + \frac{\sqrt{3}+2}{2\sqrt{3}k^2} = \frac{1}{\sqrt{3}}\left(\ln x^2 - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}+2}{2k^2}\right)\\
\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} &= \frac{\ln x^2}{\sqrt{3}} + \frac{\sqrt{3}+2}{2\sqrt{3}k^2} - \frac{\sqrt{3}+2}{2\sqrt{3}}\\
\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} &= \frac{\ln x^2}{\sqrt{3}} + \frac{\sqrt{3}+2}{2\sqrt{3}k^2} + \frac{2-\sqrt{3}}{2\sqrt{3}}
\end{aligned}\right\} \quad (1 \le x \le k_{\mathrm{j}\theta})$$

$$\left.\begin{aligned}
\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} &= \frac{k_{\mathrm{j}}^2}{\sqrt{3}k^2} = \text{constant}\\
\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} &= \frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\left(1 - \frac{k^2}{x^2}\right) = -\frac{k_{\mathrm{j}}^2}{\sqrt{3}}\left(\frac{1}{x^2} - \frac{1}{k^2}\right)\\
\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} &= \frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\left(1 + \frac{k^2}{x^2}\right) = \frac{k_{\mathrm{j}}^2}{\sqrt{3}}\left(\frac{1}{x^2} + \frac{1}{k^2}\right)
\end{aligned}\right\} \quad (k_{\mathrm{j}\theta} \le x \le k)$$

Replacing $\sqrt{3}$ in the molecule with 2 obtains the results when $k_{\mathrm{j}} = k_{\mathrm{j}*}$

2. Equivalent residual stress

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \equiv 1 \ (1 \le x \le k_{\mathrm{j}\theta}) \qquad \frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}}^2}{x^2} \ (k_{\mathrm{j}\theta} \le x \le k)$$

(1) When $k \le k_{\mathrm{r}\theta} = 2+\sqrt{3} = 3.732051$, $p \le \sigma_{\mathrm{y}}$, hence $\sigma_{\mathrm{ri}}^{\mathrm{T}} > -\sigma_{\mathrm{y}}$

(2) When $k \ge k_{\mathrm{f}\theta} = 2.105406\ldots$, $k_{\mathrm{j}\theta} < \sqrt{\sqrt{3}-1}k(=x_{\mathrm{a}})$, accordingly $\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \ge \frac{\sigma_{\theta\mathrm{j}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$; when $k < k_{\mathrm{f}\theta}$, $k_{\mathrm{j}\theta} > \sqrt{\sqrt{3}-1}k$, accordingly $\frac{\sigma_{\mathrm{ej}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} \le \frac{\sigma_{\theta\mathrm{j}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$

(3) When $x \ge \sqrt{\exp\left(\sqrt{3} - \frac{2+\sqrt{3}}{2k^2} - \frac{2-\sqrt{3}}{2}\right)} = x_{\mathrm{z}\theta}$, $\sigma_{\mathrm{e}}^{\mathrm{T}} \le \sigma_{\theta}^{\mathrm{T}}$

(4) When $k_{\mathrm{j}} = k_{\mathrm{j}\theta}$, if $2k^2\ln[(\sqrt{3}-1)k^2] - (3\sqrt{3}-2)k^2 + 2 + \sqrt{3} = 0$ or $k = k_{\mathrm{f}\theta}$, $x_{\mathrm{z}\theta} = x_{\mathrm{a}} = k_{\mathrm{j}\theta} = 1.801385\ldots$

(5) When $k_{\mathrm{j}} < \sqrt{\sqrt{3}-1}k(=x_{\mathrm{a}})$, x_{a} exists and x_{z} does not exist; When $k_{\mathrm{j}} > x_{\mathrm{a}}$, x_{a} does not exists and x_{z} exist

---

The results when $k_{\mathrm{j}} = \frac{k+1}{2}$

1. When $k \le k_{\mathrm{a}\theta} = 2.360344\ldots$, $|\sigma'_{\theta\mathrm{i}}| \le \sigma_{\mathrm{y}}$; When $k \ge k_{\mathrm{a}\theta}$, $|\sigma'_{\theta\mathrm{i}}| \ge \sigma_{\mathrm{y}}$

2. When $k = k_{\mathrm{a}}$, $k_{\mathrm{j}} = \frac{k+1}{2} = 1.680172\ldots$

3. The maximum load-bearing capacity and autofrettage pressure

$$\frac{p}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}} - \frac{(k+1)^2}{4\sqrt{3}k^2} + \frac{2}{\sqrt{3}}\ln\frac{k+1}{2} \le \tfrac{2}{\sqrt{3}}\ln k = \frac{p_{\mathrm{y}}}{\sigma_{\mathrm{y}}}$$

4. When $k \le k_{\mathrm{a}\theta}$, $\frac{1}{\sqrt{3}} - \frac{(k+1)^2}{4\sqrt{3}k^2} + \frac{2}{\sqrt{3}}\ln\frac{k+1}{2} < \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$;

When $k \ge k_{\mathrm{a}\theta}$, $\frac{1}{\sqrt{3}} - \frac{(k+1)^2}{4\sqrt{3}k^2} + \frac{2}{\sqrt{3}}\ln\frac{k+1}{2} > \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$, but when $k \ge k_{\mathrm{a}\theta}$, $|\sigma'_{\theta\mathrm{i}}| > \sigma_{\mathrm{y}}$

See also table 3.2 of chapter 3

Tab. 4.3 – (Continued).

The results when $k_{\mathrm{j}} = \sqrt{k}$ 1. When $k \leq k_{\mathrm{b}\theta} = 2.65972\ldots$, $\lvert\sigma'_{\theta\mathrm{i}}\rvert \leq \sigma_{\mathrm{y}}$; When $k \geq k_{\mathrm{b}\theta}$, $\lvert\sigma'_{\theta\mathrm{i}}\rvert \geq \sigma_{\mathrm{y}}$ 2. When $k = k_{\mathrm{b}\theta}$, $k_{\mathrm{j}} = \sqrt{k} = 1.630865\ldots$ See also table 3.2 of chapter 3
The results when $k_{\mathrm{j}} = \frac{r_{\mathrm{j}}}{r_{\mathrm{i}}} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\right)$ 1. The allowable load in order for $\sigma'_{\theta\mathrm{i}}/\sigma_{\mathrm{y}} \geq -1$ when $k \geq k_{\mathrm{c}\theta}$ $\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right) + \frac{\sqrt{3}}{2}k^2 - \sqrt{3}k^2\frac{p}{\sigma_{\mathrm{y}}} - \frac{2+\sqrt{3}}{2} \geq 0 \text{ or } k^2 \leq \frac{\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right) - \frac{2+\sqrt{3}}{2}}{\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}} - \frac{\sqrt{3}}{2}}$ Replacing $\sqrt{3}$ in the molecule with 2 obtains $\exp\left(\sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}\right) + k^2 - \sqrt{3}k^2\frac{p}{\sigma_{\mathrm{y}}} - 2 \geq 0$, the result when $k_{\mathrm{j}} = k_{\mathrm{j}*}$ See also table 3.2 of chapter 3

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Chapter 5

Mechanical Autofrettage Technology Under Low Load

5.1 Introduction

When a pressure vessel is processed with autofrettage, it must be ensured that the pressure vessel is not yielded compressively. Based on the ultimate load bearing criterion, we found the suitable $k_{\rm j}$ for a pressure vessel with k not to be yielded when processed with autofrettage, which is $k^2 \ln k_{\rm j}^2 - k^2 - k_{\rm j}^2 + 2 = 0$, or equation (2.46), and which is fit for both the maximum shear stress theory (Tresca yield criterion) and the maximum distortion strain energy theory (Mises's yield criterion). If $k_{\rm j}$ is determined by this equation, the ultimate load-bearing capacity is $2p_{\rm e}$.

It is well known that the greater $k_{\rm j}$ is, the greater the load-bearing capacity is, but the more complex the autofrettage process is. Therefore, if it is not necessary for a cylinder to bear $2p_{\rm e}$, $k_{\rm j}$ can be decreased to reduce the complexity of the autofrettage process. Accordingly, the following questions should be resolved: How to determine an appropriate plastic depth $k_{\rm j}$ for a certain p and k? What is the relationship between $k_{\rm j}$ and k for a certain p when $p \neq 2p_{\rm e}$? What results will be obtained under new conditions? How is p for a certain $k_{\rm j}$ and k determined? What are the characteristics of residual and total stresses under new conditions? In this chapter, we study the aforementioned problems and do the comparison of the results based on the maximum shear stress theory (Tresca yield criterion) and the maximum distortion strain energy theory (Mises's yield criterion) in order to bring to light essential correlation and laws contained in the results.

On the basis of the previous work[1–4], this chapter is intended to study mechanical autofrettage technology under low load with the help of the mathematical analysis of reference[5].

DOI: 10.1051/978-2-7598-3111-1.c005
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5.2 The Optimum Plastic Depth

Equation (2.21) indicates that, in general, when the load-bearing capacity $\frac{p}{\sigma_{\mathrm{y}}}$ equals autofrettage pressure p_{a}, or $\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_a}{\sigma_{\mathrm{y}}}$, it is optimum, in this case, $\sigma_{\mathrm{e}}^{\mathrm{T}} \equiv \sigma_{\mathrm{y}}$ in the whole plastic region.

Based on the maximum shear stress theory (Tresca yield criterion)[6, 7], autofrettage pressure is equation (2.1), or

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \ln k_{\mathrm{j}} + \frac{k^2 - k_{\mathrm{j}}^2}{2k^2} = \frac{k^2 \ln k_{\mathrm{j}}^2 + k^2 - k_{\mathrm{j}}^2}{2k^2} \tag{5.1}$$

Based on the maximum distortion strain energy theory (Mises's yield criterion)[6, 7], autofrettage pressure is equation (3.1), or

$$\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}} \ln k_{\mathrm{j}} + \frac{k^2 - k_{\mathrm{j}}^2}{\sqrt{3}k^2} = \frac{k^2 \ln k_{\mathrm{j}}^2 + k^2 - k_{\mathrm{j}}^2}{\sqrt{3}k^2} \tag{5.2}$$

Set

$$\frac{p}{\sigma_{\mathrm{y}}} = \lambda \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} \tag{5.3}$$

λ is called the reinforcing coefficient for load-bearing capacity.

For the maximum shear stress theory, $\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{2k^2}$, therefore,

$$\frac{p}{\sigma_{\mathrm{y}}} = \lambda \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \lambda \frac{k^2 - 1}{2k^2} \tag{5.4}$$

For the maximum distortion strain energy theory, $\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{\sqrt{3}k^2}$, therefore,

$$\frac{p}{\sigma_{\mathrm{y}}} = \lambda \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \lambda \frac{k^2 - 1}{\sqrt{3}k^2} \tag{5.5}$$

Substituting equation (5.4) into equation (5.1), substituting equation (5.5) into equation (5.2), and writing k_{j} as $k_{\mathrm{j}\lambda}$, we obtain the same result, that is

$$k^2 \ln k_{\mathrm{j}\lambda}^2 - (\lambda - 1)k^2 - k_{\mathrm{j}\lambda}^2 + \lambda = 0 \quad (e^{0.5} \leq k_{\mathrm{j}\lambda} \leq k_{\mathrm{c}\lambda} \text{ and } k \geq k_{\mathrm{c}\lambda}) \tag{5.6}$$

That is to say, the maximum shear stress theory and the maximum distortion strain energy theory result in the same equation, *i.e.*, equations (5.6). Equation (5.6) presents the optimum plastic depth, $k_{\mathrm{j}\lambda}$, for fixed k and λ. In other words, for fixed k and λ, if the plastic depth is determined by equation (5.6), then the plastic depth is the minimum; thus, the complexity of the autofrettage process is reduced, and the load-bearing capacity $p = \lambda p_{\mathrm{e}}$ can be ensured. If the actual plastic region depth $k_{\mathrm{j}} < k_{\mathrm{j}\lambda}$, that is, the pressure that a vessel can contain is lower than λp_{e} and can be calculated by equations (5.1) or (5.2), then the pressure vessel is safe. If $k_{\mathrm{j}\lambda} < k_{\mathrm{j}} < k_{\mathrm{j}*}$, *i.e.*, the pressure that a vessel can contain is higher than λp_{e} and can be calculated by equations (5.1) or (5.2), then the pressure vessel is safe, but the autofrettage process is more complex. Thus, equation (5.6) is a key formula.

When $\lambda = 2$, or $p = 2p_{\mathrm{e}}$, equation (5.6) is just equation (2.34), or $k^2 \ln k_{\mathrm{j}}^2 - k^2 - k_{\mathrm{j}}^2 + 2 = 0$.

If k_{j} and k are related by equation (5.6) and a pressure vessel contains the pressure $p = p_{\mathrm{a}}$, then, $\sigma_{\mathrm{e}}^{\mathrm{T}} \equiv \sigma_{\mathrm{y}}$ in the plastic region, $|\sigma'_{\mathrm{ei}}| \leq \sigma_{\mathrm{y}}$; When $\lambda = 2$, $\sigma'_{\mathrm{ei}} = -\sigma_{\mathrm{y}}$.

According to ref.[6], the equivalent residual stress at a general location in the plastic region is

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}} - \frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2 - k_{\mathrm{j}}^2 + k^2 \ln k_{\mathrm{j}}^2}{(k^2 - 1)x^2} \tag{5.7}$$

The equivalent residual stress at a general location in an elastic region is

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}} - \frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = \frac{k^2(k_{\mathrm{j}}^2 - 1 - \ln k_{\mathrm{j}}^2)}{(k^2 - 1)x^2} \tag{5.8}$$

Although the components of residual stresses in three directions (σ'_{θ}, σ'_{r}, and σ'_{z}) based on the maximum shear stress theory are different from those based on the maximum distortion strain energy theory, the equivalent residual stress based on both strength theories is the same.

In fact, by combining equations (5.1) and (5.7) or combining equations (5.2) and (5.7), we can determine that if $p \leq 2p_{\mathrm{e}}$, or $\lambda \leq 2$, then $|\sigma'_{\mathrm{ei}}| \leq \sigma_{\mathrm{y}}$. If the practical plastic depth is larger than $k_{\mathrm{j}*}$, then $|\sigma'_{\mathrm{ei}}| > \sigma_{\mathrm{y}}$, which is not allowed. If the practical plastic depth is larger than $k_{\mathrm{j}\lambda}$ and smaller than $k_{\mathrm{j}*}$, then $|\sigma'_{\mathrm{ei}}| < \sigma_{\mathrm{y}}$, and the load-bearing capacity can be improved to $\lambda \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} < \frac{p}{\sigma_{\mathrm{y}}} < 2\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$, but the autofrettage process is more complex. If the practical plastic depth is smaller than $k_{\mathrm{j}\lambda}$, then the autofrettage process is easier, but the load-bearing capacity is reduced to $\frac{p}{\sigma_{\mathrm{y}}} < \lambda \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$.

When $k \geq k_{\mathrm{c}}$ and $k_{\mathrm{j}} \leq k_{\mathrm{j}*}$, $|\sigma'_{\mathrm{ei}}| \leq \sigma_{\mathrm{y}}$; when $k \geq k_{\mathrm{c}}$ and $k_{\mathrm{j}} = k_{\mathrm{j}*}$, $\sigma'_{\mathrm{ei}} = -\sigma_{\mathrm{y}}$, and $p = 2p_{\mathrm{e}}$; when $k \leq k_{\mathrm{c}}$, $k_{\mathrm{j}} = k$ (entirely yielded), $|\sigma'_{\mathrm{ei}}| < \sigma_{\mathrm{y}}$ and a cylinder can bear entire yield loading, p_{y}. We focus on the case of $k \geq k_{\mathrm{c}}$.

From equation (5.6), we obtain $k = \sqrt{\frac{k_{\mathrm{j}\lambda}^2 - \lambda}{\ln k_{\mathrm{j}\lambda}^2 - \lambda + 1}}$. Therefore, the definition domain of $k_{\mathrm{j}\lambda}$ is that $k_{\mathrm{j}\lambda} > \mathrm{e}^{(\lambda-1)/2}$ and $k_{\mathrm{j}\lambda} \leq \lambda^{0.5}$. $k_{\mathrm{j}\lambda}$, *i.e.*, equation (5.6) is plotted in figure 5.1, in which the intersection points between the straight line ($k_{\mathrm{j}\lambda} = k$) and each curve are o (1, 1), a (1.2071..., 1.2071...), b (1.43005..., 1.43005...), c (1.671272..., 1.671272...), d (1.93322..., 1.93322...), m (k_{c}, k_{c}), *i.e.* m (2.218457..., 2.218457...), f (2.52973..., 2.52973...) .... Furthermore, points n (∞, $\mathrm{e}^{0.5}$) and v (∞, 0) are at infinity, and the curve mn corresponds to $\lambda = 2$, the curve gov corresponds to $\lambda = 1$. The straight line $k_{\mathrm{j}\lambda} = k$ divides the curves into two parts. The above parts of the straight line $k_{\mathrm{j}\lambda} = k$ have no significance.

Figure 5.1 is applicable to both the maximum shear stress theory and the maximum distortion strain energy theory. In figure 5.1,

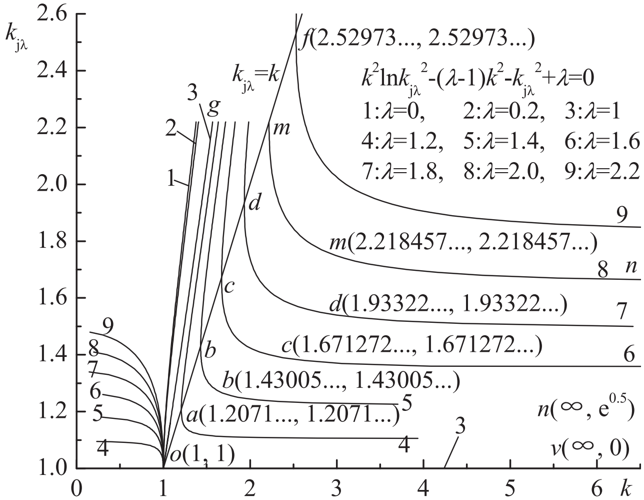

FIG. 5.1 – Curves of $k^2 \ln k_{j\lambda}^2 - (\lambda - 1)k^2 - k_{j\lambda}^2 + \lambda = 0$ for different λ.

Curve 1, which has one branch, is for $\lambda = 0$.

Curve 2, which has one branch, is for $\lambda = 0.2$.

Curve 3 is for $\lambda = 1$, which includes the curve og and the horizontal coordinate axis.

Curve 4 is for $\lambda = 1.2$, which has two parts. The intersection point of the straight line ($k_{j\lambda} = k$) and curve 4 is point a. Coordinate of point a is a ($k_{c\lambda}$, $k_{c\lambda}$), or a (1.2071…, 1.2071…).

Curve 5 is for $\lambda = 1.4$, which has two parts. The intersection point of the straight line ($k_{j\lambda} = k$) and curve 5 is point b. Coordinate of point b is b ($k_{c\lambda}$, $k_{c\lambda}$), or b (1.43005…, 1.43005…).

Curve 6 is for $\lambda = 1.6$, which has two parts. The intersection point of the straight line ($k_{j\lambda} = k$) and curve 6 is point c. Coordinate of point c is c ($k_{c\lambda}$, $k_{c\lambda}$), or c (1.671272…, 1.671272…).

Curve 7 is for $\lambda = 1.8$, which has two parts. The intersection point of the straight line ($k_{j\lambda} = k$) and curve 7 is point d. Coordinate of point d is d ($k_{c\lambda}$, $k_{c\lambda}$), or d (1.93322…, 1.93322…).

Curve 8 is for $\lambda = 2$, which has two parts. The intersection point of the straight line ($k_{j\lambda} = k$) and curve 8 is point m. Coordinate of point m is m ($k_{c\lambda}$, $k_{c\lambda}$), or m (k_c, k_c), *i.e.* m (2.218 457 489 916 7…, 2.218 457 489 916 7…).

Curve 9 is for $\lambda = 2.2$, which has two parts. The intersection point of the straight line ($k_{j\lambda} = k$) and curve 9 is point f. Coordinate of point f is f ($k_{c\lambda}$, $k_{c\lambda}$), or f (2.52973…, 2.52973…).

In conclusion, from equation (5.6) and figure 5.1, we can draw the following conclusions.

(1) When $\lambda \leq 1$ (curves $1 - 3$), $k_{j\lambda}$ is always greater than k irrespective of k. This is meaningless in practice.

(2) When $\lambda > 1$ (curves 4 − 9), the curves for equation (5.6) are divided into two branches, on the left branch, $k < 1$, and $k_{j\lambda} > k$, this is meaningless in practice; on the right branch, the straight line ($k_{j\lambda} = k$) divides the curves into two parts: above the straight line ($k_{j\lambda} = k$), $k_{j\lambda} > k$, this is meaningless in practice; below the straight line ($k_{j\lambda} = k$), $k_{j\lambda} < k$. Letting $k_{j\lambda} = k$ in equation (5.6) obtains the critical radius ratio corresponding to a certain λ, or $k_{c\lambda}$, which is

$$k_{c\lambda}^2 \ln k_{c\lambda}^2 - \lambda(k_{c\lambda}^2 - 1) = 0 \quad \text{or} \quad \lambda = \frac{k_{c\lambda}^2 \ln k_{c\lambda}^2}{k_{c\lambda}^2 - 1} \tag{5.9}$$

Equation (5.9) is the maximum radius ratio when the whole wall is yielded while $|\sigma'_{ei}| \leq \sigma_y$ under a certain λ. In this case, $k_{j\lambda} = k = k_{c\lambda}$. When $k \leq k_{c\lambda}$, $|\sigma'_{ei}\sigma_y| < 1$ irrespective of k_j even if $k_j = k$. When $k_{j\lambda} = k$, $p = p_y = \sigma_y \ln k < \lambda p_e$. Because $1 \leq \lambda \leq 2$, $1 \leq k_{c\lambda} \leq k_c$ and $1 \leq k_{j\lambda} \leq k_{j*}$. λ indicates the multiple of p_e. Equation (5.9) is applicable for both the maximum shear stress theory (Tresca yield criterion) and the maximum distortion strain energy theory (Mises's yield criterion). Equation (5.9) is plotted in figure 5.2.

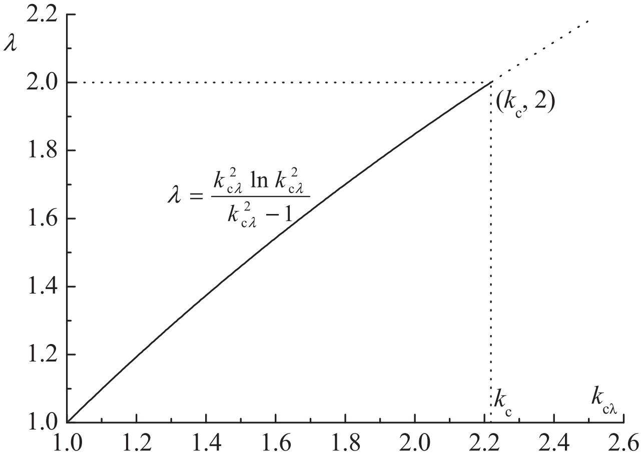

FIG. 5.2 – Correlation between λ and $k_{c\lambda}$.

From equation (5.9) and figure 5.2, it is seen that the greater $k_{c\lambda}$ is, the greater λ is.

(3) When $k \geq k_c$, the maximum of λ is 2 for any k. When $\lambda = 2$, $k_{c\lambda}$ becomes k_c, $k_{j\lambda}$ becomes k_{j*}, equation (5.6) becomes $k^2 \ln k_{j*}^2 - k^2 - k_{j*}^2 + 2 = 0$ or equations (2.34), and (5.9) becomes $\frac{k^2 \ln k}{k^2 - 1} = 1$ or equation (2.33). $k_c =$ 2.2184574899167…, called critical radius ratio, which is the solution of $k_{c\lambda}$ in equation (5.9) when $\lambda = 2$.

(4) For the right of the two branches, $k_{j\lambda}$ decreases as k increases and increases as λ increases. $k \to \infty$, $k_{j\lambda} = e^{(\lambda-1)/2} = k_{j\lambda\infty}$. When $k = \infty$,

$$\lambda = \ln k_j^2 + 1 \tag{5.10}$$

Thus, only when $\lambda > 1$, the discussion about autofrettage is significant. Because compressive yield occurs when $\lambda > 2$ (if $\lambda > 2$, $k_{j\lambda}$ is higher than the value on curve 8 where $\sigma'_{ei} = -\sigma_y$), curve 9 in figure 5.1 is meaningless. The horizontal axis can be regarded as the curve with $\lambda = 1$. Then, the meaningful part of figure 5.1, or the meaningful and possible plastic depth lies in the quasi-infinite area surrounded by the horizontal axis ($\lambda = 1$), the straight line $k_j = k$, and the curve $k^2 \ln k_{j*}^2 - k^2 - k_{j*}^2 + 2 = 0$ (curve 8 in figure 5.1). The coordinate of fourth vertexes of this quasi-infinite area (m, o, v, n) are shown in figure 5.1. The significant part of figure 5.1, or the significant and possible optimum plastic depth is re-plotted as shown in figure 5.3, which is applicable for both the maximum shear stress theory and the maximum distortion strain energy theory.

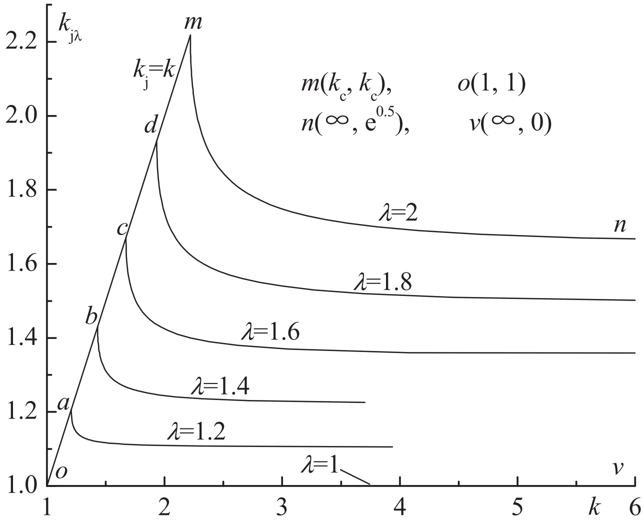

FIG. 5.3 – Significant and possible optimum plastic depth with $p = \lambda p_e$.

From equation (5.6) and figure 5.3, it is seen that the greater k is, the smaller $k_{j\lambda}$ is.

The application condition for equation (5.6) is $k \geq k_{c\lambda}$.

5.3 Analysis of Residual Stresses Under the Optimum Plastic Depth

When plastic depth $k_j = k_{j\lambda}$, whether based on the maximum shear stress theory or the maximum distortion strain energy theory, the relative location of the intersection of the three residual stress curves or the abscissa at which $\sigma'_e = 0$ is

$$x_0 = \sqrt{\lambda} \tag{5.11}$$

Based on the maximum shear stress theory, when $x = x_0$,

$$\frac{\sigma'_{\mathrm{zx0}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{\theta x0}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{rx0}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{rmin}}}{\sigma_{\mathrm{y}}} = y_0^{\mathrm{s}} = -\frac{\lambda - 1 - \ln\lambda}{2} \tag{5.12}$$

Based on the maximum distortion strain energy theory, when $x = x_0$,

$$\begin{gathered}\frac{\sigma'_{\mathrm{zx0}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{\theta x0}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{rx0}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{rmin}}}{\sigma_{\mathrm{y}}} = y_0^{\mathrm{d}} = -\frac{\lambda - 1 - \ln\lambda}{\sqrt{3}} \\ y_0^{\mathrm{d}} = \frac{2}{\sqrt{3}} y_0^{\mathrm{s}}\end{gathered} \tag{5.13}$$

Whether based on the maximum shear stress theory or the maximum distortion strain energy theory, the relative location where $\sigma'_{\mathrm{z}} = 0$ is $x_1 = \sqrt{\mathrm{e}^{\lambda-1}}$.

X_0, x_1, y_0^{s}, and y_0^{d} are not related to k or k_{j}, only related to λ.

When $k_{\mathrm{j}} = k_{\mathrm{j\lambda}}$ and $k \geq k_{\mathrm{c\lambda}}$, by means of equation (5.6), the complex expressions of residual stresses in general condition, or equations (2.2)–(2.9), can be simplified and re-expressed as follows.

The results based on the maximum shear stress theory are as follows.

In the plastic region:

$$\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = \ln\frac{r}{r_{\mathrm{i}}} - \frac{\lambda - 1}{2} = \ln x - \frac{\lambda - 1}{2} \tag{5.14}$$

$$\frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = \ln x + \frac{\lambda}{2x^2} - \frac{\lambda}{2} \tag{5.15}$$

$$\frac{\sigma'_{\mathrm{\theta}}}{\sigma_{\mathrm{y}}} = \ln x - \frac{\lambda}{2x^2} - \frac{\lambda}{2} + 1 \tag{5.16}$$

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = 1 - \frac{\lambda}{x^2} \tag{5.17}$$

σ'_{z}, σ'_{r}, $\sigma'_{\mathrm{\theta}}$, and σ'_{e} are related with λ but not with k_{j}, which means that for some λ, the curves of residual stresses in whichever direction (axial, radial and hoop direction) and equivalent residual stress under different $k_{\mathrm{j\lambda}}$ and k coincide in the plastic region.

In the elastic region:

$$\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} \equiv \frac{k_{\mathrm{j\lambda}}^2 - \lambda}{2k^2} \tag{5.18}$$

$$\frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = \left(1 - \frac{k^2}{x^2}\right)\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j\lambda}}^2 - \lambda}{2k^2} - \frac{k_{\mathrm{j\lambda}}^2 - \lambda}{2x^2} \tag{5.19}$$

$$\frac{\sigma'_\theta}{\sigma_y} = \left(1 + \frac{k^2}{x^2}\right)\frac{\sigma'_z}{\sigma_y} = \frac{k_{j\lambda}^2 - \lambda}{2k^2} + \frac{k_{j\lambda}^2 - \lambda}{2x^2} \tag{5.20}$$

$$\frac{\sigma'_e}{\sigma_y} = \frac{k_{j\lambda}^2 - \lambda}{x^2} \tag{5.21}$$

On the inner surface,

$$\frac{\sigma'_{zi}}{\sigma_y} = -\frac{\lambda - 1}{2} \tag{5.22}$$

$$\frac{\sigma'_{ri}}{\sigma_y} = 0 \tag{5.23}$$

$$\frac{\sigma'_{\theta i}}{\sigma_y} = 1 - \lambda = 2\frac{\sigma'_{zi}}{\sigma_y} = \frac{\sigma'_{ei}}{\sigma_y} \tag{5.24}$$

$$\frac{\sigma'_{ei}}{\sigma_y} = 1 - \lambda \tag{5.25}$$

On the outer surface,

$$\frac{\sigma'_{zo}}{\sigma_y} = \frac{k_{j\lambda}^2 - \lambda}{2k^2} \tag{5.26}$$

$$\sigma'_{ro} = 0 \tag{5.27}$$

$$\frac{\sigma'_{\theta o}}{\sigma_y} = 2\frac{\sigma'_{zo}}{\sigma_y} = \frac{k_{j\lambda}^2 - \lambda}{k^2} \tag{5.28}$$

$$\frac{\sigma'_e}{\sigma_y} = \frac{k_{j\lambda}^2 - \lambda}{k^2} \tag{5.29}$$

According to the DS transition rule in chapter 1, if based on the maximum distortion strain energy theory (Mises's yield criterion), the residual stresses when $k_j = k_{j\lambda}$ and $k \geq k_{c\lambda}$ are corresponding as follows:

$$\frac{\sigma'_z}{\sigma_y} = \frac{2}{\sqrt{3}}\ln\frac{r}{r_i} - \frac{\lambda - 1}{\sqrt{3}} = \frac{\ln x^2}{\sqrt{3}} - \frac{\lambda - 1}{\sqrt{3}} \tag{5.30}$$

$$\frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = \frac{\ln x^2}{\sqrt{3}} + \frac{\lambda}{\sqrt{3}x^2} - \frac{\lambda}{\sqrt{3}} \tag{5.31}$$

$$\frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}} = \frac{\ln x^2}{\sqrt{3}} - \frac{\lambda}{\sqrt{3}x^2} + \frac{2-\lambda}{\sqrt{3}} \tag{5.32}$$

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = 1 - \frac{\lambda}{x^2} \tag{5.33}$$

$$\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} \equiv \frac{k_{\mathrm{j}\lambda}^2 - \lambda}{\sqrt{3}k^2} \tag{5.34}$$

$$\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = \left(1 - \frac{k^2}{x^2}\right)\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2 - \lambda}{\sqrt{3}k^2} - \frac{k_{\mathrm{j}\lambda}^2 - \lambda}{\sqrt{3}x^2} \tag{5.35}$$

$$\frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}} = \left(1 + \frac{k^2}{x^2}\right)\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2 - \lambda}{\sqrt{3}k^2} + \frac{k_{\mathrm{j}\lambda}^2 - \lambda}{\sqrt{3}x^2} \tag{5.36}$$

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2 - \lambda}{x^2} \tag{5.37}$$

On the inner surface,

$$\frac{\sigma'_{\mathrm{zi}}}{\sigma_{\mathrm{y}}} = -\frac{\lambda - 1}{\sqrt{3}} \tag{5.38}$$

$$\frac{\sigma'_{\mathrm{ri}}}{\sigma_{\mathrm{y}}} = 0 \tag{5.39}$$

$$\frac{\sigma'_{\theta\mathrm{i}}}{\sigma_{\mathrm{y}}} = -\frac{2}{\sqrt{3}}(\lambda - 1) = 2\frac{\sigma'_{\mathrm{zi}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}} \tag{5.40}$$

$$\frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}} = 1 - \lambda \tag{5.41}$$

On the outer surface,

$$\frac{\sigma'_{\mathrm{zo}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2 - \lambda}{\sqrt{3}k^2} \tag{5.42}$$

$$\sigma'_{\mathrm{ro}} = 0 \tag{5.43}$$

$$\frac{\sigma'_{\theta\mathrm{o}}}{\sigma_{\mathrm{y}}} = 2\frac{\sigma'_{\mathrm{zo}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\frac{k_{\mathrm{j}\lambda}^2 - \lambda}{k^2} \tag{5.44}$$

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2 - \lambda}{k^2} \tag{5.45}$$

In both the plastic and elastic region, the components of residual stresses in three directions based on the maximum shear stress theory (Tresca yield criterion) are $\frac{\sqrt{3}}{2}$ times of those based on the maximum distortion strain energy theory, and the equivalent residual stress based on both strength theories is the same. The reason for this situation has been explained in chapters 2 and 3.

In the case of the maximum shear stress theory (Tresca yield criterion), when plastic depth $k_{\mathrm{j}} = k_{\mathrm{j}\lambda}$, the residual stresses are illustrated figures 5.4–5.6.

In figure 5.4, curves 1–5 are circumferential residual stresses in the plastic zone under the state of entire yielding with different λ, curves 4a–4d are circumferential residual stresses in an elastic zone with different k and $\lambda = 1.8$, where curve 1 is for $\lambda = 1.2$ and $k_{\mathrm{c}\lambda} = 1.207094$; curve 2 is for $\lambda = 1.4$ and $k_{\mathrm{c}\lambda} = 1.430048$; curve 3 is for $\lambda = 1.6$ and $k_{\mathrm{c}\lambda} = 1.671272$; curve 4 is for $\lambda = 1.8$ and $k_{\mathrm{c}\lambda} = 1.93322$; curve 5 is for $\lambda = 2$ and $k_{\mathrm{c}\lambda} = k_{\mathrm{c}} = 2.2184574899167$; curve 4a is for $\lambda = 1.8$, $k = 2$ and $k_{\mathrm{j}\lambda} = 1.736906$; curve 4b is for $\lambda = 1.8$, $k = 3$ and $k_{\mathrm{j}\lambda} = 1.539944$; curve 4c is for $\lambda = 1.8$, $k = 4$ and $k_{\mathrm{j}\lambda} = 1.515107$; curve 4d is for $\lambda = 1.8$, $k = \infty$ and $k_{\mathrm{j}\lambda} = \mathrm{e}^{0.4}$. The parameters ($k_{\mathrm{j}\lambda}$, $k_{\mathrm{c}\lambda}$, etc.) are calculated by equations (5.6) and (5.9).

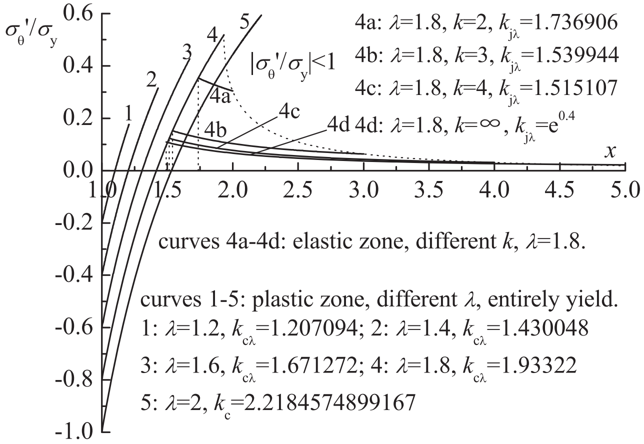

FIG. 5.4 – Circumferential residual stresses with different k and λ under Tresca yield criterion.

Figure 5.5 is circumferential residual stresses in the whole wall with different λ when $k = 4$ and $k_j = k_{j\lambda}$, where curve 1 is for $\lambda = 1.2$ and $k_{j\lambda} = 1.105972$; curve 2 is for $\lambda = 1.4$ and $k_{j\lambda} = 1.225275$; curve 3 is for $\lambda = 1.6$ and $k_{j\lambda} = 1.360485$; curve 4 is for $\lambda = 1.8$ and $k_{j\lambda} = 1.515107$; curve 5 is for $\lambda = 2$ and $k_{j\lambda} = k_{j*} = 1.694172$.

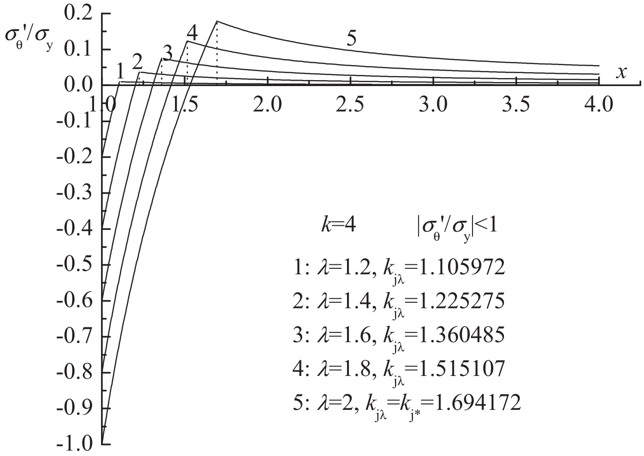

FIG. 5.5 – Circumferential residual stresses with different λ and $k = 4$ under Tresca yield criterion.

Figure 5.6 is equivalent residual stresses in the whole wall with different λ when $k = 4$ and $k_j = k_{j\lambda}$, where curve 1 is for $\lambda = 1.2$ and $k_{j\lambda} = 1.105972$; curve 2 is for $\lambda = 1.4$ and $k_{j\lambda} = 1.225275$; curve 3 is for $\lambda = 1.6$ and $k_{j\lambda} = 1.360485$; curve 4 is for $\lambda = 1.8$ and $k_{j\lambda} = 1.515107$; curve 5 is for $\lambda = 2$ and $k_{j\lambda} = k_{j*} = 1.694172$.

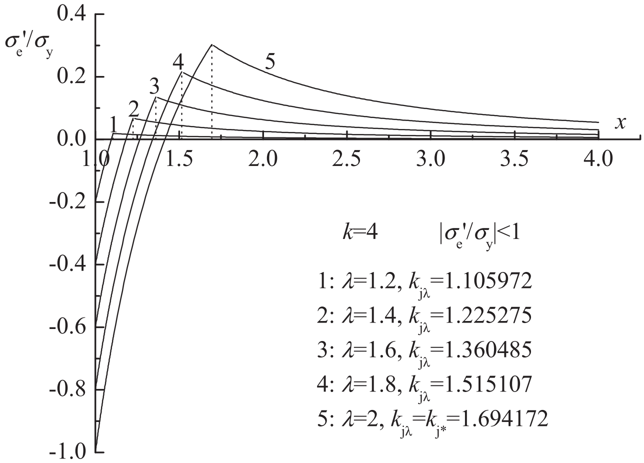

FIG. 5.6 – Equivalent residual stresses with different λ and $k = 4$ under Tresca yield criterion.

The dash curve in figure 5.4 is the connection of $\frac{\sigma'_{\theta o}}{\sigma_y}$ for $\lambda = 1.8$ and all k. This dash curve is obtained as follows: for some λ, from equation (5.9), $k_{c\lambda}$ is obtained; $k_{c\lambda}$ is the maximum $k_{j\lambda}$ under this λ; for $k_{j\lambda} = e^{(\lambda-1)/2}$ (corresponding to $k = \infty$) to $k_{c\lambda}$, from equation (5.6), k^2 is obtained for this λ; substituting k^2 into equation (5.28) obtains $\frac{\sigma'_{\theta o}}{\sigma_y}$; taking k ($=k_{c\lambda}$ to ∞) as the horizontal coordinate and $\frac{\sigma'_{\theta o}}{\sigma_y}$ as the vertical coordinate, this dash curve is formed. For example, for $\lambda = 1.8$, from equation (5.9), $k_{c\lambda} = 1.93322$; for $k_{j\lambda} = e^{(\lambda-1)/2} = e^{0.4} = 1.491825$ (corresponding to $k = \infty$) to 1.93322, from equation (5.6), k^2 is obtained for this $\lambda = 1.8$; substituting k^2 into equation (5.28) obtains $\frac{\sigma'_{\theta o}}{\sigma_y}$; taking $k = k_{c\lambda} = 1.93322$ to ∞ as the horizontal coordinate and $\frac{\sigma'_{\theta o}}{\sigma_y}$ as the vertical coordinate, the dash curve in figure 5.4 is formed.

Figure 5.7 shows the axial residual stress together with radial residual stress and circumferential residual stress for $k = 4$, $\lambda = 1.8$ and $k_{j\lambda} = 1.515107$, that is curve 4 in figures 5.4–5.6. The coordinate of the crossover point of the three curves for residual stresses is $w\,(x_0,\, y_0^s) = w\left(\sqrt{\lambda},\, -\frac{\lambda-1-\ln\lambda}{2}\right) = w\,(1.341641\ldots,\, -0.10611\ldots)$.

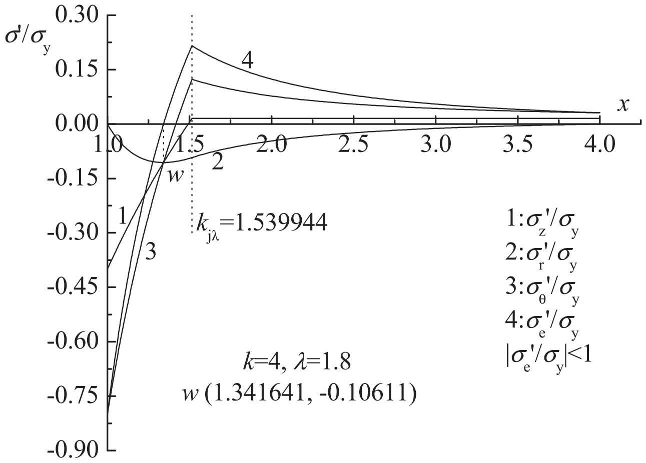

FIG. 5.7 – The distribution of residual stresses in three directions for $k = 4$ and $\lambda = 1.8$ under Tresca yield criterion.

In the case of the maximum distortion strain energy theory (Mises's yield criterion), when plastic depth $k_j = k_{j\lambda}$, the residual stresses are illustrated figures 5.8 and 5.10.

In figure 5.8, curves 1 to 5 are circumferential residual stresses in a plastic zone under the state of entire yielding with different λ, curves 4a–4d are circumferential residual stresses in an elastic zone with different k and $\lambda = 1.8$, where curve 1 is for $\lambda = 1.2$ and $k_{c\lambda} = 1.207094$; curve 2 is for $\lambda = 1.4$ and $k_{c\lambda} = 1.430048$; curve 3 is for $\lambda = 1.6$ and $k_{c\lambda} = 1.671272$; curve 4 is for $\lambda = 1.8$ and $k_{c\lambda} = 1.93322$; curve 5 is for $\lambda = 2$ and $k_{c\lambda} = k_c = 2.2184574899167$; curve 4a is for $\lambda = 1.8$, $k = 2$ and

$k_{j\lambda} = 1.736906$; curve 4b is for $\lambda = 1.8$, $k = 3$ and $k_{j\lambda} = 1.539944$; curve 4c is for $\lambda = 1.8$, $k = 4$ and $k_{j\lambda} = 1.515107$; curve 4d is for $\lambda = 1.8$, $k = \infty$ and $k_{j\lambda} = e^{0.4}$. The parameters ($k_{j\lambda}$, $k_{c\lambda}$, etc.) are calculated by equations (5.6) and (5.9).

Figure 5.9 is circumferential residual stresses in the whole wall with different λ when $k = 4$ and $k_j = k_{j\lambda}$, where curve 1 is for $\lambda = 1.2$ and $k_{j\lambda} = 1.105972$; curve 2 is for $\lambda = 1.4$ and $k_{j\lambda} = 1.225275$; curve 3 is for $\lambda = 1.6$ and $k_{c\lambda} = 1.360485$; curve 4 is for $\lambda = 1.8$ and $k_{j\lambda} = 1.515107$; curve 5 is for $\lambda = 2$ and $k_{j\lambda} = k_{j*} = 1.694172$.

Figure 5.10 is equivalent residual stresses in the whole wall with different λ when $k = 4$ and $k_j = k_{j\lambda}$, where curve 1 is for $\lambda = 1.2$ and $k_{j\lambda} = 1.105972$; curve 2 is for $\lambda = 1.4$ and $k_{j\lambda} = 1.225275$; curve 3 is for $\lambda = 1.6$ and $k_{c\lambda} = 1.360485$; curve 4 is for $\lambda = 1.8$ and $k_{j\lambda} = 1.515107$; curve 5 is for $\lambda = 2$ and $k_{j\lambda} = k_{j*} = 1.694172$.

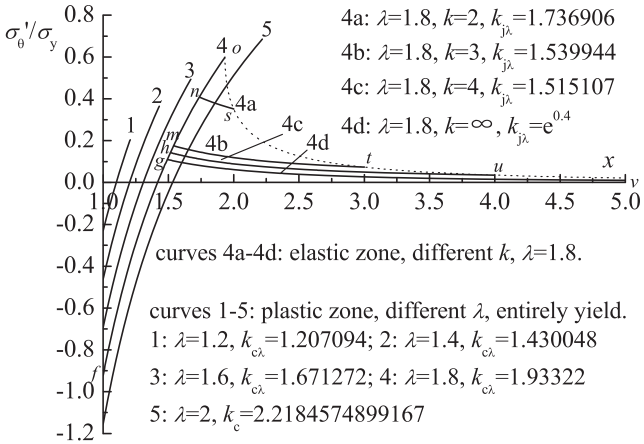

FIG. 5.8 – Circumferential residual stresses with different k and λ under Mises yield criterion.

The dash curve in figure 5.8 is the connection of $\frac{\sigma'_{\theta o}}{\sigma_y}$ for $\lambda = 1.8$ and all k. This dash curve is made out as follows: for some λ, from equation (5.9), $k_{c\lambda}$ is obtained; $k_{c\lambda}$ is the maximum $k_{j\lambda}$ under this λ; for $k_{j\lambda} = e^{(\lambda-1)/2}$ (corresponding to $k = \infty$) to $k_{c\lambda}$, from equation (5.6), k^2 is obtained for this λ; substituting k^2 into equation (5.44) obtains $\frac{\sigma'_{\theta o}}{\sigma_y}$; taking k ($=k_{c\lambda}$ to ∞) as the horizontal coordinate and $\frac{\sigma'_{\theta o}}{\sigma_y}$ as the vertical coordinate, the dash curve is formed. For example, for $\lambda = 1.8$, from equation (5.9), $k_{c\lambda} = 1.93322$; for $k_{j\lambda} = e^{(\lambda-1)/2} = e^{0.4}$ (corresponding to $k = \infty$) to 1.93322, from equation (5.6), k^2 is obtained for this $\lambda = 1.8$; substituting k^2 into equation (5.44) obtains $\frac{\sigma'_{\theta o}}{\sigma_y}$; taking $k = k_{c\lambda} = 1.93322$ to ∞ as the horizontal coordinate and $\frac{\sigma'_{\theta o}}{\sigma_y}$ as the vertical coordinate, the dash curve in figure 5.8 is formed. $\frac{\sigma'_{\theta o}}{\sigma_y}$ based on the maximum distortion strain energy theory is $\frac{2}{\sqrt{3}}$ times of $\frac{\sigma'_{\theta o}}{\sigma_y}$ based on the maximum shear stress theory.

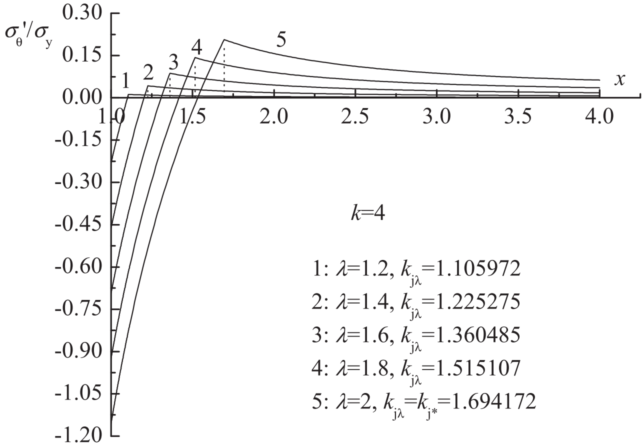

FIG. 5.9 – Circumferential residual stresses with different λ and $k = 4$ under Mises yield criterion.

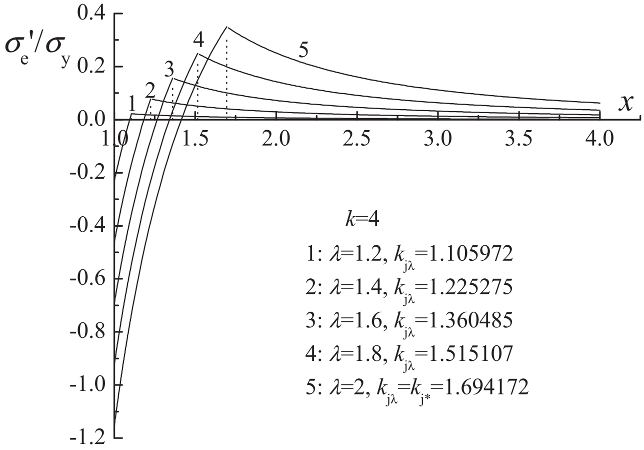

FIG. 5.10 – Equivalent residual stresses with different λ and $k = 4$ under Mises yield criterion.

Figure 5.11 shows the axial residual stress together with radial residual stress and circumferential residual stress for $k = 4$, $\lambda = 1.8$ and $k_{j\lambda} = 1.515107$, that is curve 4 in figures 5.8–5.10. The coordinate of the crossover point of the three curves for residual stresses is $s\left(x_0, y_0^{d}\right) = s\left(\sqrt{\lambda}, -\frac{\lambda-1-\ln\lambda}{\sqrt{3}}\right) = s\left(1.341641\ldots, -0.12252\ldots\right)$.

It is easy to prove that, if λ is fixed, the three curves for residual stress $\left(\frac{\sigma_z'}{\sigma_y}, \frac{\sigma_r'}{\sigma_y}\right.$ and $\left.\frac{\sigma_\theta'}{\sigma_y}\right)$ collect at a fixed point within the plastic zone: $\left(\sqrt{\lambda}, -\frac{\lambda-1-\ln\lambda}{2}\right)$ (based on the maximum shear stress theory) or $\left(\sqrt{\lambda}, -\frac{\lambda-1-\ln\lambda}{\sqrt{3}}\right)$ (based on the maximum distortion strain energy theory) for any k, k_j and λ, and the coordinate of the intersection is

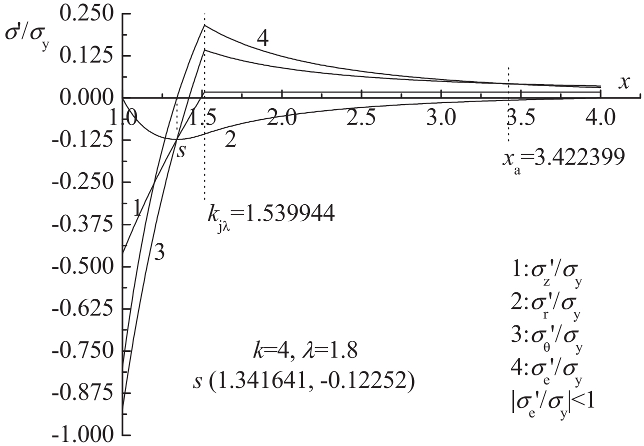

FIG. 5.11 – The distribution of residual stresses in three directions for $k = 4$ and $\lambda = 1.8$ under Mises yield criterion.

not related with k and k_{j} but only with λ. If plastic $k_{\mathrm{j}} \neq k_{\mathrm{j}\lambda}$, this situation does not happen, the coordinate of the intersection is related not only with λ but also with k and k_{j}.

According to equations (5.25) or (5.41), at the inner surface, $\frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}} = 1 - \lambda$ and σ'_{ei} is the most dangerous. When $\lambda \geq 1$, $\sigma'_{\mathrm{ei}} \leq 0$, this indicates compressive stress; when $\lambda \leq 2$, $\sigma'_{\mathrm{ei}} \geq -\sigma_{\mathrm{y}}$. Thus when $1 \leq \lambda \leq 2$, σ'_{ei} is safe. Since $k_{\mathrm{j}} \geq \lambda^{0.5}$ ($k_{\mathrm{j}} \geq \mathrm{e}^{(\lambda-1)/2}$ at the same time), when $\lambda \leq 2$, $k_{\mathrm{j}}^2 – \lambda > 0$ in the elastic zone, *i.e.* $\sigma'_{\mathrm{e}} > 0$ (tension) in an elastic zone. At an elastic-plastic juncture where $x = k_{\mathrm{j}\lambda}$, equivalent residual stress is the maximum (algebraic value, not absolute value) in the whole elastic zone, or $\frac{\sigma'_{\mathrm{ej}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2 - \lambda}{k_{\mathrm{j}\lambda}^2} = 1 - \frac{\lambda}{k_{\mathrm{j}\lambda}^2}$. Clearly, $0 < \sigma'_{\mathrm{ej}} < \sigma_{\mathrm{y}}$.

Since $k = \infty$, $k_{\mathrm{j}\lambda} = \mathrm{e}^{(\lambda-1)/2} = k_{\mathrm{j}\lambda\infty}$, then from equations (5.18)–(5.21), in the whole elastic region,

$$\sigma'_{\mathrm{z}} \equiv 0, \text{ for } x = \mathrm{e}^{(\lambda-1)/2} \sim \infty \tag{5.46}$$

$$\frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} = -\frac{\mathrm{e}^{\lambda-1} - \lambda}{2x^2}, \text{ for } x = \mathrm{e}^{(\lambda-1)/2} \sim \infty \tag{5.47}$$

$$\frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}} = \frac{\mathrm{e}^{\lambda-1} - \lambda}{2x^2} = -\frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}}, \text{ for } x = \mathrm{e}^{(\lambda-1)/2} \sim \infty \tag{5.48}$$

$$\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{\mathrm{e}^{\lambda-1} - \lambda}{x^2}, \text{ for } x = \mathrm{e}^{(\lambda-1)/2} \sim \infty \tag{5.49}$$

In elastic zone, $x = \mathrm{e}^{(\lambda-1)/2} \sim \infty$, therefore, when $k = \infty$, in elastic zone, with $x = \mathrm{e}^{(\lambda-1)/2}$ to ∞, $\sigma'_\mathrm{z} \equiv 0$; $\frac{\sigma'_\mathrm{r}}{\sigma_\mathrm{y}} = -\frac{\mathrm{e}^{\lambda-1}-\lambda}{2\mathrm{e}^{\lambda-1}}$ to 0; $\frac{\sigma'_\theta}{\sigma_\mathrm{y}} = \frac{\mathrm{e}^{\lambda-1}-\lambda}{2\mathrm{e}^{\lambda-1}}$ to 0; $\frac{\sigma'_\mathrm{e}}{\sigma_\mathrm{y}} = \frac{\mathrm{e}^{\lambda-1}-\lambda}{\mathrm{e}^{\lambda-1}}$ to 0.

When $k = \infty$, based on the maximum distortion strain energy theory (Mises's yield criterion), the residual stresses in the elastic region are as follows.

$$\sigma'_\mathrm{z} \equiv 0, \text{ for } x = \mathrm{e}^{(\lambda-1)/2} \sim \infty \tag{5.50}$$

$$\frac{\sigma'_\mathrm{r}}{\sigma_\mathrm{y}} = -\frac{\mathrm{e}^{\lambda-1}-\lambda}{\sqrt{3}x^2}, \text{ for } x = \mathrm{e}^{(\lambda-1)/2} \sim \infty \tag{5.51}$$

$$\frac{\sigma'_\theta}{\sigma_\mathrm{y}} = \frac{\mathrm{e}^{\lambda-1}-\lambda}{\sqrt{3}x^2} = -\frac{\sigma'_\mathrm{r}}{\sigma_\mathrm{y}}, \text{ for } x = \mathrm{e}^{(\lambda-1)/2} \sim \infty \tag{5.52}$$

$$\frac{\sigma'_\mathrm{e}}{\sigma_\mathrm{y}} = \frac{\mathrm{e}^{\lambda-1}-\lambda}{x^2}, \text{ for } x = \mathrm{e}^{(\lambda-1)/2} \sim \infty \tag{5.53}$$

In elastic zone, $x = \mathrm{e}^{(\lambda-1)/2}$ to ∞, therefore, when $k = \infty$, in elastic zone, from $x = \mathrm{e}^{(\lambda-1)/2}$ to ∞, $\sigma'_\mathrm{z} \equiv 0$; $\frac{\sigma'_\mathrm{r}}{\sigma_\mathrm{y}} = -\frac{\mathrm{e}^{\lambda-1}-\lambda}{\sqrt{3}\mathrm{e}^{\lambda-1}}$ to 0; $\frac{\sigma'_\theta}{\sigma_\mathrm{y}} = \frac{\mathrm{e}^{\lambda-1}-\lambda}{\sqrt{3}\mathrm{e}^{\lambda-1}}$ to 0; $\frac{\sigma'_\mathrm{e}}{\sigma_\mathrm{y}} = \frac{\mathrm{e}^{\lambda-1}-\lambda}{\mathrm{e}^{\lambda-1}}$ to 0.

The residual stresses in the plastic region are still equations (5.14)–(5.17) and equations (5.30)–(5.33) when $k = \infty$.

When $\sigma'_\mathrm{e} = 0$, $x = \lambda^{0.5}$, which is just the abscissa of the intersection of the three curves for the residual stress and depends on λ only.

Further description of curve 4 and curves 4a–4d in figure 5.8 is as follows, the other curves are the same.

When $\lambda = 1.8$, from equation (5.13), we obtain $k_{\mathrm{c}\lambda} = 1.93322$. $k_{\mathrm{c}\lambda}$ is the maximum plastic depth k_j for $\lambda = 1.8$. If plastic depth $k_\mathrm{j} = k$, (entirely yielded), then $k = k_{\mathrm{c}\lambda} = k_\mathrm{j} = k_{\mathrm{j}\lambda}$. In this case, there is only a plastic region and no elastic region, or the whole wall is the plastic region, the curve for residual stresses is curve 4 or curve *fo*, which is the longest for all k under $\lambda = 1.8$. The greater k is, the smaller $k_{\mathrm{j}\lambda}$ is. If $k > k_{\mathrm{c}\lambda} = 1.93322$, the corresponding $k_{\mathrm{j}\lambda} < 1.93322$.

(1) $k = 2$ ($\lambda = 1.8$), from equation (5.6), we obtain $k_{\mathrm{j}\lambda} = 1.736906\ldots$. In this case, the curve for residual stresses in the plastic region is curve *fn* and the curve for residual stresses in the elastic region is curve *ns*, the section *no* is dropped out. Within the plastic region, x varies from 1 to $k_{\mathrm{j}\lambda} = 1.7369065$ (point f to n), within the elastic region, x varies from $k_{\mathrm{j}\lambda}$ to k (=2) (point n to s).

(2) $k = 3$ ($\lambda = 1.8$), from equation (5.6), we obtain $k_{\mathrm{j}\lambda} = 1.539944\ldots$. In this case, the curve for residual stresses in the plastic region is curve *fm* and the curve for residual stresses in the elastic region is curve *mt*, the section *mo* is dropped out. Within the plastic region, x varies from 1 to $k_{\mathrm{j}\lambda} = 1.539944$ (point f to m), within the elastic region, x varies from $k_{\mathrm{j}\lambda}$ to k (=3) (point m to t).

(3) $k = 4$ ($\lambda = 1.8$), from equation (5.6), we obtain $k_{\mathrm{j}\lambda} = 1.515107\ldots$. In this case, the curve for residual stresses in the plastic region is curve *fh* and the curve for

residual stresses in the elastic region is curve *hu*, the section ho is dropped out. Within the plastic region, x varies from 1 to $k_{\mathrm{j}\lambda} = 1.515107$ (point *f* to *h*), within the elastic region, x varies from $k_{\mathrm{j}\lambda}$ to k (=4) (point *h* to *u*).

(4) $k = \infty$ ($\lambda = 1.8$), from equation (5.6), we obtain $k_{\mathrm{j}\lambda} = \mathrm{e}^{0.4} = 1.491825\ldots$ In this case, the curve for residual stresses in the plastic region is curve *fg* and the curve for residual stresses in the elastic region is curve *gv*, the section *go* is dropped out. Within the plastic region, x varies from 1 to $k_{\mathrm{j}\lambda} = \mathrm{e}^{0.4}$ (point *f* to *g*), within the elastic region, x varies from $k_{\mathrm{j}\lambda}$ to k (=∞) (point *g* to *v*). Point v is at infinity, the coordinate is v (∞, 0).

From figures 5.4 and 5.8. Equations (5.16), and (5.32), it is known that all curves of the circumferential residual stresses under a fixed λ for any k and $k_{\mathrm{j}\lambda}$ within the plastic region are located on the identical curve, such as, when $\lambda = 1.8$, all curves of the circumferential residual stresses under a fixed $\lambda = 1.8$ for any k and $k_{\mathrm{j}\lambda}$ is *fo*, except that the different curve for different k and $k_{\mathrm{j}\lambda}$ is located on a different section of curve *fo*. Saying, the above curves in the plastic zone, *fo* ($k = k_{\mathrm{j}\lambda} = k_{\mathrm{c}\lambda}$), *fn* ($k = 2$, $k_{\mathrm{j}\lambda} = 1.736906$), *fm* ($k = 3$, $k_{\mathrm{j}\lambda} = 1.539944$), *fh* ($k = 4$, $k_{\mathrm{j}\lambda} = 1.515107$), *fg* ($k = \infty$, $k_{\mathrm{j}\lambda} = \mathrm{e}^{0.4}$) are all on curve *fo*, or they partly coincide with each other. The greater k is, the shorter the curves for the circumferential residual stresses within the plastic region are, with the curve under $k = k_{\mathrm{c}\lambda} = k_{\mathrm{j}} = k_{\mathrm{j}\lambda}$ being the longest, such as, when $k = 4$ and $\lambda = 1.8$, the longest curve for the circumferential residual stresses within the plastic region is curve *fo*, where $k = k_{\mathrm{c}\lambda} = k_{\mathrm{j}} = k_{\mathrm{j}\lambda} = 1.93322$.

When based on the maximum shear stress theory (Tresca yield criterion), if $\lambda \leq 1$, $\sigma'_{\theta\mathrm{i}} \geq -\sigma_{\mathrm{y}}$ and $\sigma'_{\mathrm{ei}} \geq -\sigma_{\mathrm{y}}$ can be guaranteed meanwhile. When based on the maximum distortion strain energy theory (Mises's yield criterion), if $\lambda \leq 1$, $\sigma'_{\mathrm{ei}} \geq -\sigma_{\mathrm{y}}$ can be guaranteed but $\sigma'_{\theta\mathrm{i}} < -\sigma_{\mathrm{y}}$. In order for $\sigma'_{\theta\mathrm{i}} \geq -\sigma_{\mathrm{y}}$, from equation (5.40), $\lambda \leq \frac{2+\sqrt{3}}{2}$. This confirms the fact that the load-bearing capacity is $\frac{p}{\sigma_{\mathrm{y}}} = \frac{\sqrt{3}+2}{2}\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \lambda\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$ when the circumferential residual stress $\sigma'_{\theta\mathrm{i}}$ is controlled in chapter 4.

When $x < x_{\mathrm{a}}$, $\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} > \frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}}$; when $x > x_{\mathrm{a}}$, $\frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} < \frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}}$.

5.4 Analysis of the Stresses Caused by Internal Pressure and Total Stresses

When $p = \lambda p_{\mathrm{e}}$, the stresses caused by internal pressure p at a general location, called operation stresses, are:

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{1}{k^2-1}\frac{p}{\sigma_{\mathrm{y}}} = \frac{\lambda}{2k^2} \tag{5.54}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \left(1-\frac{k^2}{x^2}\right)\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{\lambda}{2}\left(\frac{1}{k^2}-\frac{1}{x^2}\right) \tag{5.55}$$

$$\frac{\sigma_\theta^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \left(1 + \frac{k^2}{x^2}\right)\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{\lambda}{2}\left(\frac{1}{k^2} + \frac{1}{x^2}\right) \tag{5.56}$$

Equivalent operation stress based on the maximum shear stress theory (Tresca criterion) is

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{ps}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_\theta^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{\lambda}{x^2} \tag{5.57}$$

Total stresses $\boldsymbol{\sigma}^{\mathrm{T}}$ include residual stresses and operation stress. When $p = \lambda p_{\mathrm{e}}$ and $k_{\mathrm{j}} = k_{\mathrm{j}\lambda}$, the total stresses based on the maximum shear stress theory (Tresca yield criterion) are as follows.

In the plastic region,

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \ln x - \frac{\lambda - 1}{2} + \frac{\lambda}{2k^2} \tag{5.58}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \ln x - \frac{\lambda}{2} + \frac{\lambda}{2k^2} \tag{5.59}$$

$$\frac{\sigma_\theta^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \ln x - \frac{\lambda}{2} + \frac{\lambda}{2k^2} + 1 \tag{5.60}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_\theta^{\mathrm{T}}}{\sigma_{\mathrm{y}}} - \frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} \equiv 1 \tag{5.61}$$

Therefore, when a pressure vessel is subject to a pressure $p = \lambda p_{\mathrm{e}}$ and its plastic depth is determined by equation (5.6), then, equivalent total stress everywhere in plastic zone $\sigma_{\mathrm{e}} \equiv \sigma_{\mathrm{y}}$.

In the elastic region,

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2}{2k^2} \tag{5.62}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2}{2}\left(\frac{1}{k^2} - \frac{1}{x^2}\right) \tag{5.63}$$

$$\frac{\sigma_\theta^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2}{2}\left(\frac{1}{k^2} + \frac{1}{x^2}\right) \tag{5.64}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_\theta^{\mathrm{T}}}{\sigma_{\mathrm{y}}} - \frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2}{x^2} \tag{5.65}$$

On the other hand, when based on the maximum shear stress theory (Tresca yield criterion), if $\frac{p}{\sigma_\mathrm{y}} = \frac{p_\mathrm{a}}{\sigma_\mathrm{y}} = \frac{k^2 \ln k_\mathrm{j}^2 + k^2 - k_\mathrm{j}^2}{2k^2}$ (equation (5.1)), regardless of k_j, there is

$$\frac{\sigma_\mathrm{z}^\mathrm{p}}{\sigma_\mathrm{y}} = \frac{1}{k^2-1}\frac{p}{\sigma_\mathrm{y}} = \frac{k^2 \ln k_\mathrm{j}^2 + k^2 - k_\mathrm{j}^2}{2k^2(k^2-1)} \tag{5.66}$$

Using equations (5.66), the general equations of residual stresses or equations (2.2)–(2.9) are rewritten as follows:

In the plastic region,

$$\frac{\sigma_\mathrm{z}'}{\sigma_\mathrm{y}} = \frac{1}{2}\left(\frac{k_\mathrm{j}^2}{k^2} + \ln x^2 - \ln k_\mathrm{j}^2\right) - \frac{\sigma_\mathrm{z}^\mathrm{p}}{\sigma_\mathrm{y}} \tag{5.67}$$

$$\frac{\sigma_\mathrm{r}'}{\sigma_\mathrm{y}} = \frac{1}{2}\left(\frac{k_\mathrm{j}^2}{k^2} - 1 + \ln x^2 - \ln k_\mathrm{j}^2\right) - \frac{\sigma_\mathrm{r}^\mathrm{p}}{\sigma_\mathrm{y}} \tag{5.68}$$

$$\frac{\sigma_\theta'}{\sigma_\mathrm{y}} = \frac{1}{2}\left(\frac{k_\mathrm{j}^2}{k^2} + 1 + \ln x^2 - \ln k_\mathrm{j}^2\right) - \frac{\sigma_\theta^\mathrm{p}}{\sigma_\mathrm{y}} \tag{5.69}$$

$$\frac{\sigma_\mathrm{e}'}{\sigma_\mathrm{y}} = \frac{\sigma_\theta'}{\sigma_\mathrm{y}} - \frac{\sigma_\mathrm{r}'}{\sigma_\mathrm{y}} = 1 - \left(\frac{\sigma_\theta^\mathrm{p}}{\sigma_\mathrm{y}} - \frac{\sigma_\mathrm{r}^\mathrm{p}}{\sigma_\mathrm{y}}\right) \tag{5.70}$$

In the elastic region,

$$\frac{\sigma_\mathrm{z}'}{\sigma_\mathrm{y}} = \frac{1}{2}\frac{k_\mathrm{j}^2}{k^2} - \frac{\sigma_\mathrm{z}^\mathrm{p}}{\sigma_\mathrm{y}} \tag{5.71}$$

$$\frac{\sigma_\mathrm{r}'}{\sigma_\mathrm{y}} = \frac{1}{2}\left(\frac{k_\mathrm{j}^2}{k^2} - \frac{k_\mathrm{j}^2}{x^2}\right) - \frac{\sigma_\mathrm{r}^\mathrm{p}}{\sigma_\mathrm{y}} \tag{5.72}$$

$$\frac{\sigma_\theta'}{\sigma_\mathrm{y}} = \frac{1}{2}\left(\frac{k_\mathrm{j}^2}{k^2} + \frac{k_\mathrm{j}^2}{x^2}\right) - \frac{\sigma_\theta^\mathrm{p}}{\sigma_\mathrm{y}} \tag{5.73}$$

$$\frac{\sigma_\mathrm{e}'}{\sigma_\mathrm{y}} = \frac{\sigma_\theta'}{\sigma_\mathrm{y}} - \frac{\sigma_\mathrm{r}'}{\sigma_\mathrm{y}} = \frac{k_\mathrm{j}^2}{x^2} - \left(\frac{\sigma_\theta^\mathrm{p}}{\sigma_\mathrm{y}} - \frac{\sigma_\mathrm{r}^\mathrm{p}}{\sigma_\mathrm{y}}\right) \tag{5.74}$$

Therefore, when $p = p_\mathrm{a}$, irrespective of k_j, total stresses are as follows.

In the plastic region,

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\sigma_{\mathrm{z}}'}{\sigma_{\mathrm{y}}}+\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}=\frac{1}{2}\left(\frac{k_{\mathrm{j}}^2}{k^2}+\ln x^2-\ln k_{\mathrm{j}}^2\right) \tag{5.75}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\sigma_{\mathrm{r}}'}{\sigma_{\mathrm{y}}}+\frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}=\frac{1}{2}\left(\frac{k_{\mathrm{j}}^2}{k^2}-1+\ln x^2-\ln k_{\mathrm{j}}^2\right) \tag{5.76}$$

$$\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\sigma_{\theta}'}{\sigma_{\mathrm{y}}}+\frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}=\frac{1}{2}\left(\frac{k_{\mathrm{j}}^2}{k^2}+1+\ln x^2-\ln k_{\mathrm{j}}^2\right) \tag{5.77}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}-\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\equiv 1 \tag{5.78}$$

In the elastic region,

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\sigma_{\mathrm{z}}'}{\sigma_{\mathrm{y}}}+\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}=\frac{1}{2}\frac{k_{\mathrm{j}}^2}{k^2} \tag{5.79}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\sigma_{\mathrm{r}}'}{\sigma_{\mathrm{y}}}+\frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}=\frac{1}{2}\left(\frac{k_{\mathrm{j}}^2}{k^2}-\frac{k_{\mathrm{j}}^2}{x^2}\right) \tag{5.80}$$

$$\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\sigma_{\theta}'}{\sigma_{\mathrm{y}}}+\frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}=\frac{1}{2}\left(\frac{k_{\mathrm{j}}^2}{k^2}+\frac{k_{\mathrm{j}}^2}{x^2}\right) \tag{5.81}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}-\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{k_{\mathrm{j}}^2}{x^2} \tag{5.82}$$

When $p=p_{\mathrm{a}}$, irrespective of k_{j}, the equivalent total stress is always in a safe range. If plastic depth $k_{\mathrm{j}}=k_{\mathrm{j}\lambda}$, the components of the total stresses and the equivalent total stress are in a safe range. If $k_{\mathrm{j}}\neq k_{\mathrm{j}\lambda}$, the equivalent total stress is in a safe range, but the components of the total stresses are not necessarily within a safe range.

When plastic depth $k_{\mathrm{j}}=k_{\mathrm{j}\lambda}$, equations (5.75)–(5.82) just becomes equations (5.58)–(5.65), respectively.

When $x<x_{\mathrm{a}}$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}>\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$; when $x>x_{\mathrm{a}}$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}<\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$.

When based on the maximum distortion strain energy theory (Mises's yield criterion), accordingly we have the following results by replacing σ_{y} in the results based on the maximum shear stress theory with $\frac{2}{\sqrt{3}}\sigma_{\mathrm{y}}$.

When $p=\lambda p_{\mathrm{e}}$, the stresses caused by internal pressure p at a general location, called operation stresses, are:

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{1}{k^2 - 1}\frac{p}{\sigma_{\mathrm{y}}} = \frac{\lambda}{\sqrt{3}k^2} \tag{5.83}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \left(1 - \frac{k^2}{x^2}\right)\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{\lambda}{\sqrt{3}}\left(\frac{1}{k^2} - \frac{1}{x^2}\right) \tag{5.84}$$

$$\frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \left(1 + \frac{k^2}{x^2}\right)\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{\lambda}{\sqrt{3}}\left(\frac{1}{k^2} + \frac{1}{x^2}\right) \tag{5.85}$$

Equivalent operation stress

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{pd}}}{\sigma_{\mathrm{y}}} = \frac{\sqrt{3}}{2}\left(\frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} - \frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}\right) = \frac{\lambda}{x^2} = \frac{\sigma_{\mathrm{e}}^{\mathrm{ps}}}{\sigma_{\mathrm{y}}} \tag{5.86}$$

When $p = \lambda p_{\mathrm{e}}$ and $k_{\mathrm{j}} = k_{\mathrm{j}\lambda}$, the total stresses are as follows.

In the plastic region,

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\ln x^2}{\sqrt{3}} - \frac{\lambda - 1}{\sqrt{3}} + \frac{\lambda}{\sqrt{3}k^2} \tag{5.87}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\ln x^2}{\sqrt{3}} - \frac{\lambda}{\sqrt{3}} + \frac{\lambda}{\sqrt{3}k^2} \tag{5.88}$$

$$\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\ln x^2}{\sqrt{3}} - \frac{\lambda}{\sqrt{3}} + \frac{\lambda}{\sqrt{3}k^2} + \frac{2}{\sqrt{3}} \tag{5.89}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sqrt{3}}{2}\left(\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} - \frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\right) = \frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} \equiv 1 \tag{5.90}$$

Therefore, similarly, when a pressure vessel is subject to a pressure $p = \lambda p_{\mathrm{e}}$ and its plastic depth is determined by equation (5.6), then, equivalent total stress everywhere in plastic zone $\sigma_{\mathrm{e}} \equiv \sigma_{\mathrm{y}}$.

In the elastic region,

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2}{\sqrt{3}k^2} \tag{5.91}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2}{\sqrt{3}}\left(\frac{1}{k^2} - \frac{1}{x^2}\right) \tag{5.92}$$

$$\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2}{\sqrt{3}}\left(\frac{1}{k^2} + \frac{1}{x^2}\right) \tag{5.93}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{\sqrt{3}}{2}\left(\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}-\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\right)=\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}}+\frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}=\frac{k_{\mathrm{j}\lambda}^{2}}{x^{2}} \tag{5.94}$$

On the other hand, when based on the maximum distortion strain energy theory (Mises's criterion), if $\frac{p}{\sigma_{\mathrm{y}}}=\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}=\frac{k^{2}\ln k_{\mathrm{j}}^{2}+k^{2}-k_{\mathrm{j}}^{2}}{\sqrt{3}k^{2}}$ (equation (5.2)), regardless of k_{j}, there is

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}=\frac{1}{k^{2}-1}\frac{p}{\sigma_{\mathrm{y}}}=\frac{k^{2}\ln k_{\mathrm{j}}^{2}+k^{2}-k_{\mathrm{j}}^{2}}{\sqrt{3}k^{2}(k^{2}-1)} \tag{5.95}$$

Using equations (5.66), the general equations of residual stresses or equations (2.2)–(2.9) are rewritten as follows:

In the plastic region,

$$\frac{\sigma_{\mathrm{z}}'}{\sigma_{\mathrm{y}}}=\frac{1}{\sqrt{3}}\left(\frac{k_{\mathrm{j}}^{2}}{k^{2}}+\ln x^{2}-\ln k_{\mathrm{j}}^{2}\right)-\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} \tag{5.96}$$

$$\frac{\sigma_{\mathrm{r}}'}{\sigma_{\mathrm{y}}}=\frac{1}{\sqrt{3}}\left(\frac{k_{\mathrm{j}}^{2}}{k^{2}}-1+\ln x^{2}-\ln k_{\mathrm{j}}^{2}\right)-\frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} \tag{5.97}$$

$$\frac{\sigma_{\theta}'}{\sigma_{\mathrm{y}}}=\frac{1}{\sqrt{3}}\left(\frac{k_{\mathrm{j}}^{2}}{k^{2}}+1+\ln x^{2}-\ln k_{\mathrm{j}}^{2}\right)-\frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} \tag{5.98}$$

$$\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}}=\frac{\sqrt{3}}{2}\left(\frac{\sigma_{\theta}'}{\sigma_{\mathrm{y}}}-\frac{\sigma_{\mathrm{r}}'}{\sigma_{\mathrm{y}}}\right)=1-\frac{\sqrt{3}}{2}\left(\frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}-\frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}\right)=1-\frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} \tag{5.99}$$

In the elastic region,

$$\frac{\sigma_{\mathrm{z}}'}{\sigma_{\mathrm{y}}}=\frac{1}{\sqrt{3}}\frac{k_{\mathrm{j}}^{2}}{k^{2}}-\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} \tag{5.100}$$

$$\frac{\sigma_{\mathrm{r}}'}{\sigma_{\mathrm{y}}}=\frac{1}{\sqrt{3}}\left(\frac{k_{\mathrm{j}}^{2}}{k^{2}}-\frac{k_{\mathrm{j}}^{2}}{x^{2}}\right)-\frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} \tag{5.101}$$

$$\frac{\sigma_{\theta}'}{\sigma_{\mathrm{y}}}=\frac{1}{\sqrt{3}}\left(\frac{k_{\mathrm{j}}^{2}}{k^{2}}+\frac{k_{\mathrm{j}}^{2}}{x^{2}}\right)-\frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} \tag{5.102}$$

$$\frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}}=\frac{\sqrt{3}}{2}\left(\frac{\sigma_{\theta}'}{\sigma_{\mathrm{y}}}-\frac{\sigma_{\mathrm{r}}'}{\sigma_{\mathrm{y}}}\right)=\frac{k_{\mathrm{j}}^{2}}{x^{2}}-\frac{\sqrt{3}}{2}\left(\frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}-\frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}\right)=\frac{k_{\mathrm{j}}^{2}}{x^{2}}-\frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} \tag{5.103}$$

Therefore, when $p = p_{\mathrm{a}}$, irrespective of k_{j}, total stresses are as follows.

In the plastic region,

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{z}}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}\left(\frac{k_{\mathrm{j}}^2}{k^2} + \ln x^2 - \ln k_{\mathrm{j}}^2\right) \tag{5.104}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{r}}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}\left(\frac{k_{\mathrm{j}}^2}{k^2} - 1 + \ln x^2 - \ln k_{\mathrm{j}}^2\right) \tag{5.105}$$

$$\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\theta}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}\left(\frac{k_{\mathrm{j}}^2}{k^2} + 1 + \ln x^2 - \ln k_{\mathrm{j}}^2\right) \tag{5.106}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sqrt{3}}{2}\left(\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} - \frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\right) \equiv 1 \tag{5.107}$$

In the elastic region,

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{z}}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}\frac{k_{\mathrm{j}}^2}{k^2} \tag{5.108}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{r}}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}\left(\frac{k_{\mathrm{j}}^2}{k^2} - \frac{k_{\mathrm{j}}^2}{x^2}\right) \tag{5.109}$$

$$\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\theta}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}}\left(\frac{k_{\mathrm{j}}^2}{k^2} + \frac{k_{\mathrm{j}}^2}{x^2}\right) \tag{5.110}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sqrt{3}}{2}\left(\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} - \frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\right) = \frac{k_{\mathrm{j}}^2}{x^2} \tag{5.111}$$

When $p = p_{\mathrm{a}}$, irrespective of k_{j}, the equivalent total stress is always in a safe range. If plastic depth $k_{\mathrm{j}} = k_{\mathrm{j}\lambda}$, the components of the total stresses and the equivalent total stress are in a safe range. If $k_{\mathrm{j}} \neq k_{\mathrm{j}\lambda}$, the equivalent total stress is in a safe range, but the components of the total stresses are not necessarily within a safe range.

When $k_{\mathrm{j}} = k_{\mathrm{j}\lambda}$, equations (5.104)–(5.111) just becomes equations (5.87)–(5.94), respectively.

When $x < x_{\mathrm{a}}$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} > \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$; when $x > x_{\mathrm{a}}$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} < \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$.

When $k_{\mathrm{j}} = k_{\mathrm{j}\lambda}$ ($p = p_{\mathrm{a}}$) and $\lambda = 1.2$, total stresses based on the maximum shear stress theory (Tresca yield criterion) are plotted in figure 5.12; When $k_{\mathrm{j}} = k_{\mathrm{j}\lambda}$

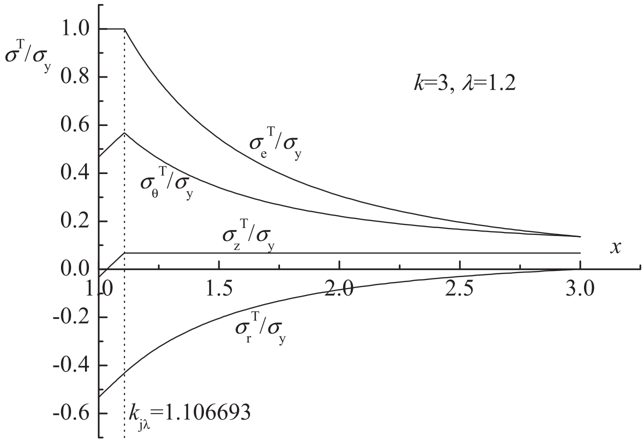

FIG. 5.12 – The total stresses when $k_{\mathrm{j}} = k_{\mathrm{j}\lambda}$ and $k = 3$, $\lambda = 1.2$ under Tresca yield criterion.

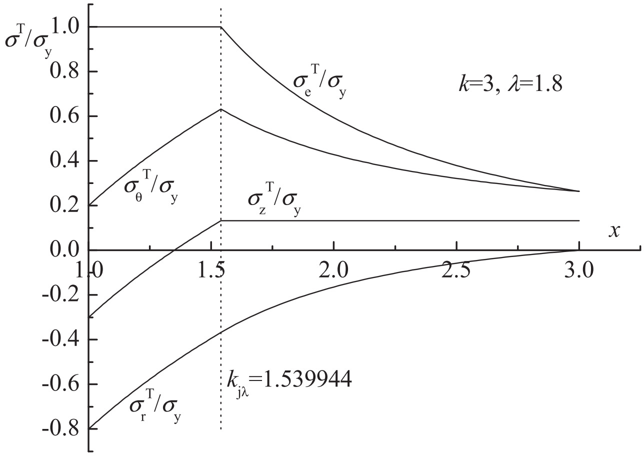

FIG. 5.13 – The total stresses when $k_{\mathrm{j}} = k_{\mathrm{j}\lambda}$ and $k = 3$, $\lambda = 1.8$ under Tresca yield criterion.

($p = p_{\mathrm{a}}$) and $\lambda = 1.8$, total stresses based on the maximum shear stress theory (Tresca yield criterion) are plotted in figure 5.13.

When $k_{\mathrm{j}} = k_{\mathrm{j}\lambda}$ ($p = p_{\mathrm{a}}$) and $\lambda = 1.2$, total stresses based on the maximum distortion strain energy theory (Mises's yield criterion) are plotted in figure 5.14; When $k_{\mathrm{j}} = k_{\mathrm{j}\lambda}$ ($p = p_{\mathrm{a}}$) and $\lambda = 1.8$, total stresses based on the maximum distortion strain energy theory (Mises's yield criterion) are plotted in figure 5.15.

The components of the total stresses based on the maximum distortion strain energy theory (Mises yield criterion) are $\frac{2}{\sqrt{3}}$ times of those based on the maximum shear stress theory (Tresca yield criterion), the equivalent total stress is unchanged, $k_{\mathrm{j}\lambda}$ and x_{a} are unchanged.

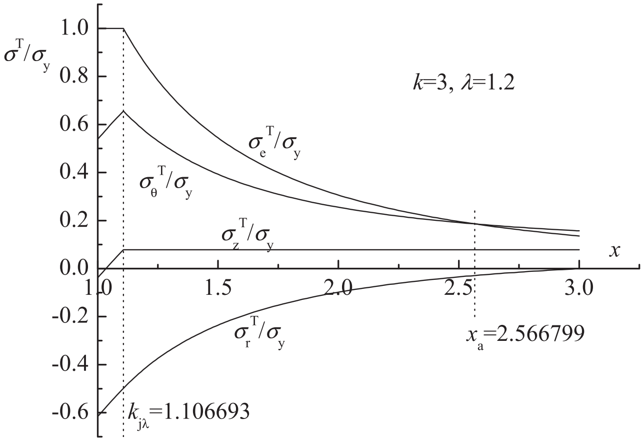

FIG. 5.14 – The total stresses when $k_j = k_{j\lambda}$ and $k = 3$, $\lambda = 1.2$ under Mises yield criterion.

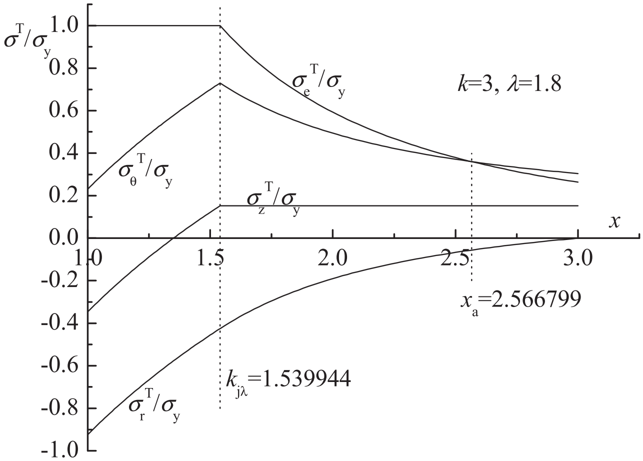

FIG. 5.15 – The total stresses when $k_j = k_{j\lambda}$ and $k = 3$, $\lambda = 1.8$ under Mises yield criterion.

From equations (5.62)–(5.65) and equations (5.91)–(5.94), seemingly, total stresses in the elastic zone have nothing to do with λ. However, $k_{j\lambda}$ depends on λ as seen in equation (5.6).

It is thus obvious that it is equations (5.1) or (5.2) that ensures $\sigma_e^T/\sigma_y \equiv 1$ everywhere in the plastic zone and $\sigma_e^T/\sigma_y = k_j^2/x^2$ in elastic zone irrespective of k_j and k. Substituting equation (5.1) into the equation of the equivalent operation stresses, or equation (1.9), obtains the equivalent operation stresses based on the maximum shear stress theory (Tresca yield criterion), σ_e^{ps}

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{ps}}}{\sigma_{\mathrm{y}}}=\frac{2k^2}{k^2-1}\frac{p}{\sigma_{\mathrm{y}}}x^{-2}=\frac{k^2-k_{\mathrm{j}}^2+k^2\ln k_{\mathrm{j}}^2}{(k^2-1)x^2} \tag{5.112}$$

Substituting equation (5.2) into the equation of the equivalent operation stresses, or equation (1.10), obtains the equivalent operation stresses based on the maximum distortion strain energy theory (Mises's yield criterion), $\sigma_{\mathrm{e}}^{\mathrm{pd}}$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{pd}}}{\sigma_{\mathrm{y}}}=\frac{\sqrt{3}}{2}\left(\frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}-\frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}\right)=\frac{\sqrt{3}k^2}{x^2(k^2-1)}\frac{p}{\sigma_{\mathrm{y}}}=\frac{k^2-k_{\mathrm{j}}^2+k^2\ln k_{\mathrm{j}}^2}{(k^2-1)x^2} \tag{5.113}$$

Equations (5.112) and (5.113) are the same, the reason is that $\frac{\sigma_{\mathrm{e}}^{\mathrm{pd}}}{\sigma_{\mathrm{y}}}=\frac{\sqrt{3}}{2}\frac{\sigma_{\mathrm{e}}^{\mathrm{ps}}}{\sigma_{\mathrm{y}}}$, but $\frac{p_{\mathrm{a}}^{\mathrm{d}}}{\sigma_{\mathrm{y}}}=\frac{2}{\sqrt{3}}\frac{p_{\mathrm{a}}^{\mathrm{s}}}{\sigma_{\mathrm{y}}}$.

Substituting equations (5.112) and (5.7) into equation (5.61) or substituting equations (5.113) and (5.7) into equation (5.90) just obtains $\sigma_{\mathrm{e}}^{\mathrm{T}}\equiv\sigma_{\mathrm{y}}$; Substituting equations (5.112) and (5.8) into equation (5.65) or substituting equations (5.113) and (5.8) into equation (5.94) just obtains $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{k_{\mathrm{j}}^2}{x^2}$. So, as long as $p=p_{\mathrm{a}}$, $\sigma_{\mathrm{e}}^{\mathrm{T}}\equiv\sigma_{\mathrm{y}}$ in the plastic zone and $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{k_{\mathrm{j}}^2}{x^2}$ in the elastic zone are inexorable law irrespective of k_{j} and k. In other words, providing $p=p_{\mathrm{a}}$, for any k_{j} and k, which need not be related by equation (5.6), $\sigma_{\mathrm{e}}^{\mathrm{T}}\equiv\sigma_{\mathrm{y}}$ in the plastic zone and $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}=\frac{k_{\mathrm{j}}^2}{x^2}$ $(0<k_{\mathrm{j}}^2/x^2<1$, for $k_{\mathrm{j}}\leq x\leq k)$ in the elastic zone are inevitable. However, k_{j} and k affect residual stresses. Inadequate k_{j} for a certain k may cause compressive yield, and it is necessary for k_{j} to be smaller than the value determined by equation (5.6), otherwise, compressive yield occurs.

5.5 Analysis of the Effect of Load Ratio (λ) and Plastic Depth ($k_{\mathrm{j}\lambda}$)

According to equation (5.6), the effect of λ on $k_{\mathrm{j}\lambda}$ is concluded as follows.

When $k\leq k_{\mathrm{c}}$, $k_{\mathrm{j}\lambda\max}=k=k_{\mathrm{c}\lambda}$, and $\lambda_{\max}\leq 2$; When $k\geq k_{\mathrm{c}}$, $k_{\mathrm{j}\lambda\max}<k$, $k_{\mathrm{j}\lambda\max}$ is determined by $k^2\ln k_{\mathrm{j}\lambda}^2-k^2-k_{\mathrm{j}\lambda}^2+2=0$ (This equation is corresponding to $\lambda=2$), and $\lambda_{\max}=2$.

For a fixed k, there are two values of $k_{\mathrm{j}\lambda}$ corresponding to the same λ according to equation (5.6): one is greater than k, which is of no practical significance; another is smaller than k. $\frac{\mathrm{d}k_{\mathrm{j}\lambda}}{\mathrm{d}\lambda}=\frac{k_{\mathrm{j}\lambda}(k^2-1)}{2(k^2-k_{\mathrm{j}\lambda}^2)}$. When $k_{\mathrm{j}\lambda}=k$, $\frac{\mathrm{d}k_{\mathrm{j}\lambda}}{\mathrm{d}\lambda}=\infty$; when $k_{\mathrm{j}\lambda}>k$, $\frac{\mathrm{d}k_{\mathrm{j}\lambda}}{\mathrm{d}\lambda}<0$, which is of no significance; when $k_{\mathrm{j}\lambda}<k$, $\frac{\mathrm{d}k_{\mathrm{j}\lambda}}{\mathrm{d}\lambda}>0$, which is of great significance. Thus, for a certain k, when $k_{\mathrm{j}\lambda}<k$, the greater λ is, the greater $k_{\mathrm{j}\lambda}$ is. When $k_{\mathrm{j}\lambda}=k$, equation (5.6) becomes equation (5.7). When $\lambda=2$, equation (5.7) becomes $\frac{k^2\ln k}{k^2-1}=1$, the solution of which is $k=k_{\mathrm{c}}=2.2184574899167\ldots$.

When $k > k_{j\lambda}$, as k continuously increases, $k_{j\lambda}$ corresponding to various k becomes increasingly closer, or when k is sufficiently large, $k_{j\lambda}$ is almost unchangeable. For $k = \infty$, if $\lambda = 2$, $k_j^{\infty} = e^{0.5}$.

Equation (5.6) applies to both the maximum shear stress theory and the maximum distortion strain energy theory. In order to intuitively understand the effect of λ on $k_{j\lambda}$, we plot equation (5.6) to show the correlation between λ and $k_{j\lambda}$. To plot equation (5.6), we first rewrite the equation as follows.

$$\lambda = \frac{k^2 \ln k_{j\lambda}^2 + k^2 - k_{j\lambda}^2}{k^2 - 1} \tag{5.114}$$

Setting $\frac{d\lambda}{dk_{j\lambda}} = 0$ leads to $k_{j\lambda} = k$. That is to say, when $k_{j\lambda} = k$, λ reaches a maximum.

Equation (5.114) is plotted as shown in figures 5.16 and 5.17.

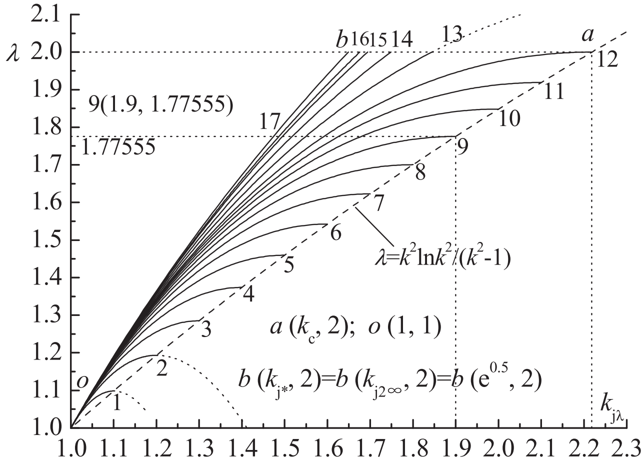

FIG. 5.16 – The correlation between λ and $k_{j\lambda}$ under a certain k.

In figure 5.16,
curve 1: $k = 1.1$, $\lambda_{\max} = 1.098336$;
curve 2: $k = 1.2$, $\lambda_{\max} = 1.193377$;
curve 3: $k = 1.3$, $\lambda_{\max} = 1.285205$;
curve 4: $k = 1.4$, $\lambda_{\max} = 1.373928$;
curve 5: $k = 1.5$, $\lambda_{\max} = 1.459674$;
curve 6: $k = 1.6$, $\lambda_{\max} = 1.542576$;
curve 7: $k = 1.7$, $\lambda_{\max} = 1.622768$;
curve 8: $k = 1.8$, $\lambda_{\max} = 1.700383$;
curve 9: $k = 1.9$, $\lambda_{\max} = 1.77555$;
curve 10: $k = 2$, $\lambda_{\max} = 1.848392$;

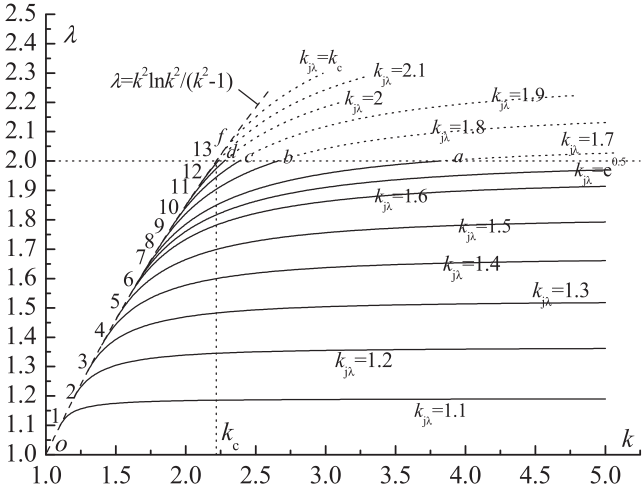

FIG. 5.17 – The correlation between λ and k under a certain $k_{j\lambda}$.

curve 11: $k = 2.1$, $\lambda_{\max} = 1.919029$;
curve 12: $k = k_c$, $\lambda_{\max} = 2$;
curve 13: $k = 2.5$, $\lambda_{\max} = 2$;
curve 14: $k = 3$, $\lambda_{\max} = 2$;
curve 15: $k = 4$, $\lambda_{\max} = 2$;
curve 16: $k = 5$, $\lambda_{\max} = 2$;
curve 17: $k = \infty$, $\lambda_{\max} = 2$.

Points 1–12 and a are peak points, points 13, 14, …, b are not peak points.

The dash curve *oa* is equation (5.9), or $\lambda = \frac{k^2 \ln k^2}{k^2-1} = \frac{k_{c\lambda}^2 \ln k_{c\lambda}^2}{k_{c\lambda}^2-1} = \lambda_{\max}$.

When $k \le k_c$, $k_{j\lambda\max} = k = k_{c\lambda}$, $\lambda_{\max} = \frac{k^2 \ln k^2}{k^2-1} = \frac{k_{c\lambda}^2 \ln k_{c\lambda}^2}{k_{c\lambda}^2-1}$, such as curves 1–12 in figure 5.16.

When $k \ge k_c$, $k_{j\lambda\max} = k_{j*} < k$, $\lambda_{\max} = 2$, such as curves 12–17 in figure 5.16.

When $k \le k_c$, equation (5.6) still holds because $\lambda \frac{p_e}{\sigma_y} = \frac{p_a}{\sigma_y} < \ln k < \frac{k^2-1}{k^2}$ (based on the maximum shear stress theory) or $\lambda \frac{p_e}{\sigma_y} = \frac{p_a}{\sigma_y} < \frac{2}{\sqrt{3}} \ln k < \frac{2}{\sqrt{3}} \frac{k^2-1}{k^2}$ (based on the maximum distortion strain energy theory). This is proved as follows.

When based on the maximum distortion strain energy theory, substituting equation (5.6) into equations (5.1) or (5.2) results in $\lambda \frac{p_e}{\sigma_y} = \frac{p_a}{\sigma_y}$. Substituting equation (5.114) into $\lambda \frac{p_e}{\sigma_y}$ also results in $\lambda \frac{p_e}{\sigma_y} = \frac{p_a}{\sigma_y}$.

From $\lambda \frac{p_e}{\sigma_y} = \lambda \frac{k^2-1}{2k^2} < \ln k$, we obtain

$$\frac{k^2 \ln k^2}{k^2 - 1} > \lambda \tag{5.115}$$

Substituting equation (5.114) into equation (5.115) obtains

$$k^2 \ln k^2 > k^2 \ln k_{\mathrm{j}}^2 + k^2 - k_{\mathrm{j}}^2 = a \tag{5.116}$$

$\frac{\mathrm{d}a}{\mathrm{d}k_{\mathrm{j}}} = 2k_{\mathrm{j}}\left(\frac{k^2}{k_{\mathrm{j}}^2} - 1\right) > 0$. When $k_{\mathrm{j}} = 1$, $a = k^2 - 1 < k^2 \ln k^2$; When $k_{\mathrm{j}} = k$, $a = k^2 \ln k^2$; In a word, when $1 \leq k_{\mathrm{j}} \leq k$, $\lambda \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \lambda \frac{k^2-1}{2k^2} < \ln k$. It is known that when $k \leq k_{\mathrm{c}}$, $\ln k < \frac{k^2-1}{k^2}$, and even if $k_{\mathrm{j}} = k$, $\sigma'_{\mathrm{ei}} \geq -\sigma_{\mathrm{y}}$. Therefore, $\lambda \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} < \ln k < \frac{k^2-1}{k^2}$. On the other hand, $\lambda \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \lambda \frac{k^2-1}{2k^2} \leq \lambda_{\max} \frac{k^2-1}{2k^2} = \frac{k^2 \ln k^2}{k^2-1} \frac{k^2-1}{2k^2} = \ln k$.

When based on the maximum distortion strain energy theory, we can also prove that $\lambda \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} < \frac{2}{\sqrt{3}} \ln k < \frac{2}{\sqrt{3}} \frac{k^2-1}{k^2}$.

Figure 5.17 is plotted from $k_{\mathrm{j}\lambda} = 1.1$ to $k_{\mathrm{j}\lambda} = k_{\mathrm{c}}$. In figure 5.17,
for $k_{\mathrm{j}\lambda} = 1.1$ to $k_{\mathrm{j}\lambda} = \mathrm{e}^{0.5}$, $k_{\max} = \infty$, $\lambda_{\max} = \ln k_{\mathrm{j}\lambda}^2 + 1$;
$k_{\mathrm{j}\lambda} = 1.7$, $k_{\max} = 3.811702$, $\lambda_{\max} = 2$;
$k_{\mathrm{j}\lambda} = 1.8$, $k_{\max} = 2.657551$, $\lambda_{\max} = 2$;
$k_{\mathrm{j}\lambda} = 1.9$, $k_{\max} = 2.382195$, $\lambda_{\max} = 2$;
$k_{\mathrm{j}\lambda} = 2$, $k_{\max} = 2.27539$, $\lambda_{\max} = 2$;
$k_{\mathrm{j}\lambda} = 2.1$, $k_{\max} = 2.231732$, $\lambda_{\max} = 2$;
$k_{\mathrm{j}\lambda} = k_{\mathrm{c}}$, $k_{\max} = k_{\mathrm{c}} = k_{\mathrm{j}\lambda}$, $\lambda_{\max} = 2$.

The dash curve o13 is equation (5.9), or $\lambda = \frac{k^2 \ln k^2}{k^2-1} = \frac{k_{\mathrm{c}\lambda}^2 \ln k_{\mathrm{c}\lambda}^2}{k_{\mathrm{c}\lambda}^2-1} = \lambda_{\max}$.

The coordinates of key points are.

1 (1.1, 1.098336); 2 (1.2, 1.193377); 3 (1.3, 1.285205); 4 (1.4, 1.373928); 5 (1.5, 1.459674);
6 (1.6, 1.542576); 7 ($\mathrm{e}^{0.5}$, 1.581977); 8 (1.7, 1.622768); 9 (1.8, 1.700383); 10 (1.9, 1.77555);
11 (2, 1.848392); 12 (2.1, 1.919029); 13 (k_{c}, 2); a (3.811702, 2), $k_{\mathrm{j}\lambda} = 1.7$;
b (2.657551, 2), $k_{\mathrm{j}\lambda} = 1.8$; c (2.382195, 2), $k_{\mathrm{j}\lambda} = 1.9$;
d (2.27539, 2), $k_{\mathrm{j}\lambda} = 2$; f (2.231732, 2), $k_{\mathrm{j}\lambda} = 2.1$; o (1, 1).

In figure 5.17, the dash curves have no practical significance, they are an extension of solid curves, or only the elongation of the mathematical relation expressed by equation (5.114). For curve $k_{\mathrm{j}\lambda} = k_{\mathrm{c}}$, the solid part is only a point, *i.e.*, point 13 (k_{c}, 2).

If based on the maximum shear stress theory (Tresca yield criterion), when $k = 1.6$ and $\lambda = 1.2$, from equation (5.6), $k_{\mathrm{j}\lambda} = 1.114108$; from equation (5.9), $k_{\mathrm{c}\lambda} = 1.207098 < k$ and $\lambda_{\max} = \frac{k^2 \ln k^2}{k^2-1} = \frac{k_{\mathrm{c}\lambda}^2 \ln k_{\mathrm{c}\lambda}^2}{k_{\mathrm{c}\lambda}^2-1} = 1.542576 > 1.2$. $\lambda \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = 1.2 \frac{1.6^2-1}{2\times 1.6^2} = 0.365625$. $\frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = \ln k_{\mathrm{j}} + \frac{k^2-k_{\mathrm{j}}^2}{2k^2} = \ln 1.114108 + \frac{1.6^2-1.114108^2}{2\times 1.6^2} = 0.365625$. $\ln k = \ln 1.6 = 0.470004$, $\frac{k^2-1}{k^2} = 0.609375$. When $k = 1.6$, $k_{\mathrm{j}} = k_{\mathrm{j}\lambda} = 1.114108$, $p = p_{\mathrm{a}}$ and $\lambda = 1.2$, total stresses based on the maximum shear stress theory (Tresca yield criterion) are plotted in figure 5.18.

If based on the maximum distortion strain energy theory (Mises's yield criterion), when $k = 1.6$ and $\lambda = 1.2$, from equation (5.6), $k_{\mathrm{j}\lambda} = 1.114108$; from

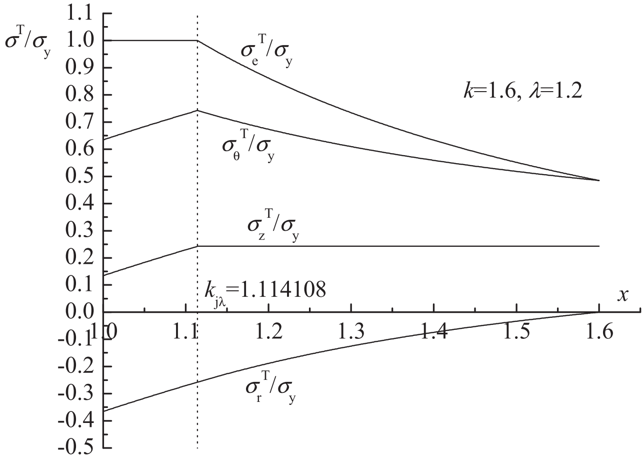

FIG. 5.18 – The total stresses when $k_j = k_{j\lambda}$ and $k = 1.6$, $\lambda = 1.2$ under Tresca yield criterion.

equation (5.9), $k_{c\lambda} = 1.207098 < k$. $\lambda\frac{p_e}{\sigma_y} = 1.2\frac{1.6^2-1}{\sqrt{3}\times 1.6^2} = 0.422187$. $\frac{p_a}{\sigma_y} = \frac{2}{\sqrt{3}}\ln k_j + \frac{k^2-k_j^2}{\sqrt{3}k^2} = \frac{2}{\sqrt{3}}\ln 1.114108 + \frac{1.6^2-1.114108^2}{\sqrt{3}\times 1.6^2} = 0.422187$. $\frac{2}{\sqrt{3}}\ln k = 0.542713$, $\frac{2}{\sqrt{3}}\frac{k^2-1}{k^2} = 0.703646$. When $k = 1.6$, $k_j = k_{j\lambda} = 1.114108$, $p = p_a$ and $\lambda = 1.2$, total stresses based on the maximum distortion strain energy theory (Mises's yield criterion) are plotted in figure 5.19.

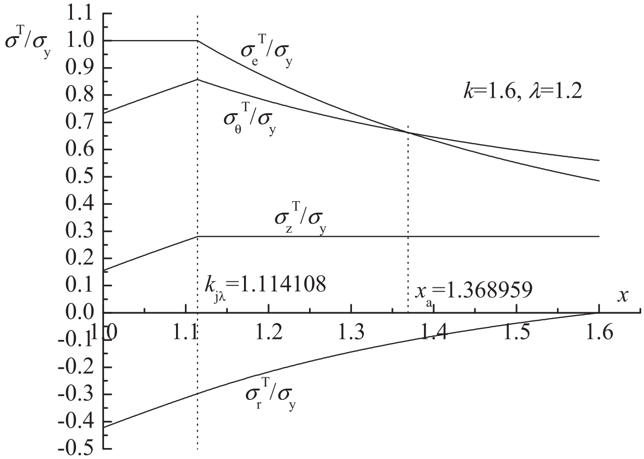

FIG. 5.19 – The total stresses when $k_j = k_{j\lambda}$ and $k = 1.6$, $\lambda = 1.2$ under Mises yield criterion.

If based on the maximum shear stress theory (Tresca yield criterion), when $k = 2$ and $\lambda = 1.2$, from equation (5.6), $k_{j\lambda} = 1.109443$; from equation (5.9), $k_{c\lambda} = 1.207098 < k$. $\lambda\frac{p_e}{\sigma_y} = 1.2\frac{2^2-1}{2\times 2^2} = 0.45$. $\frac{p_a}{\sigma_y} = 0.45$. $\ln k = \ln 2 = 0.693147$,

$\frac{k^2-1}{k^2} = 0.75$. When $k = 2$, $k_\mathrm{j} = k_{\mathrm{j}\lambda} = 1.109443$, $p = p_\mathrm{a}$ and $\lambda = 1.2$, total stresses based on the maximum shear stress theory (Tresca yield criterion) are plotted in figure 5.20.

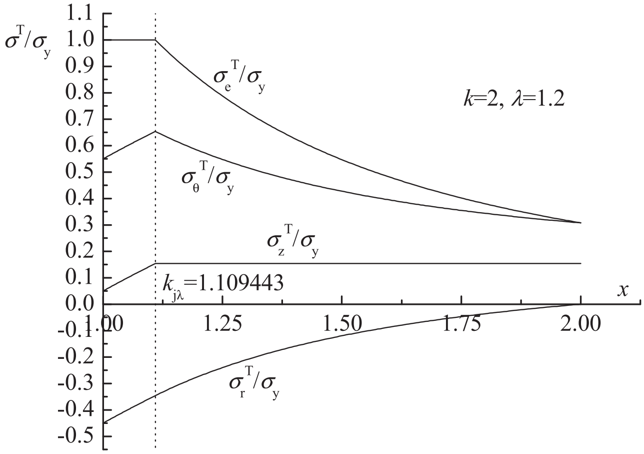

FIG. 5.20 – The total stresses when $k_\mathrm{j} = k_{\mathrm{j}\lambda}$ and $k = 2$, $\lambda = 1.2$ under Tresca yield criterion.

If based on the maximum distortion strain energy theory (Mises's yield criterion), when $k = 2$ and $\lambda = 1.2$, from equation (5.6), $k_{\mathrm{j}\lambda} = 1.109443$; from equation (5.9), $k_{\mathrm{c}\lambda} = 1.207098 < k$. $\lambda\frac{p_\mathrm{e}}{\sigma_\mathrm{y}} = 1.2\frac{2^2-1}{\sqrt{3}\times 2^2} = 0.519615$. $\frac{p_\mathrm{a}}{\sigma_\mathrm{y}} = 0.519615$. $\frac{2}{\sqrt{3}}\ln k = 0.800377$, $\frac{2}{\sqrt{3}}\frac{k^2-1}{k^2} = 0.866025$. When $k = 2$, $k_\mathrm{j} = k_{\mathrm{j}\lambda} = 1.109443$, $p = p_\mathrm{a}$ and $\lambda = 1.2$, total stresses based on the maximum s distortion strain energy theory (Mises's yield criterion) are plotted in figure 5.21.

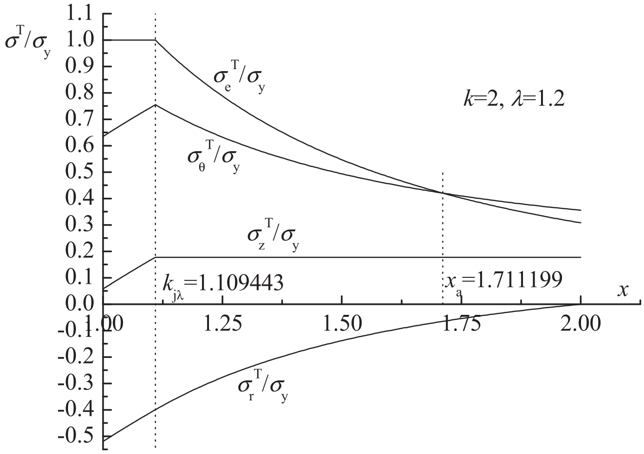

FIG. 5.21 – The total stresses when $k_\mathrm{j} = k_{\mathrm{j}\lambda}$ and $k = 2$, $\lambda = 1.2$ under Mises yield criterion.

5.6 Analysis of Load-Bearing Capacity

5.6.1 Based on Tresca Criterion

For a pressure vessel with k and $k_{\mathrm{j}\lambda}$, its load-bearing capacity $p = \lambda p_{\mathrm{e}}$. When based on the maximum shear stress theory (Tresca criterion), the load-bearing capacity is plotted in figure 5.22, where the coordinates of relevant points are o (1, 0), a (1.2071…, 0.18822…), b (1.43005…, 0.357709…), c (1.671272…, 0.513585…), d (1.93322…, 0.659187…), e (2.2184574899167…, 0.796812…).

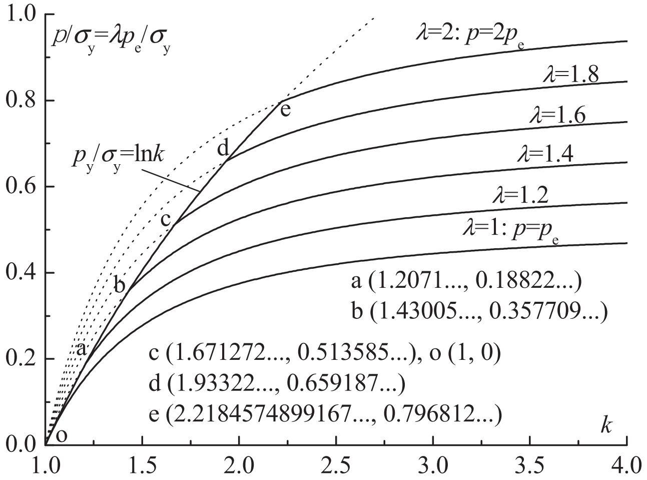

FIG. 5.22 – Load-bearing capacity for a pressure vessel with k and $k_{\mathrm{j}\lambda}$ based on the maximum shear stress theory.

Given the data in figure 5.22, the following may be drawn to show the significance and applications of figure 5.22.

When $\lambda = 1.2$, $k_{\mathrm{c}\lambda} = 1.2071\ldots$, if $k \le k_{\mathrm{c}\lambda}$, $|\sigma'_{\mathrm{ei}}| \le \sigma_{\mathrm{y}}$ irrespective of k_{j} even if $k_{\mathrm{j}} = k$; if $k_{\mathrm{j}} = k < k_{\mathrm{c}\lambda}$, $p = p_{\mathrm{y}} = \sigma_{\mathrm{y}} \ln k$ $(<\lambda p_{\mathrm{e}})$; if $k_{\mathrm{j}} = k = k_{\mathrm{c}\lambda}$, $p = p_{\mathrm{y}} = \sigma_{\mathrm{y}} \ln k = \lambda p_{\mathrm{e}}$; if $k \ge k_{\mathrm{c}\lambda}$, $|\sigma'_{\mathrm{ei}}| \le \sigma_{\mathrm{y}}$ for $k_{\mathrm{j}} \le k_{\mathrm{j}\lambda}$, if $k_{\mathrm{j}} = k_{\mathrm{j}\lambda}$, $p = \lambda p_{\mathrm{e}} = 1.2 p_{\mathrm{e}}$ $(<\sigma_{\mathrm{y}} \ln k)$ and $\sigma^{\mathrm{T}}_{\mathrm{ej}} \equiv \sigma_{\mathrm{y}}$ in the whole plastic region.

When $\lambda = 1.4$, $k_{\mathrm{c}\lambda} = 1.43005\ldots$, if $k \le k_{\mathrm{c}\lambda}$, $|\sigma'_{\mathrm{ei}}| \le \sigma_{\mathrm{y}}$ irrespective of k_{j} even if $k_{\mathrm{j}} = k$; if $k_{\mathrm{j}} = k < k_{\mathrm{c}\lambda}$, $p = p_{\mathrm{y}} = \sigma_{\mathrm{y}} \ln k$ $(<\lambda p_{\mathrm{e}})$; if $k_{\mathrm{j}} = k = k_{\mathrm{c}\lambda}$, $p = p_{\mathrm{y}} = \sigma_{\mathrm{y}} \ln k = \lambda p_{\mathrm{e}}$; if $k \ge k_{\mathrm{c}\lambda}$, $|\sigma'_{\mathrm{ei}}| \le \sigma_{\mathrm{y}}$ for $k_{\mathrm{j}} \le k_{\mathrm{j}\lambda}$, if $k_{\mathrm{j}} = k_{\mathrm{j}\lambda}$, $p = \lambda p_{\mathrm{e}} = 1.4 p_{\mathrm{e}}$ $(<\sigma_{\mathrm{y}} \ln k)$ and $\sigma^{\mathrm{T}}_{\mathrm{ej}} \equiv \sigma_{\mathrm{y}}$ in the whole plastic region.

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When $\lambda = 2$, $k_{\mathrm{c}\lambda} = k_{\mathrm{c}}$, if $k \le k_{\mathrm{c}}$, $|\sigma'_{\mathrm{ei}}| \le \sigma_{\mathrm{y}}$ irrespective of k_{j} even if $k_{\mathrm{j}} = k$; if $k_{\mathrm{j}} = k < k_{\mathrm{c}}$, $p = p_{\mathrm{y}} = \sigma_{\mathrm{y}} \ln k$ $(<2 p_{\mathrm{e}} = \sigma_{\mathrm{y}}\ (k^2 - 1)/k^2)$; if $k_{\mathrm{j}} = k = k_{\mathrm{c}}$, $p = p_{\mathrm{y}} =$

$\sigma_y \ln k = 2p_e$; if $k \geq k_c$, $|\sigma'_{ei}| \leq \sigma_y$ for $k_j \leq k_{j*}$, if $k_j = k_{j*}$, $\frac{p}{\sigma_y} = \frac{k^2-1}{k^2} = 2\frac{p_e}{\sigma_y}$ ($<\ln k$) and $\sigma_{ej}^{T} \equiv \sigma_y$ in the whole plastic region.

Substituting equation (5.9) into $\frac{p}{\sigma_y} = \frac{k^2-1}{2k^2} = \frac{p_e}{\sigma_y}$ results in $p = \sigma_y \ln k_{c\lambda}$. It is easy to prove that when $k \leq k_{c\lambda}$, $\ln k \leq \frac{k^2-1}{k^2}$.

When $k \leq k_c$, $\lambda_{\max} = \frac{k^2 \ln k^2}{k^2-1}$ (This is equation (5.9), and this can also be seen from figure 5.16), from equation (5.4), $k^2 = \frac{\lambda\sigma_y}{\lambda\sigma_y - 2p}$, combining these two equations obtains the maximum reinforcing coefficient for a certain p when $k = k_{c\lambda}$:

$$\lambda_{\max} = \frac{e^{2p/\sigma_y}}{e^{2p/\sigma_y} - 1} \frac{2p}{\sigma_y} \tag{5.117}$$

For a certain p, when $\lambda < \lambda_{\max}$, $k_{c\lambda} < k$. When $\lambda > \lambda_{\max}$, $k < k_{c\lambda}$, which is of meaningless in practice. $\lambda_{\max}$ increases as p increases. When $p = 0$, $\lambda_{\max} = 1$. Letting $\lambda_{\max} \leq 2$ obtains

$$\frac{p}{\sigma_y} e^{2p/\sigma_y} - e^{2p/\sigma_y} + 1 \leq 0 \tag{5.118}$$

The solution of equation (5.118) is $\frac{p}{\sigma_y} \leq 0.796812$, which is just the vertical coordinate of point e in figure 5.14. When $\frac{p}{\sigma_y} > 0.796812$, there is always $k < k_{c\lambda}$. Thus, $0 \leq \frac{p}{\sigma_y} \leq 0.796812$.

(1) Given k and λ

From equation (5.9), $\lambda_{\max} = \frac{k^2 \ln k^2}{k^2-1}$. When $k > k_c$, $\lambda_{\max} = 2$. If the given $\lambda > \lambda_{\max}$, there is no solution for $k_{j\lambda}$ (see figure 5.16); if the given $\lambda < \lambda_{\max}$, then, the optimum load is $\frac{p}{\sigma_y} = \lambda \frac{p_e}{\sigma_y} = \lambda \frac{k^2-1}{2k^2} = \frac{p_a}{\sigma_y}$. $\frac{d\lambda_{\max}}{dk} > 0$, so, if the given $\lambda > \lambda_{\max}$, then, $k < k_{c\lambda}$. If $\frac{p}{\sigma_y} = \ln k$ is taken as the load ($k_j = k$ at the same time), $\sigma_e^{T} \equiv \sigma_y$ in the plastic region, but $\frac{p}{\sigma_y} = \ln k < \lambda \frac{k^2-1}{2k^2}$; If $\lambda \frac{k^2-1}{2k^2}$ is taken as the load, $\sigma_e^{T} > \sigma_y$.

(I) $k = 1.6 < k_c$, $\lambda = 1.2$

$\lambda_{\max} = 1.542576 > 1.2$. Then, $\frac{p}{\sigma_y} = \lambda \frac{p_e}{\sigma_y} = \lambda \frac{k^2-1}{2k^2} = 0.365625 < \frac{p}{\sigma_y} = \ln k = 0.470004$. When $k = 1.6$, $\lambda = 1.2$, from equation (5.6), $k_{j\lambda} = 1.114108$; When $\lambda = 1.2$, from equation (5.9), $k_{c\lambda} = 1.207098 < k$.

When $k = 1.6$, $k_j = k_{j\lambda} = 1.114108$, $p = p_a$ and $\lambda = 1.2$, total stresses based on the maximum shear stress theory (Tresca yield criterion) has been plotted in figure 5.18.

(II) $k = 1.6 < k_c$, $\lambda = 1.55$

$\lambda_{\max} = 1.542576 < 1.55$. $\frac{p}{\sigma_y} = \ln k = 0.470004$; $\lambda \frac{k^2-1}{2k^2} = 1.55 \frac{1.6^2-1}{2 \times 1.6^2} = 0.472266$. Letting $\frac{p}{\sigma_y} = \ln k = 0.470004 = \lambda \frac{k^2-1}{2k^2}$ obtains $\lambda = \lambda_{\max} = \frac{k^2 \ln k^2}{k^2-1} = 1.542576$.

When $\lambda = 1.55$, from equation (5.9), $k_{c\lambda} = 1.60912 > k$.

When $k_{\mathrm{j}} = k$ and $p = p_{\mathrm{y}} = \sigma_{\mathrm{y}}\ln k$, the total stresses are equations (2.70)–(2.73), where $\sigma_{\mathrm{e}}^{\mathrm{T}} \equiv \sigma_{\mathrm{y}}$ in the plastic region. Since $\lambda\frac{k^2-1}{2k^2} = 1.55\frac{1.6^2-1}{2\times1.6^2} > \frac{p}{\sigma_{\mathrm{y}}} = \ln k$, if $\lambda\frac{k^2-1}{2k^2}$ is taken as the load, $\sigma_{\mathrm{e}}^{\mathrm{T}} > \sigma_{\mathrm{y}}$.
In fact, when $\lambda > \lambda_{\max}$, the maximum plastic depth is $k_{\mathrm{j}} = k$. When $k_{\mathrm{j}} = k$, from equation (2.60), the equivalent residual stress on inner surface is $\frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2\ln k^2}{k^2-1} = -0.542576$. When $\frac{p}{\sigma_{\mathrm{y}}} = \lambda\frac{k^2-1}{2k^2}$, from equation (5.57), the equivalent operation stress on inner surface is $\frac{\sigma_{\mathrm{ei}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \lambda = 1.55$. Then, the equivalent total stress on inner surface is $\frac{\sigma_{\mathrm{ei}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1.55–0.542576 = 1.007424 > 1$.

(III) $k = 2.5 > k_{\mathrm{c}}$.
$\lambda_{\max} = \frac{k^2\ln k^2}{k^2-1} = 2.181645$. When $k > k_{\mathrm{c}}$, $\lambda_{\max} = 2$. When $k = 2.5$, $\lambda = \lambda_{\max} = 2$, from equation (5.6), $k_{\mathrm{j}\lambda} = k_{\mathrm{j}*} = 1.844363$. When $k_{\mathrm{j}} = k_{\mathrm{j}*}$, the optimum load is $\frac{p}{\sigma_{\mathrm{y}}} = \lambda\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \lambda\frac{k^2-1}{2k^2} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = 0.84 > 0.796812$.

(2) Given k

If $k > k_{\mathrm{c}}$, $\lambda_{\max} = 2$. If $k < k_{\mathrm{c}}$, from equation (5.9), $\lambda_{\max} = \frac{k^2\ln k^2}{k^2-1} = \frac{k_{\mathrm{c}\lambda}^2\ln k_{\mathrm{c}\lambda}^2}{k_{\mathrm{c}\lambda}^2-1}$ (When $k > k_{\mathrm{c}}$, $\lambda_{\max} > 2$; When $k < k_{\mathrm{c}}$, $\lambda_{\max} < 2$.). Between $1 < \lambda < \lambda_{\max}$, select a λ to meet design requirement. When $k > k_{\mathrm{c}}$, if the selected $\lambda > 2$, $\sigma'_{\mathrm{ei}} < -\sigma_{\mathrm{y}}$. When $k < k_{\mathrm{c}}$, if $\lambda > \lambda_{\max}$ is selected, the load-bearing capacity can not reach $\frac{p}{\sigma_{\mathrm{y}}} = \lambda\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \lambda\frac{k^2-1}{2k^2}$ under the given k, because when $k < k_{\mathrm{c}\lambda}$, the maximum allowable load is $\frac{p}{\sigma_{\mathrm{y}}} = \ln k$, and when $k < k_{\mathrm{c}}$, $\ln k \le \lambda\frac{k^2-1}{2k^2}$.

(I) $k = 1.6 < k_{\mathrm{c}}$
$\lambda_{\max} = 1.542576$.

(A) Select $\lambda = \lambda_{\max} = 1.542576$, $\ln k = \lambda\frac{k^2-1}{2k^2} = 0.470004$. $\lambda = \lambda_{\max}$ indicates $k_{\mathrm{j}} = k_{\mathrm{j}\lambda} = k$.

(B) Select $\lambda = 1.4$, $\ln k = 0.470004 > \lambda\frac{k^2-1}{2k^2} = 0.426563$.

(C) Select $\lambda = 1.6$, $\ln k = 0.470004 < \lambda\frac{k^2-1}{2k^2} = 0.4875$. When $\lambda > \lambda_{\max}$, the maximum plastic depth is $k_{\mathrm{j}} = k$. When $k_{\mathrm{j}} = k$, from equation (2.60), the equivalent residual stress on inner surface is $\frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2\ln k^2}{k^2-1} = -0.542576$. When $\frac{p}{\sigma_{\mathrm{y}}} = \lambda\frac{k^2-1}{2k^2}$, from equation (5.57), the equivalent operation stress on inner surface is $\frac{\sigma_{\mathrm{ei}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \lambda = 1.6$. Then, the equivalent total stress on inner surface is $\frac{\sigma_{\mathrm{ei}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1.6–0.542576 = 1.057424 > 1$.

(II) $k = 2.5 > k_{\mathrm{c}}$
$\lambda_{\max} = \frac{k^2\ln k^2}{k^2-1} = 2.181645$. When $k > k_{\mathrm{c}}$, $\lambda_{\max} = 2$. When $k = 2.5$, $\lambda = \lambda_{\max} = 2$, from equation (5.6), $k_{\mathrm{j}\lambda} = k_{\mathrm{j}*} = 1.844363$. When $k_{\mathrm{j}} = k_{\mathrm{j}*}$, the optimum load is $\frac{p}{\sigma_{\mathrm{y}}} = \lambda\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \lambda\frac{k^2-1}{2k^2} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = 0.84 > 0.796812$.

If $\lambda = 1.6$ is selected, from equation (5.6), $k_{\mathrm{j}\lambda} = 1.384516 < k_{\mathrm{j}*} = 1.844363$. $\frac{p}{\sigma_{\mathrm{y}}} = \lambda \frac{k^2-1}{2k^2} = 0.672 < 0.84$.

(3) Given p and λ

From equation (5.117), $\lambda_{\max} = \frac{\mathrm{e}^{2p/\sigma_{\mathrm{y}}}}{\mathrm{e}^{2p/\sigma_{\mathrm{y}}}-1} \frac{2p}{\sigma_{\mathrm{y}}}$. If the given $\lambda > \lambda_{\max}$, there is no solution; if the given $\lambda < \lambda_{\max}$, use $\frac{p}{\sigma_{\mathrm{y}}} = \lambda \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \lambda \frac{k^2-1}{2k^2}$ to determine k. When the given $p = 0.796812\sigma_{\mathrm{y}}$, $\lambda_{\max} = 2$; When the given $p < 0.796812\sigma_{\mathrm{y}}$, $\lambda_{\max} < 2$; When the given $p > 0.796812\sigma_{\mathrm{y}}$, $\lambda_{\max} > 2$.

(4) Given p

From equation (5.117), $\lambda_{\max} = \frac{\mathrm{e}^{2p/\sigma_{\mathrm{y}}}}{\mathrm{e}^{2p/\sigma_{\mathrm{y}}}-1} \frac{2p}{\sigma_{\mathrm{y}}}$. Between $1 < \lambda < \lambda_{\max}$, select a λ to meet design requirement.

5.6.2 Based on Mises Criterion

When based on the maximum distortion strain energy theory (Mises's yield criterion), the load-bearing capacity is plotted in figure 5.23, where the coordinates of relevant points are o (1, 0), a (1.2071…, 0.217338…), b (1.43005…, 0.413047…), c (1.671272…, 0.593037…), d (1.93322…, 0.761164…), e (2.2184574899167…, 0.920079…).

Given the data in figure 5.23, the following may be drawn to show the significance and applications of figure 5.23.

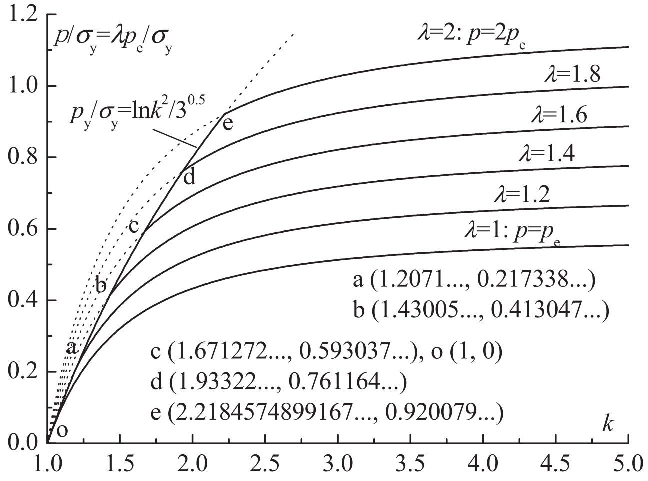

FIG. 5.23 – Load-bearing capacity for a pressure vessel with k and $k_{\mathrm{j}\lambda}$ based on the maximum distortion strain energy theory.

When $\lambda = 1.2$, $k_{\mathrm{c}\lambda} = 1.2071\ldots$, if $k \leq 1.2071\ldots = k_{\mathrm{c}\lambda}$, $|\sigma'_{\mathrm{ei}}| \leq \sigma_{\mathrm{y}}|$ irrespective of k_{j} even if $k_{\mathrm{j}} = k$; if $k_{\mathrm{j}} = k < k_{\mathrm{c}\lambda}$, $p = p_{\mathrm{y}} = \sigma_{\mathrm{y}}\ln k$ ($<\lambda p_{\mathrm{e}}$); if $k_{\mathrm{j}} = k = k_{\mathrm{c}\lambda}$, $p = p_{\mathrm{y}} = \sigma_{\mathrm{y}}\ln k = \lambda p_{\mathrm{e}}$; if $k \geq 1.2071\ldots$, $|\sigma'_{\mathrm{ei}}| \leq \sigma_{\mathrm{y}}$ for $k_{\mathrm{j}} \leq k_{\mathrm{j}\lambda}$, if $k_{\mathrm{j}} = k_{\mathrm{j}\lambda}$, $p = \lambda p_{\mathrm{e}} = 1.2p_{\mathrm{e}}$ ($<\sigma_{\mathrm{y}}\ln k$) and $\sigma^{\mathrm{T}}_{\mathrm{ej}} \equiv \sigma_{\mathrm{y}}$ in the whole plastic region.

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When $\lambda = 1.4$, $k_{\mathrm{c}\lambda} = 1.43005\ldots$, if $k \leq k_{\mathrm{c}\lambda}$, $|\sigma'_{\mathrm{ei}}| \leq \sigma_{\mathrm{y}}$ irrespective of k_{j} even if $k_{\mathrm{j}} = k$; if $k_{\mathrm{j}} = k < k_{\mathrm{c}\lambda}$, $p = p_{\mathrm{y}} = \sigma_{\mathrm{y}}\ln k$ ($<\lambda p_{\mathrm{e}}$); if $k_{\mathrm{j}} = k = k_{\mathrm{c}\lambda}$, $p = p_{\mathrm{y}} = \sigma_{\mathrm{y}}\ln k = \lambda p_{\mathrm{e}}$; if $k \geq 1.43005\ldots$, $|\sigma'_{\mathrm{ei}}| \leq \sigma_{\mathrm{y}}$ for $k_{\mathrm{j}} \leq k_{\mathrm{j}\lambda}$, if $k_{\mathrm{j}} = k_{\mathrm{j}\lambda}$, $p = \lambda p_{\mathrm{e}} = 1.4p_{\mathrm{e}}$ ($<\sigma_{\mathrm{y}}\ln k$) and $\sigma^{\mathrm{T}}_{\mathrm{ej}} \equiv \sigma_{\mathrm{y}}$ in the whole plastic region.

……

When $\lambda = 2$, k_{c} is $k_{\mathrm{c}\lambda}$, if $k \leq k_{\mathrm{c}}$, $|\sigma'_{\mathrm{ei}}| \leq \sigma_{\mathrm{y}}$ irrespective of k_{j} even if $k_{\mathrm{j}} = k$; if $k_{\mathrm{j}} = k < k_{\mathrm{c}}$, $p = p_{\mathrm{y}} = \sigma_{\mathrm{y}}\ln k$ ($<2p_{\mathrm{e}} = \sigma_{\mathrm{y}} \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$); if $k_{\mathrm{j}} = k = k_{\mathrm{c}}$, $p = p_{\mathrm{y}} = \sigma_{\mathrm{y}}\ln k = 2$ p_{e}; if $k \geq k_{\mathrm{c}}$, $|\sigma'_{\mathrm{ei}}| \leq \sigma_{\mathrm{y}}$ for $k_{\mathrm{j}} \leq k_{\mathrm{j}*}$, if $k_{\mathrm{j}} = k_{\mathrm{j}*}$, $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2} = 2\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$ ($<\ln k$) and $\sigma^{\mathrm{T}}_{\mathrm{ej}} \equiv \sigma_{\mathrm{y}}$ in the whole plastic region.

Substituting equation (5.9) into $\frac{p}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{\sqrt{3}k^2} = \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$ results in $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\ln k_{\mathrm{c}\lambda}$. It is easy to prove that when $k \leq k_{\mathrm{c}\lambda}$, $\frac{2}{\sqrt{3}}\ln k \leq \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2}$.

When $k \leq k_{\mathrm{c}}$, $\lambda_{\max} = \frac{k^2 \ln k^2}{k^2-1}$ (This is equation (5.9), and this can also be seen from figure 5.16). When $k = k_{\mathrm{c}\lambda}$, $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\ln k_{\mathrm{c}\lambda} = \frac{\ln k^2}{\sqrt{3}}$, or $\ln k^2 = \sqrt{3}\frac{p}{\sigma_{\mathrm{y}}}$ or $k^2 = \mathrm{e}^{\frac{\sqrt{3}p}{\sigma_{\mathrm{y}}}}$. Thus, the maximum reinforcing coefficient for a certain p when $k = k_{\mathrm{c}\lambda}$ is

$$\lambda_{\max} = \frac{\mathrm{e}^{\sqrt{3}p/\sigma_{\mathrm{y}}}}{\mathrm{e}^{\sqrt{3}p/\sigma_{\mathrm{y}}} - 1}\frac{\sqrt{3}p}{\sigma_{\mathrm{y}}} \tag{5.119}$$

For a certain p, when $\lambda < \lambda_{\max}$, $k > k_{\mathrm{c}\lambda}$; when $\lambda > \lambda_{\max}$, $k < k_{\mathrm{c}\lambda}$, which is of meaningless in practice. $\lambda_{\max}$ increases as p increases. When $p = 0$, $\lambda_{\max} = 1$. Letting $\lambda_{\max} \leq 2$ obtains

$$\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}}\mathrm{e}^{\sqrt{3}p/\sigma_{\mathrm{y}}} - \mathrm{e}^{\sqrt{3}p/\sigma_{\mathrm{y}}} + 1 \leq 0 \tag{5.120}$$

The solution of equation (5.120) is $\frac{p}{\sigma_{\mathrm{y}}} \leq 0.920079$, which is just the vertical coordinate of point e in figure 5.14a. When $\frac{p}{\sigma_{\mathrm{y}}} > 0.920079$, there is always $k < k_{\mathrm{c}\lambda}$. Thus, $0 \leq \frac{p}{\sigma_{\mathrm{y}}} \leq 0.920079$.

(1) Given k and λ

From equation (5.9), $\lambda_{\max} = \frac{k^2 \ln k^2}{k^2-1}$. When $k > k_{\mathrm{c}}$, $\lambda_{\max} = 2$. If the given $\lambda > \lambda_{\max}$, there is no solution for $k_{\mathrm{j}\lambda}$ (see figure 5.16); if the given $\lambda < \lambda_{\max}$, then, the optimum load is $\frac{p}{\sigma_{\mathrm{y}}} = \lambda\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \lambda\frac{k^2-1}{\sqrt{3}k^2} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}}$. $\frac{\mathrm{d}\lambda_{\max}}{\mathrm{d}k} > 0$, so, if the given $\lambda > \lambda_{\max}$, then, $k < k_{\mathrm{c}\lambda}$. If $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\ln k$ is taken as the load ($k_{\mathrm{j}} = k$ at the same time), $\sigma^{\mathrm{T}}_{\mathrm{e}} \equiv \sigma_{\mathrm{y}}$ in the plastic region, but $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\ln k < \lambda\frac{k^2-1}{\sqrt{3}k^2}$; If $\lambda\frac{k^2-1}{\sqrt{3}k^2}$ is taken as the load, $\sigma^{\mathrm{T}}_{\mathrm{e}} > \sigma_{\mathrm{y}}$.

(I) $k = 1.6 < k_{\mathrm{c}}$, $\lambda = 1.2$

$\lambda_{\max} = 1.542576 > 1.2$. Then, $\frac{p}{\sigma_{\mathrm{y}}} = \lambda \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \lambda \frac{k^2-1}{\sqrt{3}k^2} = 0.422187 < \frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}} \ln k = 0.542713$. When $k = 1.6$, $\lambda = 1.2$, from equation (5.6), $k_{\mathrm{j}\lambda} = 1.114108$; when $\lambda = 1.2$, from equation (5.9), $k_{\mathrm{c}\lambda} = 1.207098 < k$.

When $k = 1.6$, $k_{\mathrm{j}} = k_{\mathrm{j}\lambda} = 1.114108$, $p = p_{\mathrm{a}}$ and $\lambda = 1.2$, total stresses based on the maximum distortion strain energy theory (Mises's yield criterion) has been plotted in figure 5.19.

(II) $k = 1.6 < k_{\mathrm{c}}$, $\lambda = 1.55$

$\lambda_{\max} = 1.542576 < 1.55$. $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}} \ln k = 0.542713$; $\lambda \frac{k^2-1}{\sqrt{3}k^2} = 1.55 \frac{1.6^2-1}{\sqrt{3}\times 1.6^2} = 0.545325$. Letting $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}} \ln k = 0.542713 = \lambda \frac{k^2-1}{\sqrt{3}k^2}$ just obtains $\lambda = \lambda_{\max} = \frac{k^2 \ln k^2}{k^2-1} = 1.542576$.

When $\lambda = 1.55$, from equation (5.9), $k_{\mathrm{c}\lambda} = 1.60912 > k$.

When $k_{\mathrm{j}} = k$ and $p = p_{\mathrm{y}} = \sigma_{\mathrm{y}} \ln k$, the total stresses are equations (2.70)–(2.73), where $\sigma_{\mathrm{e}}^{\mathrm{T}} \equiv \sigma_{\mathrm{y}}$ in the plastic region. Since $\lambda \frac{k^2-1}{\sqrt{3}k^2} = 1.55 \frac{1.6^2-1}{\sqrt{3}\times 1.6^2} > \frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}} \ln k$, if $\lambda \frac{k^2-1}{\sqrt{3}k^2}$ is taken as the load, $\sigma_{\mathrm{e}}^{\mathrm{T}} > \sigma_{\mathrm{y}}$. In fact, when $\lambda > \lambda_{\max}$, the maximum plastic depth is $k_{\mathrm{j}} = k$. When $k_{\mathrm{j}} = k$, from equation (3.46), the equivalent residual stress on inner surface is $\frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2 \ln k^2}{k^2-1} = -0.542576$. When $\frac{p}{\sigma_{\mathrm{y}}} = \lambda \frac{k^2-1}{\sqrt{3}k^2}$, from equation (5.86), the equivalent operation stress on inner surface is $\frac{\sigma_{\mathrm{ei}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \lambda = 1.55$. Then, the equivalent total stress on inner surface is $\frac{\sigma_{\mathrm{ei}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = 1.55 - 0.542576 = 1.007424 > 1$.

(III) $k = 2.5 > k_{\mathrm{c}}$

$\lambda_{\max} = \frac{k^2 \ln k^2}{k^2-1} = 2.181645$. When $k > k_{\mathrm{c}}$, $\lambda_{\max} = 2$. When $k = 2.5$, $\lambda = \lambda_{\max} = 2$, from equation (5.6), $k_{\mathrm{j}\lambda} = k_{\mathrm{j}*} = 1.844363$. When $k_{\mathrm{j}} = k_{\mathrm{j}*}$, the optimum load is $\frac{p}{\sigma_{\mathrm{y}}} = \lambda \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \lambda \frac{k^2-1}{\sqrt{3}k^2} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = 0.969948 > 0.920079$.

(2) Given k

If $k > k_{\mathrm{c}}$, $\lambda_{\max} = 2$. If $k < k_{\mathrm{c}}$, from equation (5.9), $\lambda_{\max} = \frac{k^2 \ln k^2}{k^2-1} = \frac{k_{\mathrm{c}\lambda}^2 \ln k_{\mathrm{c}\lambda}^2}{k_{\mathrm{c}\lambda}^2-1}$ (When $k > k_{\mathrm{c}}$, $\lambda_{\max} > 2$; When $k < k_{\mathrm{c}}$, $\lambda_{\max} < 2$.). Between $1 < \lambda < \lambda_{\max}$, select a λ to meet design requirement. When $k > k_{\mathrm{c}}$, if the selected $\lambda > 2$, $\sigma'_{\mathrm{ei}} < -\sigma_{\mathrm{y}}$. When $k < k_{\mathrm{c}}$, if $\lambda > \lambda_{\max}$ is selected, the load-bearing capacity can not reach $\frac{p}{\sigma_{\mathrm{y}}} = \lambda \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \lambda \frac{k^2-1}{\sqrt{3}k^2}$ under the given k, because when $k < k_{\mathrm{c}\lambda}$, the maximum allowable load is $\frac{p}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}} \ln k$, and when $k < k_{\mathrm{c}}$, $\frac{2}{\sqrt{3}} \ln k \leq \lambda \frac{k^2-1}{\sqrt{3}k^2}$.

(I) $k = 1.6 < k_{\mathrm{c}}$

$\lambda_{\max} = 1.542576$.

(A) Select $\lambda = \lambda_{\max} = 1.542576$, $\frac{2}{\sqrt{3}} \ln k = \lambda \frac{k^2-1}{\sqrt{3}k^2} = 0.542713$. $\lambda = \lambda_{\max}$ indicates $k_{\mathrm{j}} = k_{\mathrm{j}\lambda} = k$.

(B) Select $\lambda = 1.4$, $\frac{2}{\sqrt{3}} \ln k > \lambda \frac{k^2-1}{\sqrt{3}k^2} = 0.492552$.

(C) Select $\lambda = 1.6$, $\frac{2}{\sqrt{3}}\ln k < \lambda\frac{k^2-1}{\sqrt{3}k^2} = 0.562917$. When $\lambda > \lambda_{\max}$, the maximum plastic depth is $k_{\mathrm{j}} = k$. When $k_{\mathrm{j}} = k$, from equation (3.46), the equivalent residual stress on inner surface is $\frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}} = 1 - \frac{k^2\ln k^2}{k^2-1} = -0.542576$. When $\frac{p}{\sigma_{\mathrm{y}}} = \lambda\frac{k^2-1}{\sqrt{3}k^2}$, from equation (5.86), the equivalent operation stress on inner surface is $\frac{\sigma^{\mathrm{p}}_{\mathrm{ei}}}{\sigma_{\mathrm{y}}} = \lambda = 1.6$. Then, the equivalent total stress on inner surface is $\frac{\sigma^{\mathrm{T}}_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = 1.6 - 0.542576 = 1.057424 > 1$.

(II) $k = 2.5 > k_{\mathrm{c}}$

$\lambda_{\max} = \frac{k^2\ln k^2}{k^2-1} = 2.181645$. When $k > k_{\mathrm{c}}$, $\lambda_{\max} = 2$. When $k = 2.5$, $\lambda = \lambda_{\max} = 2$, from equation (5.6), $k_{\mathrm{j}\lambda} = k_{\mathrm{j}*} = 1.844363$. When $k_{\mathrm{j}} = k_{\mathrm{j}*}$, the optimum load is $\frac{p}{\sigma_{\mathrm{y}}} = \lambda\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \lambda\frac{k^2-1}{\sqrt{3}k^2} = \frac{p_{\mathrm{a}}}{\sigma_{\mathrm{y}}} = 0.969948 > 0.920079$.

If $\lambda = 1.6$ is selected, from equation (5.6), $k_{\mathrm{j}\lambda} = 1.384516 < k_{\mathrm{j}*} = 1.844363$. $\frac{p}{\sigma_{\mathrm{y}}} = \lambda\frac{k^2-1}{\sqrt{3}k^2} = 0.775959 < 0.969948$.

(3) Given p and λ

From equation (5.119), $\lambda_{\max} = \frac{\mathrm{e}^{\sqrt{3}p/\sigma_{\mathrm{y}}}}{\mathrm{e}^{\sqrt{3}p/\sigma_{\mathrm{y}}}-1}\frac{\sqrt{3}p}{\sigma_{\mathrm{y}}}$. If the given $\lambda > \lambda_{\max}$, there is no solution; if the given $\lambda < \lambda_{\max}$, then, use $\frac{p}{\sigma_{\mathrm{y}}} = \lambda\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \lambda\frac{k^2-1}{\sqrt{3}k^2}$ to determine k. When the given $p = 0.920079\sigma_{\mathrm{y}}$, $\lambda_{\max} = 2$; When the given $p < 0.920079\sigma_{\mathrm{y}}$, $\lambda_{\max} < 2$; When the given $p > 0.920079\sigma_{\mathrm{y}}$, $\lambda_{\max} > 2$.

(4) Given p

From equation (5.119), $\lambda_{\max} = \frac{\mathrm{e}^{\sqrt{3}p/\sigma_{\mathrm{y}}}}{\mathrm{e}^{\sqrt{3}p/\sigma_{\mathrm{y}}}-1}\frac{\sqrt{3}p}{\sigma_{\mathrm{y}}}$. Between $1 < \lambda < \lambda_{\max}$, select a λ to meet design requirement.

5.7 Chapter Summary

(1) $\lambda \leq 2$ is required, otherwise, $|\sigma'_{\mathrm{ei}}/\sigma_{\mathrm{y}}| > 1$.

(2) When $k \leq k_{\mathrm{c}\lambda}$, there is always $|\sigma'_{\mathrm{ei}}\sigma_{\mathrm{y}}| < 1$ irrespective of k_{j} even if $k_{\mathrm{j}} = k$, if $k_{\mathrm{j}} = k$, $p = p_{\mathrm{y}} = \sigma_{\mathrm{y}}\ln k\,(<\lambda p_{\mathrm{e}})$; when $k \geq k_{\mathrm{c}\lambda}$, if $k_{\mathrm{j}} \leq k_{\mathrm{j}\lambda}$, $|\sigma'_{\mathrm{ei}}\sigma_{\mathrm{y}}| \leq 1$, when $k_{\mathrm{j}} = k_{\mathrm{j}\lambda}$, $p = \lambda p_{\mathrm{e}}$ $(<\ln k)$. When $\lambda = 2$, $k_{\mathrm{j}\lambda}$ and p reach the maximums, or $k^2\ln k_{\mathrm{j}}^2 - k^2 - k_{\mathrm{j}}^2 + 2 = 0$ and $p = 2p_{\mathrm{e}}$.

(3) When $k \leq k_{\mathrm{c}}$ (or $k_{\mathrm{c}\lambda}$ for $\lambda = 2$), $k_{\mathrm{j}\lambda\max} = k = k_{\mathrm{c}\lambda}$; When $k \geq k_{\mathrm{c}}$, $k_{\mathrm{j}\lambda\max} < k(k_{\mathrm{c}\lambda})$, $k_{\mathrm{j}\lambda\max}$ is determined by $k^2\ln k_{\mathrm{j}\lambda}^2 - k^2 - k_{\mathrm{j}\lambda}^2 + 2 = 0$ ($\lambda = 2$ in equation (5.6)). The greater λ is, the greater $k_{\mathrm{j}\lambda}$ is, if $k_{\mathrm{j}\lambda} < k$.

(4) So long as $p = p_{\mathrm{a}}$, irrespective of k_{j}, $\sigma_{\mathrm{e}} \equiv \sigma_{\mathrm{y}}$ in the whole plastic zone, and σ_{e} in an elastic zone is lower than σ_{y}. However, if k_{j} is outside the quasi-infinite area of sense and possible plastic depth, compressive yield occurs.

(5) Relation between various parameters and their varying tendency become concise and clearer, and the equations concerned with autofrettage are simplified greatly because of the equation $k^2 \ln k_{j\lambda}^2 - (\lambda - 1)k^2 - k_{j\lambda}^2 + \lambda = 0$.

(6) The components of residual stresses based on the maximum shear stress theory (Tresca yield criterion or the 3rd strength theory) are $3^{0.5}/2$ times of those based on the maximum distortion strain energy theory (Mises's yield criterion or the 4th strength theory), but equivalent residual stress based on both strength theories is the same; load-bearing capacity based on the 3rd strength theory is $3^{0.5}/2$ times of those based on the 4th strength theory, because p_e based on the 3rd strength theory is $3^{0.5}/2$ times of p_e based on the 4th strength theory, the ratio to p_e based on both strength theories under the identical λ is the same.

The main equations and conclusion are listed in table 5.1.

Tab. 5.1 – The main equations and conclusion of this chapter.

$$p = \lambda p_e \text{ or } \frac{p}{\sigma_y} = \lambda \frac{p_e}{\sigma_y} = \lambda \frac{k^2 - 1}{2k^2}$$

$$\text{When } k \leq k_c,\ \lambda_{\max} = \frac{k^2 \ln k^2}{k^2 - 1} \text{ or } \lambda_{\max} = \frac{e^{2p/\sigma_y}}{e^{2p/\sigma_y} - 1} \frac{2p}{\sigma_y}$$

$$1 \leq \lambda_{\max} \leq 2, \text{ then,}$$

$$\frac{p}{\sigma_y} e^{2p/\sigma_y} - e^{2p/\sigma_y} + 1 \leq 0$$

$$\text{or} \quad 0 \leq \frac{p}{\sigma_y} \leq 0.796812$$

The optimum plastic depth $k_{j\lambda}$

Based on both the maximum shear stress theory (Tresca criterion) and the maximum distortion strain energy theory (Mises's criterion)

$$k^2 \ln k_{j\lambda}^2 - (\lambda - 1)k^2 - k_{j\lambda}^2 + \lambda = 0 \ (e^{(\lambda-1)/2} \leq k_{j\lambda} \leq k_{c\lambda} \text{ and } k \geq k_{c\lambda})$$

$$\text{or } k^2 = \frac{k_{j\lambda}^2 - \lambda}{\ln k_{j\lambda}^2 - \lambda + 1} \text{ or } \lambda = \frac{k^2 \ln k_{j\lambda}^2 + k^2 - k_{j\lambda}^2}{k^2 - 1}$$

Critical radius ratio ($k_{c\lambda}$) under a certain λ

Based on both the maximum shear stress theory (Tresca criterion) and the maximum distortion strain energy theory (Mises's criterion)

$$k_{c\lambda}^2 \ln k_{c\lambda}^2 - \lambda(k_{c\lambda}^2 - 1) = 0 \text{ or } \lambda = \frac{k_{c\lambda}^2 \ln k_{c\lambda}^2}{k_{c\lambda}^2 - 1}$$

Residual stresses

Based on the maximum shear stress theory (Tresca criterion)

$$\begin{cases} \dfrac{\sigma'_z}{\sigma_y} = \ln x - \dfrac{\lambda - 1}{2} \\ \dfrac{\sigma'_r}{\sigma_y} = \ln x + \dfrac{\lambda}{2x^2} - \dfrac{\lambda}{2} \\ \dfrac{\sigma'_\theta}{\sigma_y} = \ln x - \dfrac{\lambda}{2x^2} - \dfrac{\lambda}{2} + 1 \\ \dfrac{\sigma'_e}{\sigma_y} = 1 - \dfrac{\lambda}{x^2} \end{cases} (1 \leq x \leq k_{j\lambda}) \begin{cases} \dfrac{\sigma'_z}{\sigma_y} \equiv \dfrac{k_{j\lambda}^2 - \lambda}{2k^2} = \text{constant} \\ \dfrac{\sigma'_r}{\sigma_y} = \left(1 - \dfrac{k^2}{x^2}\right) \dfrac{\sigma'_z}{\sigma_y} = \dfrac{k_{j\lambda}^2 - \lambda}{2k^2} - \dfrac{k_{j\lambda}^2 - \lambda}{2x^2} \\ \dfrac{\sigma'_\theta}{\sigma_y} = \left(1 + \dfrac{k^2}{x^2}\right) \dfrac{\sigma'_z}{\sigma_y} = \dfrac{k_{j\lambda}^2 - \lambda}{2k^2} + \dfrac{k_{j\lambda}^2 - \lambda}{2x^2} \\ \dfrac{\sigma'_e}{\sigma_y} = \dfrac{k_{j\lambda}^2 - \lambda}{x^2} \end{cases} (k_{j\lambda} \leq x \leq k)$$

Tab. 5.1 – (continued).

On the inner surface,

$$\frac{\sigma'_{\mathrm{zi}}}{\sigma_{\mathrm{y}}} = -\frac{\lambda-1}{2}, \quad \frac{\sigma'_{\mathrm{ri}}}{\sigma_{\mathrm{y}}} = 0, \quad \frac{\sigma'_{\theta\mathrm{i}}}{\sigma_{\mathrm{y}}} = 1-\lambda = 2\frac{\sigma'_{\mathrm{zi}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}}, \quad \frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}} = 1-\lambda$$

On the outer surface,

$$\frac{\sigma'_{\mathrm{zo}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2-\lambda}{2k^2}, \quad \sigma'_{\mathrm{ro}} = 0, \quad \frac{\sigma'_{\theta\mathrm{o}}}{\sigma_{\mathrm{y}}} = 2\frac{\sigma'_{\mathrm{zo}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2-\lambda}{k^2}, \quad \frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2-\lambda}{k^2}$$

Based on the maximum distortion strain energy theory (Mises criterion)

$$\left\{\begin{aligned} \frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} &= \frac{\ln x^2}{\sqrt{3}} - \frac{\lambda-1}{\sqrt{3}} \\ \frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} &= \frac{\ln x^2}{\sqrt{3}} + \frac{\lambda}{\sqrt{3}x^2} - \frac{\lambda}{\sqrt{3}} \\ \frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}} &= \frac{\ln x^2}{\sqrt{3}} - \frac{\lambda}{\sqrt{3}x^2} - \frac{\lambda-2}{\sqrt{3}} \\ \frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} &= 1 - \frac{\lambda}{x^2} \end{aligned}\right. \quad (1 \leq x \leq k_{\mathrm{j}\lambda}) \quad \left\{\begin{aligned} \frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} &= \frac{k_{\mathrm{j}\lambda}^2-\lambda}{\sqrt{3}k^2} = \text{constant} \\ \frac{\sigma'_{\mathrm{r}}}{\sigma_{\mathrm{y}}} &= \left(1-\frac{k^2}{x^2}\right)\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} \\ \frac{\sigma'_{\theta}}{\sigma_{\mathrm{y}}} &= \left(1+\frac{k^2}{x^2}\right)\frac{\sigma'_{\mathrm{z}}}{\sigma_{\mathrm{y}}} \\ \frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} &= \frac{k_{\mathrm{j}\lambda}^2-\lambda}{x^2} \end{aligned}\right. \quad (k_{\mathrm{j}\lambda} \leq x \leq k)$$

On the inner surface,

$$\frac{\sigma'_{\mathrm{zi}}}{\sigma_{\mathrm{y}}} = -\frac{\lambda-1}{\sqrt{3}}, \quad \frac{\sigma'_{\mathrm{ri}}}{\sigma_{\mathrm{y}}} = 0, \quad \frac{\sigma'_{\theta\mathrm{i}}}{\sigma_{\mathrm{y}}} = -\frac{2}{\sqrt{3}}(\lambda-1) = 2\frac{\sigma'_{\mathrm{zi}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}}, \quad \frac{\sigma'_{\mathrm{ei}}}{\sigma_{\mathrm{y}}} = 1-\lambda$$

On the outer surface,

$$\frac{\sigma'_{\mathrm{zo}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2-\lambda}{\sqrt{3}k^2}, \quad \sigma'_{\mathrm{ro}} = 0, \quad \frac{\sigma'_{\theta\mathrm{o}}}{\sigma_{\mathrm{y}}} = 2\frac{\sigma'_{\mathrm{zo}}}{\sigma_{\mathrm{y}}} = \frac{2}{\sqrt{3}}\frac{k_{\mathrm{j}\lambda}^2-\lambda}{k^2}, \quad \frac{\sigma'_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2-\lambda}{k^2}$$

The elastic stresses caused by internal pressure $p = \lambda p_{\mathrm{e}}$

Based on the maximum shear stress theory (Tresca criterion)

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{1}{k^2-1}\frac{p}{\sigma_{\mathrm{y}}} = \frac{\lambda}{2k^2}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \left(1-\frac{k^2}{x^2}\right)\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{\lambda}{2}\left(\frac{1}{k^2}-\frac{1}{x^2}\right)$$

$$\frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \left(1+\frac{k^2}{x^2}\right)\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{\lambda}{2}\left(\frac{1}{k^2}+\frac{1}{x^2}\right)$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{ps}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{\lambda}{x^2}$$

Based on the maximum distortion strain energy theory (Mises's criterion)

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{1}{k^2-1}\frac{p}{\sigma_{\mathrm{y}}} = \frac{\lambda}{\sqrt{3}k^2}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \left(1-\frac{k^2}{x^2}\right)\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = -\frac{\lambda}{\sqrt{3}}\left(\frac{1}{x^2}-\frac{1}{k^2}\right)$$

$$\frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \left(1+\frac{k^2}{x^2}\right)\frac{\sigma_{\mathrm{z}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{\lambda}{\sqrt{3}}\left(\frac{1}{k^2}+\frac{1}{x^2}\right)$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{pd}}}{\sigma_{\mathrm{y}}} = \frac{\sqrt{3}}{2}\left(\frac{\sigma_{\theta}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} - \frac{\sigma_{\mathrm{r}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}}\right) = \frac{\lambda}{x^2} = \frac{\sigma_{\mathrm{e}}^{\mathrm{ps}}}{\sigma_{\mathrm{y}}}$$

Tab. 5.1 – (continued).

The total stresses when $p = \lambda p_{\mathrm{e}}$ and $k_{\mathrm{j}} = k_{\mathrm{j}\lambda}$

Based on the maximum shear stress theory (Tresca yield criterion)

Within the plastic region,

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \ln x - \frac{\lambda-1}{2} + \frac{\lambda}{2k^2}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \ln x - \frac{\lambda}{2} + \frac{\lambda}{2k^2}$$

$$\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \ln x - \frac{\lambda}{2} + \frac{\lambda}{2k^2} + 1$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} \equiv 1$$

Within the elastic region,

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2}{2k^2}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2}{2}\left(\frac{1}{k^2} - \frac{1}{x^2}\right)$$

$$\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2}{2}\left(\frac{1}{k^2} + \frac{1}{x^2}\right)$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2}{x^2}$$

Based on the maximum distortion strain energy theory (Mises's yield criterion)

Within the plastic region,

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\ln x^2}{\sqrt{3}} - \frac{\lambda-1}{\sqrt{3}} + \frac{\lambda}{\sqrt{3}k^2}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\ln x^2}{\sqrt{3}} - \frac{\lambda}{\sqrt{3}} + \frac{\lambda}{\sqrt{3}k^2}$$

$$\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\ln x^2}{\sqrt{3}} - \frac{\lambda}{\sqrt{3}} + \frac{\lambda}{\sqrt{3}k^2} + \frac{2}{\sqrt{3}}$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sqrt{3}}{2}\left(\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} - \frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\right) = \frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} \equiv 1$$

Within the elastic region,

$$\frac{\sigma_{\mathrm{z}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2}{\sqrt{3}k^2}$$

$$\frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = -\frac{k_{\mathrm{j}\lambda}^2}{\sqrt{3}}\left(\frac{1}{x^2} - \frac{1}{k^2}\right)$$

$$\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2}{\sqrt{3}}\left(\frac{1}{k^2} + \frac{1}{x^2}\right)$$

$$\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} = \frac{\sqrt{3}}{2}\left(\frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} - \frac{\sigma_{\mathrm{r}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}\right) = \frac{\sigma_{\mathrm{e}}'}{\sigma_{\mathrm{y}}} + \frac{\sigma_{\mathrm{e}}^{\mathrm{p}}}{\sigma_{\mathrm{y}}} = \frac{k_{\mathrm{j}\lambda}^2}{x^2}$$

When $x < x_{\mathrm{a}}$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} > \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$; when $x > x_{\mathrm{a}}$, $\frac{\sigma_{\mathrm{e}}^{\mathrm{T}}}{\sigma_{\mathrm{y}}} < \frac{\sigma_{\theta}^{\mathrm{T}}}{\sigma_{\mathrm{y}}}$

Tab. 5.1 – (continued).

Based on both the maximum shear stress theory and the maximum distortion strain energy theory $x_0 = \sqrt{\lambda} = \text{constant}$
Based on both the maximum shear stress theory and the maximum distortion strain energy theory $x_1 = \sqrt{\mathrm{e}^{\lambda-1}} = \text{constant}$
Based on the maximum shear stress theory $\frac{\sigma'_{\mathrm{zxo}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{\theta xo}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{rxo}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{rmin}}}{\sigma_{\mathrm{y}}} = y_0^{\mathrm{s}} = -\frac{\lambda - 1 - \ln\lambda}{2} = \text{constant}$ Based on the maximum distortion strain energy theory $\frac{\sigma'_{\mathrm{zxo}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{\theta xo}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{rxo}}}{\sigma_{\mathrm{y}}} = \frac{\sigma'_{\mathrm{rmin}}}{\sigma_{\mathrm{y}}} = y_0^{\mathrm{d}} = -\frac{\lambda - 1 - \ln\lambda}{\sqrt{3}} = \text{constant}$ $y_0^{\mathrm{d}} = \frac{2}{\sqrt{3}} y_0^{\mathrm{s}}$ x_0, x_1, y_0^{s} and y_0^{d} are not related to k or k_{j}, only related with λ

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Chapter 6

Summary of Implement Methods and Their Characteristics of Mechanical and Thermal Autofrettage Technology

6.1 Implement Methods and Characteristics of Mechanical Autofrettage Technology

It is well known that cylindrical pressure vessels have unfavorable stress distributions under an elastic state, and the thicker the wall is, the more unfavorable the stress distribution is, as is seen in figure 1.1. To improve the stress distributions, autofrettage technology is usually adopted. The key to autofrettage technology is to introduce prestresses. Traditionally, the autofrettage is performed by applying very high pressure (such as hydraulic pressure, extrusion force, etc.) to the thick-walled pressure vessel so as to deform it plastically, after removing the very high pressure, the residual stresses on the wall of the pressure vessel are produced, enabling the pressure vessel to withstand higher working pressure. We call this method mechanical autofrettage. Currently, the common methods to obtain prestresses include the mechanical extrusion method, direct static pressure method (direct hydrostatic method), explosive expanding pressure method, and solid autofrettage method. The principles and characteristics of the above methods are as follows[1, 2].

1. Mechanical extrusion method

The principle of the mechanical extrusion method is pulling or pushing a conical interference mandrel through the inner surface of the cylinder for the inner surface to be extruded and deformed plastically to obtain residual stresses. For the mechanical extrusion method, there are currently the following three ways to pull or push a mandrel to slide through the cylinder.

(1) Press the mandrel through the cylinder by a punch or hydraulic press, which is called the indirect hydraulic pressing method.

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(2) Push the mandrel on its back through the cylinder directly by hydraulic pressure, which is called the direct hydraulic pushing method, which is shown in figure 6.1.
(3) Pull the mandrel through the cylinder by mechanical force, which is called the mechanical power traction method, which is shown in figure 6.2.

The mechanical extrusion method is mainly used in open cylinders or piping, it is economical and sealing is easy to achieve. In the mechanical extrusion method, residual stresses can reach 1500 to 2000 MPa. For materials with certain yield strength, the mechanical extrusion method requires much lower pressure than the hydrostatic method.

2. Direct hydrostatic method

The principle of the direct hydrostatic method is directly applying hydraulic pressure to the cylinder for it to be deformed plastically, after removing the hydraulic pressure, residual stresses are produced. This is the earliest method and the most commonly used method, and it is widely used for autofrettage of various gun barrels and high or ultra-high-pressure vessels and piping.

The direct hydrostatic method is basically the same as the hydraulic test process of the pressure vessel, and it is also basically the same as the operation process of the pressure vessel. The direct hydrostatic method is simple and flexible to use, in which no special pressure components are required, and the wall can obtain uniform plastic deformation. The direct hydrostatic method is especially suitable for autofrettage treatment of closed pressure vessels. For open cylinders (such as gun barrels, pipe, compressor cylinders, etc.), an end seal device can be used. In the direct hydrostatic method autofrettage, extra high-pressure power source, extra high-pressure pump and extra high-pressure pipeline, and pipe accessories are needed. Therefore, the sealing problem of ultra-high pressure needs to be solved.

3. Explosive expanding pressure method

The principle of the explosive expanding pressure method is to produce high pressure in a very short time (10^{-5} s order of magnitude) by using high-energy dynamite so that the cylinder or pipe produces plastic deformation rapidly under the action of high-pressure gas and shock wave. Since the plastic deformation of the inner wall is a function of the explosive strength, that is, the weight of the dynamite on the unit length, as long as the proper number of explosives is controlled and the pressure produced by the explosion is consistent with the pressure required for a overstrain, the cylinder or pipe can produce certain plastic deformation. In the explosive expanding pressure method, the influence of the end effect should be considered or the plastic deformation of the end of the cylinder or pipe decreases, and the influence of the radial piston effect should also be considered.

4. Solid autofrettage method

The solid autofrettage method is that the solid medium, such as lead, which has good plasticity and low melting point, is melted and poured into the cylinder, then, the solid medium is extruded by the press bar, which causes the solid medium to

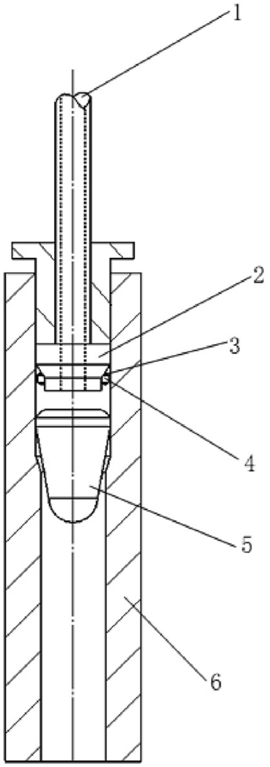

FIG. 6.1 – Autofrettage processing equipment for direct hydraulic pushing method. 1. Pressured liquid inlet; 2. Plug; 3. Soft steel filling ring; 4. Rubber “O” shaped sealing ring; 5. Mandrel; 6. Cylinder.

deform plastically and transfer the pressure to the cylinder wall, as a result, residual stresses are produced because of the plastic deformation of the cylinder wall.

In the solid autofrettage method, liquid medium consolidation should be avoided at extremely high pressure. Solid medium has high viscosity after it is plasticized, so the cylinder end seal is easy to solve. Even if a weak part of the wall breaks down, there will be no danger. This method has been applied to the high-pressure pump head with unequal thickness and special-shaped structure.

Cylinders after autofrettaged treatment must undergo low-temperature heat treatment in order to stabilize the metallographic structure of the cylinder after plastic deformation. The temperature for low-temperature heat treatment is generally 250 °C to 300 °C, sometimes 360 °C. After about 3 h of heat preservation, the heated cylinder is cooled with the furnace or cooled in the air. Attention should be paid to avoiding stress concentration due to uneven cooling speed.

It was once reported that autofrettage residual stress would be attenuated at a high temperature for a long time, and at the same time, corresponding solutions are put forward for this problem. For example, (1) choosing materials with good creep

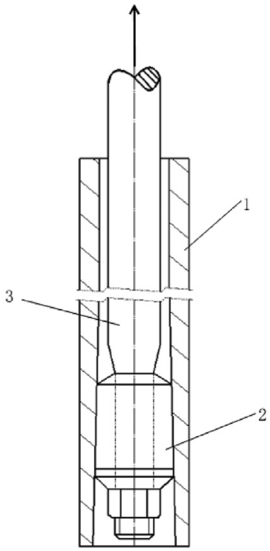

FIG. 6.2 – Autofrettage processing equipment for mechanical power traction method. 1. Cylinder; 2. Mandrel; 3. Pulling rod.

resistance properties; (2) reasonable selection of overstrain; (3) strictly observing the operation procedures; (4) choosing suitable heat treatment temperature; (5) re-autofrettage, as is an effective way to solve the autofrettage residual stress attenuation. Therefore, the problem of autofrettage residual stress attenuation will no longer be discussed in this book.

6.2 Implement Methods and Characteristics of Thermal Autofrettage Technology

Since the temperature difference in the wall of a cylinder can cause thermal stresses, we put forward an autofrettage technology by the temperature difference between the inner and outer surfaces of a cylindrical pressure vessel. We call this technology temperature difference autofrettage or thermal autofrettage. In thermal autofrettage, there are no dangerous pressured fluids and no expensive hydrostatic installations needed, consequently, once the temperature difference is too high, catastrophic accidents like pressure vessel explosions cannot be caused as in mechanical autofrettage. In addition, the magnitude and regularities of distribution of temperature difference stresses are closely related to temperature difference Δt, thus, various operating stress states can be obtained by changing the temperature difference Δt according to operation conditions, as a result, we can acquire flexible design projects for one design assignment by using thermal autofrettage. Therefore, thermal autofrettage is safe, convenient, and economical and is a potential autofrettage technology. Moreover, there sometimes naturally is a temperature difference in many industrial production processes.

When the temperature on the inner surface of a cylinder is higher than that on the outer surface, it is called internal heating; on the contrary, when the temperature on the inner surface of a cylinder is lower than that on the outer surface, it is called external heating. There are many ways to generate temperature differences. The temperature difference under the inner heating case can be generated by heating the inner surface and/or cooling the outer surface, as is shown in figure 6.3; the temperature difference under the outer heating case can be generated by cooling the inner surface and/or heating outer surface, as is shown in figure 6.4. Heating can be carried out by means of electric heating or steam heating and cooling can be carried out by means of liquid ammonia, etc.

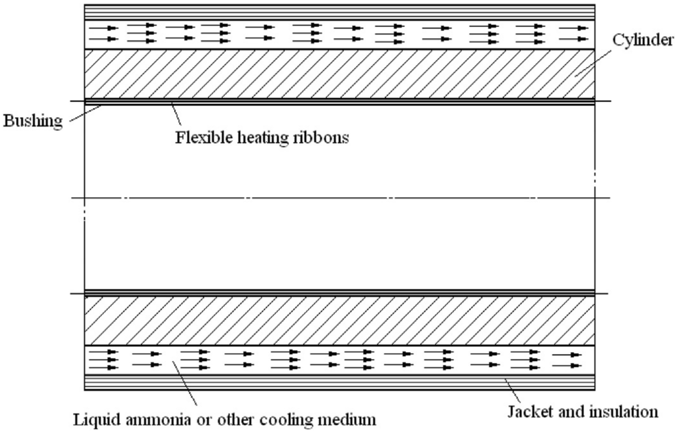

Fig. 6.3 – Temperature difference generated for inner heating.

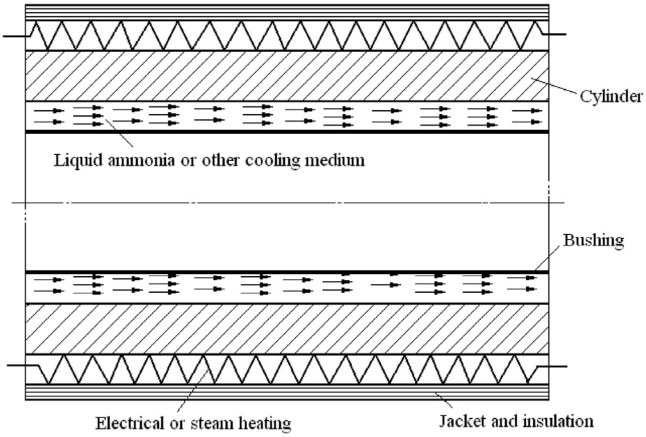
Cylinder
Liquid ammonia or other cooling medium
Bushing
Electrical or steam heating
Jacket and insulation

Fig. 6.4 – Temperature difference generated for outer heating.

For the inner heating shown in figure 6.3, the heating element is a kind of flexible heating flat ribbon, that is a resistance wire wrapped by high-temperature resistance cotton rope, as is shown in figure 6.5. The flexible heating flat ribbons are wound around the bushing, as is shown in figure 6.6, and the ends of the flexible heating flat ribbons are connected to the power source and temperature controller, etc.

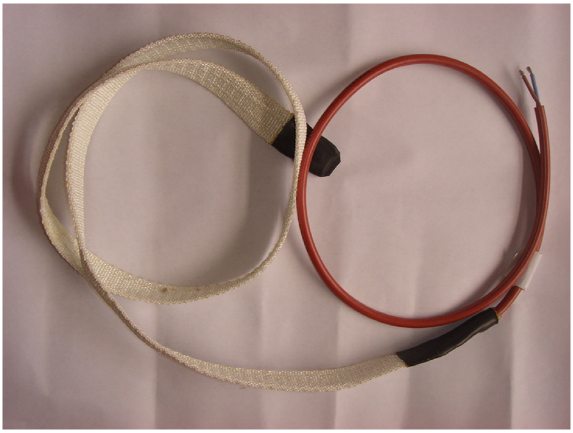

Fig. 6.5 – A section of flexible heating flat ribbon.

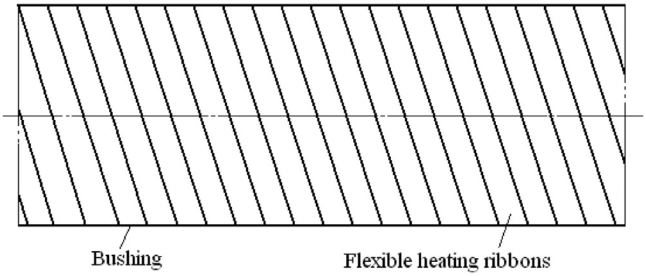

Fig. 6.6 – Bushing wound with flexible heating flat ribbons.

The heating and cooling system can be retained or dismantled in the operation of the pressure vessel, especially after the thermal stresses have caused the wall of the pressure vessel to yield and the residual thermal stresses have remained in the wall, the heating and cooling system can be dismantled.

When the temperature difference reaches the critical temperature difference, Δt_{c}, the inner surface of the pressure vessel initiates yielding, where Δt_{c} is as follows,

$$\Delta t_{\mathrm{c}} = \frac{(k^2-1)\ln k^2}{k^2\ln k^2 - k^2 + 1}\frac{(1-\mu)\sigma_{\mathrm{y}}}{E\alpha} = A\frac{(1-\mu)\sigma_{\mathrm{y}}}{E\alpha} \tag{6.1}$$

According to equation (6.1), the temperature difference does not have to be quite large, the inner surface of the pressure vessel can be yielded under the temperature difference. Therefore, after thermal autofrettage treatment, the metallic material of the pressure vessel will not change or deteriorate the metallographic structure, and reduce its mechanical properties and service life due to heating or cooling during thermal autofrettage treatment. For example, when $E = 2 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, $k = 2.5$, $\sigma_{\mathrm{y}} = 500$ MPa, from equation (6.1), $\Delta t_{\mathrm{c}} = 226.17$ °C. This is just the temperature difference between the inner and outer walls, not the temperature at every point in the wall. Even if the temperature of each point in the wall is 226.17 °C, there will be no change in the metallographic structure under such a low temperature. The temperature difference can also be generated by heating on one side and cooling by another side, saying, heating the inner surface to $t_{\mathrm{i}} = 126.17$ °C and cooling the outer surface to $t_{\mathrm{o}} = -100$ °C (this is internal heating, external heating is the other way round). According to equation (6.1), it is known that Δt_{c} decreases with the increase of the wall thickness (reflected by radius ratio k). That is, the thicker the cylinder is, the smaller the critical temperature difference, Δt_{c} is. In equation (6.1), the part $A = \frac{(k^2-1)\ln k^2}{k^2\ln k^2 - k^2 + 1}$ decreases as k (reflecting wall thickness) increases, or the critical temperature difference, Δt_{c} decreases as k increases. When $k \rightarrow 1$, A = 2; when $k \rightarrow \infty$, A = 1. For the metallic material, there is little change in E, μ and α, if $\sigma_{\mathrm{y}} = 500$ MPa ($E = 2 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}$ °C^{-1}), $\frac{(1-\mu)\sigma_{\mathrm{y}}}{E\alpha} = 145.8$ °C. Then, under the case of $\sigma_{\mathrm{y}} = 500$ MPa, the maximum critical temperature difference, Δt_{c} will not exceed 292 °C.

The degradation of the metallographic structure depends mainly on the temperature of the metal rather than the temperature difference. Furthermore, for the metallographic structure of the pressure vessel to deteriorate, it is necessary for the pressure vessel to undergo high temperatures over a long period of time. Moreover, when there is a temperature difference between the internal surface and the external surface, the temperature inside the wall is not high. In the case of steady heat transfer, the distribution of temperature along the wall thickness is as follows:

$$t = \frac{t_{\mathrm{i}} \ln \frac{k}{x} + t_{\mathrm{o}} \ln x}{\ln k} \tag{6.2}$$

When $k = 2.5$ ($E = 2 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, $\sigma_{\mathrm{y}} = 500$ MPa), it is known from the above that $\Delta t_{\mathrm{c}} = 226.17$ °C. If the internal surface is heated to $t_{\mathrm{i}} = 246.17$ °C and the external surface is at room temperature 20 °C (then the temperature difference between the internal surface and the external surface, $\Delta t = 226.17$ °C), the distribution of temperature along the wall thickness is illustrated in figure 6.7.

From figure 6.7, it is seen that the temperature inside the wall of the cylinder is not high. Therefore, metallographic structure change/degradation can not happen, so the mechanical properties of the material and service life of the cylinder can not be reduced. If k is increased, the temperature or the temperature critical

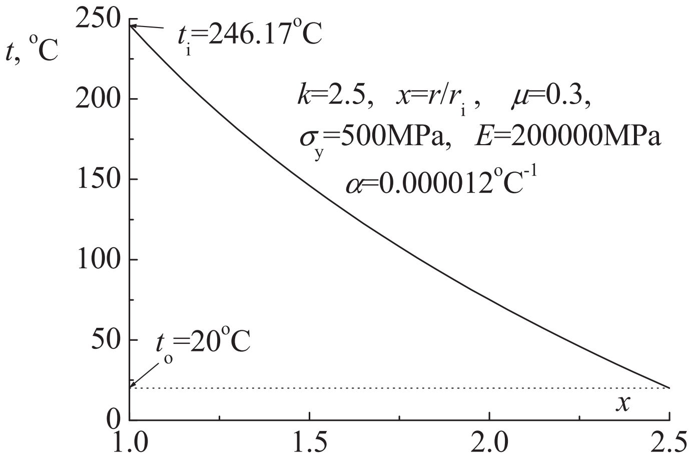

FIG. 6.7 – The distribution of temperature along the wall thickness with $k = 2.5$.

temperature difference Δt_c will be decreased. When $k = 3$ (still, $E = 2 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, $\sigma_y = 500$ MPa), it is known from the above that $\Delta t_c = 217.7$ °C. If the internal surface is heated to $t_i = 237.7$ °C and the external surface is at room temperature 20 °C (then the temperature difference between the internal surface and the external surface, $\Delta t = 217.7$ °C), the distribution of temperature along the wall thickness is illustrated in figure 6.8.

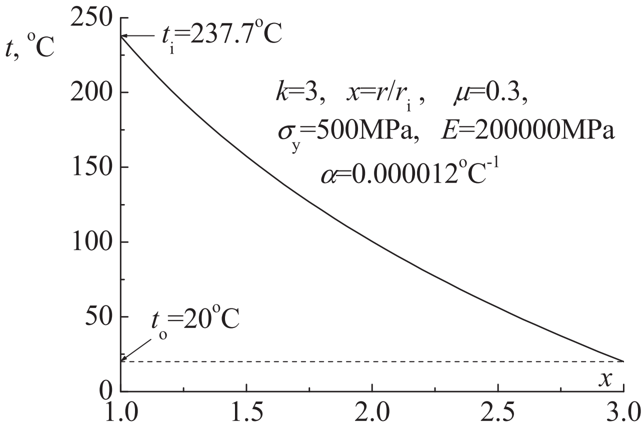

FIG. 6.8 – The distribution of temperature along the wall thickness with $k = 3$.

When $\sigma_y = 300$ MPa, $E = 2 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, the variation of the critical temperature difference Δt_c with radius ratio k is shown in figure 6.9.

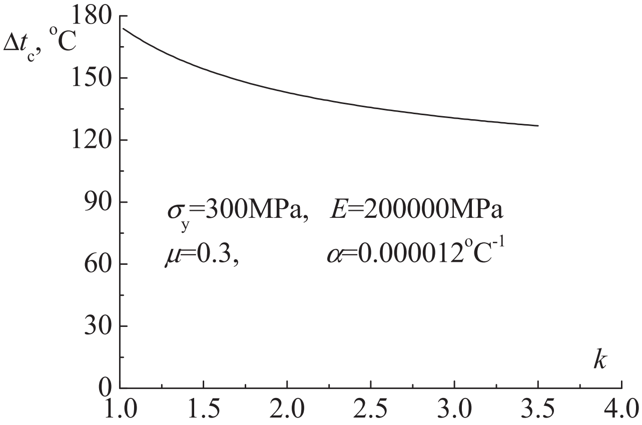

FIG. 6.9 – Variation of Δt_c with k when $\sigma_y = 300$ MPa.

When $\sigma_y = 500$ MPa, $E = 2 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, the variation of the critical temperature difference Δt_c with radius ratio k is shown in figure 6.10.

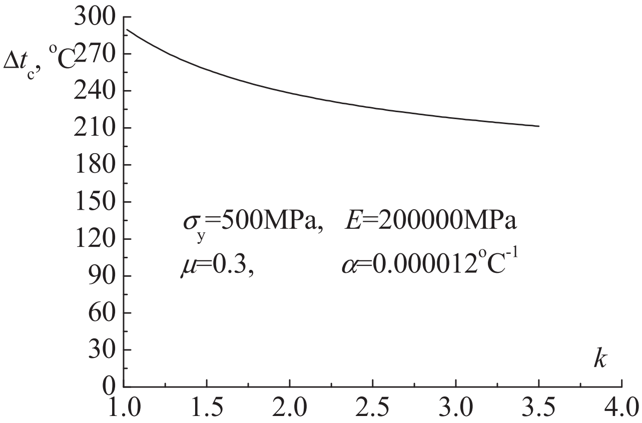

FIG. 6.10 – Variation of Δt_c with k when $\sigma_y = 500$ MPa.

When $\sigma_y = 700$ MPa, $E = 2 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, the variation of the critical temperature difference Δt_c with radius ratio k is shown in figure 6.11.

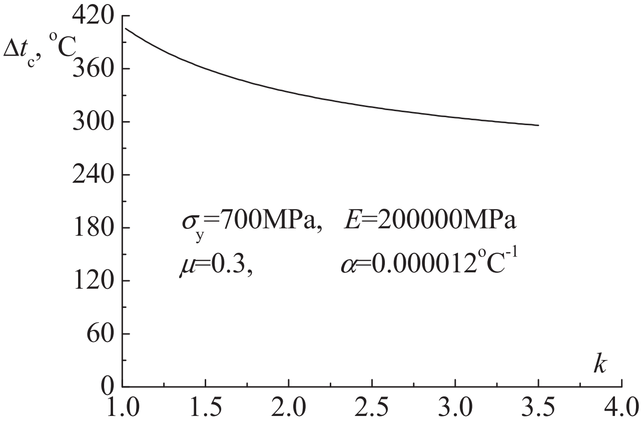

FIG. 6.11 – Variation of Δt_c with k when $\sigma_y = 700$ MPa.

When $k = 2$, $E = 2 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, the variation of the critical temperature difference Δt_c with yield strength σ_y is shown in figure 6.12.

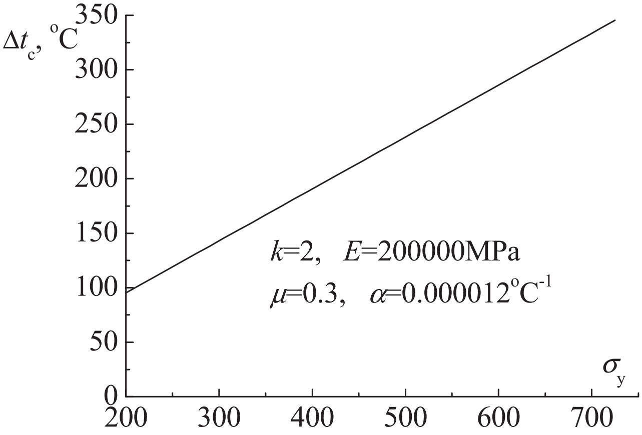

FIG. 6.12 – Variation of Δt_c with σ_y when $k = 2$.

When $k = 3$, $E = 2 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, the variation of the critical temperature difference Δt_c with yield strength σ_y is shown in figure 6.13.

When $k = 4$, $E = 2 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, the variation of the critical temperature difference Δt_c with yield strength σ_y is shown in figure 6.14.

It can be seen that the temperature in the wall is not high, so there will be no change in metallography. Under the condition of keeping temperature difference, the thermal stress can not decay. As for the attenuation of residual temperature

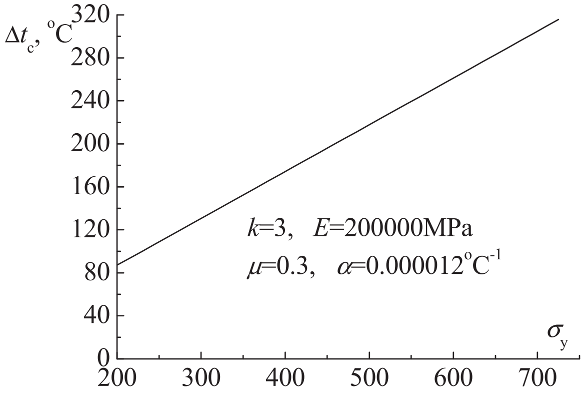

FIG. 6.13 – Variation of Δt_c with σ_y when $k = 3$.

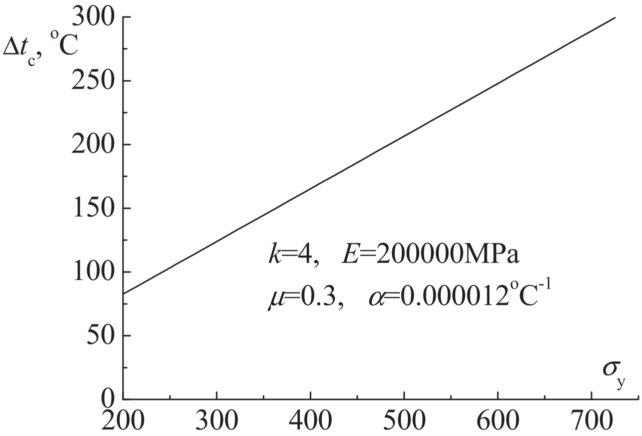

FIG. 6.14 – Variation of Δt_c with σ_y when $k = 4$.

difference stresses after removing temperature difference, it can be solved just like solving the attenuation of the above mechanical residual stresses.

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Chapter 7

Thermal Autofrettage Technology Based on Tresca Yield Criterion

7.1 Introduction

The metallic materials have the properties of thermal expansion and contraction (expand with heat and contract with cold). The media contained in high-pressure apparatuses are frequently in high or low temperatures, the materials of the high-pressure apparatuses are thus heated or cooled by the media, and thermal expansion and contraction in the materials are thereupon caused. For the thick-walled high-pressure apparatuses, deformations of the wall are not free but highly restrained. Therefore, if the thick high-pressure apparatuses are heated or cooled (whether by operation media or by artificial means), thermal stresses will be produced due to deformation constraints. Thermal stresses are often useful in the operation of high-pressure apparatuses.

The study of the thermal stresses can help us to explore the effects of material characteristics on thermal stresses and to create a unique autofrettage technology for thick high-pressure apparatuses when the thermal stresses are advantageous. It is well known that high-pressure apparatuses have unfavorable stress distributions under an elastic state. To improve stress distributions, autofrettage technology is usually adopted. Traditionally, the autofrettage is achieved by mechanical method, or by applying hydraulic pressure to the bore of the thick-walled high-pressure apparatuses to deform them plastically, the compressive residual stresses on the inner part of the high-pressure apparatuses are thus produced, the operation stresses, therefore, are partially offset by the residual stresses, enabling the high-pressure apparatuses to withstand higher working pressure. This is mechanical autofrettage technology. In the former chapters, we have carried out studies on mechanical autofrettage.

The key to autofrettage of the thick high-pressure apparatuses is pre-stresses. Since temperature differences in the wall of a high-pressure apparatus can cause pre-stresses, heating and/or cooling the thick high-pressure apparatuses to obtain pre-stresses is a promising approach for autofrettage of thick apparatuses. For this

DOI: 10.1051/978-2-7598-3111-1.c007
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reason, we first need to explore the thermal stresses in metal materials. On the basis of studying thermal stresses, we will study thermal autofrettage technology.

Most high-pressure apparatuses are usually shaped into cylinders, so we investigate the thermal stresses in cylindrically shaped pressure vessels.

7.2 Derivation of Thermal Stresses

To draw on the advantages and avoid disadvantages of the thermal stresses, it is necessary and meaningful to investigate the magnitudes and the characteristics of the thermal stresses in the materials quantitatively and in detail.

Figure 7.1 is a thick cylindrical pressure vessel with internal radius r_i and external radius r_o; the temperature on the internal wall is t_i, and t_o on the external wall; the temperature difference, $\Delta t = t_i - t_o$, °C; and the temperature at any radius r is t, and $t = t(r)$. This is an axial symmetry problem, the cross-section remains plane under thermal stresses, the radial displacement u is related only to radius r, the axial strain ε_z is constant, and clearly, there is no hoop displacement. We will derive the expressions of thermal stresses based on such a mechanical and mathematical model.

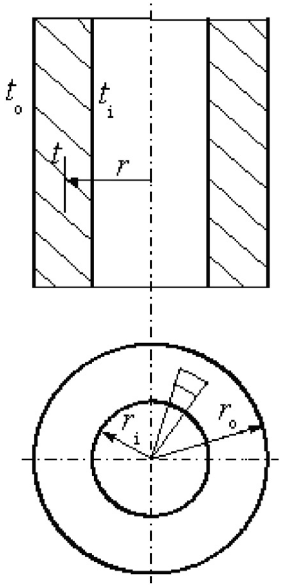

Fig. 7.1 – A thick cylindrical pressure vessel with temperature difference in the wall.

At an arbitrary radius r, cutting the cylinder with two concentric circles at a distance of $\mathrm{d}r$, two planes with included angle $\mathrm{d}\theta$, and two parallel planes at a distance of $\mathrm{d}z$, we obtain an infinitesimal volume element as is shown in figure 7.2.

Considering equilibrium of the element in r direction, we obtain the equilibrium differential equation of the cylinder at any point as follows[1],

$$\begin{cases} \dfrac{\mathrm{d}\sigma_\mathrm{r}}{\mathrm{d}r} + \dfrac{\sigma_\mathrm{r} - \sigma_\theta}{r} = 0 \\ \dfrac{\mathrm{d}\sigma_\mathrm{z}}{\mathrm{d}z} = 0 \end{cases} \tag{7.1}$$

where σ_r, σ_θ, and σ_z are radial stress, circumferential stress, and axial (thermal) stress, respectively. Equation (7.1) is fit for the materials in both elastic and plastic states.

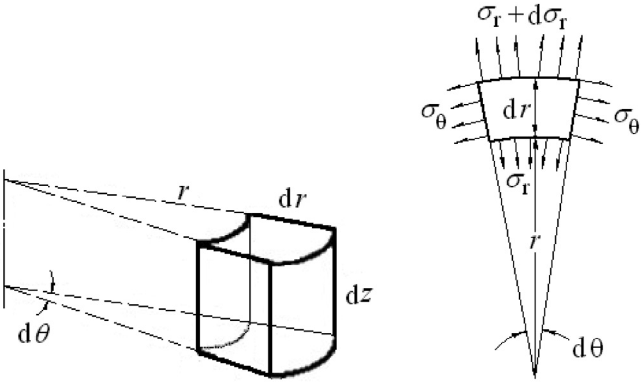

FIG. 7.2 – An element cutting out of a cylindrical pressure vessel.

When an elastic material is heated to temperature t °C and can expand freely, the thermal strains are $\varepsilon_\mathrm{r}^\mathrm{t} = \varepsilon_\theta^\mathrm{t} = \varepsilon_\mathrm{z}^\mathrm{t} = \alpha t$, where $\varepsilon_\mathrm{r}^\mathrm{t}$, $\varepsilon_\theta^\mathrm{t}$, and $\varepsilon_\mathrm{z}^\mathrm{t}$ are radial strain, circumferential strain, and axial thermal strain, respectively; α is the thermal expansion coefficient of the materials, $°\mathrm{C}^{-1}$. If the material is constrained, the thermal stresses are consequently caused in the material. When the material is constrained, at an arbitrary radius r, the total strains are thermal strains plus strains produced by thermal stresses. For the materials at elastic state, according to Hook's law[1],

$$\begin{cases} \varepsilon_\mathrm{r}^\mathrm{t} = \dfrac{1}{E}\left[\sigma_\mathrm{r}^\mathrm{t} - \mu(\sigma_\theta^\mathrm{t} + \sigma_\mathrm{z}^\mathrm{t})\right] + \alpha t \\ \varepsilon_\theta^\mathrm{t} = \dfrac{1}{E}\left[\sigma_\theta^\mathrm{t} - \mu(\sigma_\mathrm{z}^\mathrm{t} + \sigma_\mathrm{r}^\mathrm{t})\right] + \alpha t \\ \varepsilon_\mathrm{z}^\mathrm{t} = \dfrac{1}{E}\left[\sigma_\mathrm{z}^\mathrm{t} - \mu(\sigma_\mathrm{r}^\mathrm{t} + \sigma_\theta^\mathrm{t})\right] + \alpha t \\ \gamma_\mathrm{zr}^\mathrm{t} = 0 \end{cases} \text{or} \begin{cases} \sigma_\mathrm{r}^\mathrm{t} = 2G\left(\varepsilon_\mathrm{r}^\mathrm{t} + \dfrac{\mu}{1-2\mu}e^\mathrm{t} - \dfrac{1+\mu}{1-2\mu}\alpha t\right) \\ \sigma_\theta^\mathrm{t} = 2G\left(\varepsilon_\theta^\mathrm{t} + \dfrac{\mu}{1-2\mu}e^\mathrm{t} - \dfrac{1+\mu}{1-2\mu}\alpha t\right) \\ \sigma_\mathrm{z}^\mathrm{t} = 2G\left(\varepsilon_\mathrm{z}^\mathrm{t} + \dfrac{\mu}{1-2\mu}e^\mathrm{t} - \dfrac{1+\mu}{1-2\mu}\alpha t\right) \\ \tau_\mathrm{zr}^\mathrm{t} = 0 \end{cases} \tag{7.2}$$

Where $\sigma_\mathrm{r}^\mathrm{t}$, σ_θ^t, and $\sigma_\mathrm{z}^\mathrm{t}$ are thermal radial stress, thermal circumferential stress, and thermal axial stress, respectively; $e^\mathrm{t} = \varepsilon_\mathrm{r}^\mathrm{t} + \varepsilon_\theta^\mathrm{t} + \varepsilon_\mathrm{z}^\mathrm{t} = \frac{1-2\mu}{E}\left(\sigma_\mathrm{r}^\mathrm{t} + \sigma_\theta^\mathrm{t} + \sigma_\mathrm{z}^\mathrm{t}\right) + 3\alpha t$; μ is the Poisson ratio of the material; E is Young's elastic module of the material; G is shearing elastic modules of the material, $G = \frac{E}{2(1+\mu)}$.

For the axial symmetry problem, we take any circumferential fiber with radius r in the cylinder for study, as is shown in figure 7.3.

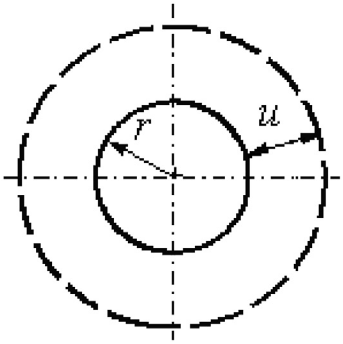

FIG. 7.3 – The radial displacement of a circumferential fiber.

Before deformation the radius of the circumferential fiber in figure 7.3 is r. After deformation, the circumferential fiber arrives at the dotted circle in figure 7.3. The radius of the circumferential fiber is $r + u$ after deformation, that is, the radial displacement is u, as is shown in figure 7.3. Therefore, after deformation, the length or the circumference of the circumferential fiber is $2\pi\ (r + u)$, and before deformation, the circumference of the circumferential fiber is $2\pi r$, thus, the circumferential strain of the circumferential fiber is $\varepsilon_{\theta}^{t} = \frac{2\pi(r+u)-2\pi r}{2\pi r} = \frac{u}{r}$.

When the radial displacement of the circumferential fiber with radius r is u, the radial displacement of the circumferential fiber with radius $r + \mathrm{d}r$ is $u + \mathrm{d}u$, as is shown in figure 7.4. Then, the deflection of a radial length $\mathrm{d}r$ is $\mathrm{d}u$. Therefore, the radial strain is $\varepsilon_{r}^{t} = \frac{\mathrm{d}u}{\mathrm{d}r}$. In a similar way, the axial strain is $\varepsilon_{z}^{t} = \frac{\mathrm{d}w}{\mathrm{d}z}$. Since the axial deformation is uniform, $\varepsilon_{z}^{t} = \frac{\mathrm{d}w}{\mathrm{d}z} = $ a constant. Clearly, $\gamma_{zr}^{t} = 0$. Therefore, we have

$$\varepsilon_{r}^{t} = \frac{\mathrm{d}u}{\mathrm{d}r}, \quad \varepsilon_{\theta}^{t} = \frac{u}{r}, \quad \varepsilon_{z}^{t} = \frac{\mathrm{d}w}{\mathrm{d}z} = \text{a constant}, \quad \gamma_{zr}^{t} = 0 \tag{7.3}$$

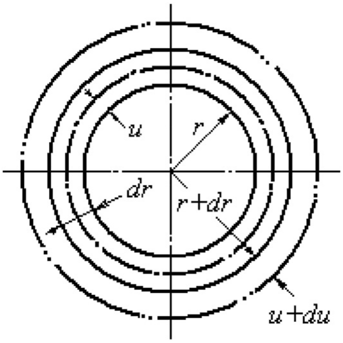

FIG. 7.4 – The radial displacements of two circumferential fiber.

where u and w are radial displacement and axial displacement respectively, γ_{zr} is the shearing strain in the r–z plane.

Then, by using equations (7.3), equation (7.2) can be rewritten as follows

$$\begin{cases} \sigma_{\mathrm{r}}^{\mathrm{t}} = 2G\left[\dfrac{\mathrm{d}u}{\mathrm{d}r} + \dfrac{\mu}{1-2\mu}\left(\dfrac{\mathrm{d}u}{\mathrm{d}r} + \dfrac{u}{r} + \varepsilon_{\mathrm{z}}\right) - \dfrac{1+\mu}{1-2\mu}\alpha t\right] \\ \sigma_{\theta}^{\mathrm{t}} = 2G\left[\dfrac{u}{r} + \dfrac{\mu}{1-2\mu}\left(\dfrac{\mathrm{d}u}{\mathrm{d}r} + \dfrac{u}{r} + \varepsilon_{\mathrm{z}}\right) - \dfrac{1+\mu}{1-2\mu}\alpha t\right] \end{cases} \tag{7.4}$$

Substituting equation (7.4) into the first one of equation (7.1) obtains

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{u}{r^2} = \frac{1+\mu}{1-\mu}\alpha\frac{\mathrm{d}t}{\mathrm{d}r} \tag{7.5}$$

Solving equation (7.5) gives

$$u = \frac{1+\mu}{1-\mu}\frac{\alpha}{r}\int_{r_{\mathrm{i}}}^{r} tr\mathrm{d}r + \frac{C_1 r}{2} + \frac{C_2}{r} \tag{7.6}$$

where C_1 and C_2 are integral constants.

Substituting equation (7.6) into equation (7.4) results in

$$\begin{cases} \sigma_{\mathrm{r}}^{\mathrm{t}} = -\dfrac{E\alpha}{(1-\mu)r^2}\displaystyle\int_{r_{\mathrm{i}}}^{r} tr\mathrm{d}r + A - \dfrac{B}{r^2} \\ \sigma_{\theta}^{\mathrm{t}} = \dfrac{E\alpha}{(1-\mu)r^2}\displaystyle\int_{r_{\mathrm{i}}}^{r} tr\mathrm{d}r - \dfrac{E\alpha t}{1-\mu} + A + \dfrac{B}{r^2} \\ \sigma_{\mathrm{z}}^{\mathrm{t}} = E(\varepsilon_{\mathrm{z}}^{\mathrm{t}} - \alpha t) + \mu(\sigma_{\mathrm{r}}^{\mathrm{t}} + \sigma_{\theta}^{\mathrm{t}}) \\ \quad = -\dfrac{E\alpha t}{1-\mu} + 2\mu A + E\varepsilon_{\mathrm{z}}^{\mathrm{t}} \\ \tau_{\mathrm{zr}}^{\mathrm{t}} = 0, \tau_{\mathrm{r}\theta}^{\mathrm{t}} = 0 \end{cases} \tag{7.7}$$

where $A = \frac{E(C_1 + 2\mu\varepsilon_{\mathrm{z}}^{\mathrm{t}})}{2(1+\mu)(1-2\mu)}$, $B = \frac{EC_2}{1+\mu}$.

When a pressure vessel is not subjected to additional axial loads, there is

$$\int_{r_{\mathrm{i}}}^{r_{\mathrm{o}}} \sigma_{\mathrm{z}}^{\mathrm{t}} 2\pi r\mathrm{d}r = 0 \tag{7.8}$$

Substituting the third one of equation (7.7) into equation (7.8) gives

$$-\frac{E\alpha}{1-\mu}\int_{r_{\mathrm{i}}}^{r_{\mathrm{o}}} tr\mathrm{d}r + \frac{1}{2}(2\mu A + E\varepsilon_{\mathrm{z}}^{\mathrm{t}})(r_{\mathrm{o}}^2 - r_{\mathrm{i}}^2) = 0$$

When a pressure vessel is not loaded with other loads but temperature difference, then

$$(\sigma_{\mathrm{r}}^{\mathrm{t}})_{r=r_{\mathrm{i}}} = (\sigma_{\mathrm{r}}^{\mathrm{t}})_{r=r_{\mathrm{o}}} = 0 \tag{7.9}$$

Substituting equation (7.9) into equation (7.7) gives

$$\begin{cases} A = \dfrac{E\alpha}{(1-\mu)(r_{\mathrm{o}}^2 - r_{\mathrm{i}}^2)} \displaystyle\int_{r_{\mathrm{i}}}^{r_{\mathrm{o}}} tr\mathrm{d}r \\ B = r_{\mathrm{i}}^2 A \\ \varepsilon_{\mathrm{z}}^{\mathrm{t}} = \dfrac{2(1-\mu)}{E} A \end{cases} \tag{7.10}$$

From equations (7.7) and (7.10), we obtain

$$\sigma_{\mathrm{z}}^{\mathrm{t}} = \sigma_{\theta}^{\mathrm{t}} + \sigma_{\mathrm{r}}^{\mathrm{t}} = -\frac{E\alpha t}{1-\mu} + 2A \tag{7.11}$$

For a cylindrical pressure vessel with internal radius r_{i}, external radius r_{o}, and internal wall temperature t_{i}, external wall temperature t_{o} as shown in figure 7.5, at any radius r, we take a thin cylinder with differential thickness dr and axial length l. Under steady heat transfer, according to Fourier's heat transfer law[2–4], the quantity of heat Q conducted through the thin cylinder in unit time is

$$Q = -2\pi r l \lambda \frac{\mathrm{d}t}{\mathrm{d}r} \tag{7.12}$$

where λ is the thermal conductivity of the pressure vessel material, W/m•°C or W/m•K. Rewriting equation (7.12) as follows

$$Q\frac{\mathrm{d}r}{r} = -2\pi l \lambda \mathrm{d}t \tag{7.13}$$

and integrating both sides of equation (7.13) from the internal wall to the external wall, *i.e.*

$$Q\int_{r_{\mathrm{i}}}^{r_{\mathrm{o}}} \frac{\mathrm{d}r}{r} = -2\pi l \lambda \int_{t_{\mathrm{i}}}^{t_{\mathrm{o}}} \mathrm{d}t$$

we obtain

$$Q = 2\pi l \lambda \frac{t_{\mathrm{i}} - t_{\mathrm{o}}}{\ln \frac{r_{\mathrm{o}}}{r_{\mathrm{i}}}} \tag{7.14}$$

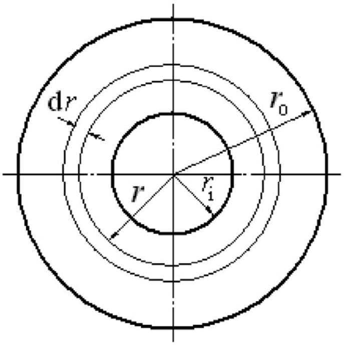

Fig. 7.5 – Heat conduction through a wall of cylinder.

On the other hand, if we integrate equation (7.13) from the inner wall to any radius r, or

$$Q\int_{r_{\mathrm{i}}}^{r}\frac{\mathrm{d}r}{r}=-2\pi l\lambda\int_{t_{\mathrm{i}}}^{t}\mathrm{d}t$$

we obtain

$$Q=2\pi l\lambda\frac{t_{\mathrm{i}}-t}{\ln\frac{r}{r_{\mathrm{i}}}}\tag{7.15}$$

Setting equation (7.15) equaling equation (7.14) gives

$$t=t(r)=\frac{t_{\mathrm{i}}\ln\frac{r_{\mathrm{o}}}{r}+t_{\mathrm{o}}\ln\frac{r}{r_{\mathrm{i}}}}{\ln\frac{r_{\mathrm{o}}}{r_{\mathrm{i}}}}=\frac{t_{\mathrm{i}}\ln\frac{k}{x}+t_{\mathrm{o}}\ln x}{\ln k}=\frac{t_{\mathrm{i}}\ln k-\Delta t\ln x}{\ln k}=t_{\mathrm{i}}-\Delta t\frac{\ln x}{\ln k}\tag{7.16}$$

where k is the radius ratio, or ratio of outside radius to an inside radius, $k=r_{\mathrm{o}}/r_{\mathrm{i}}$; x is relative location, $x=r/r_{\mathrm{i}}$.

The temperature difference, $\Delta t=t_{\mathrm{i}}-t_{\mathrm{o}}>0$ is called internal heating, $\Delta t<0$ is called external heating.

The distribution of wall temperature in the case of internal heating for $k=3$ is shown in figure 7.6, where the solid curves are for $t_{\mathrm{i}}=200$ °C, the dash curves are for $t_{\mathrm{i}}=300$ °C, and the parameters are as follows:

(1) $t_{\mathrm{i}}=200$ °C, $t_{\mathrm{o}}=0$ °C, $\Delta t=t_{\mathrm{i}}-t_{\mathrm{o}}=200$ °C;
(2) $t_{\mathrm{i}}=200$ °C, $t_{\mathrm{o}}=50$ °C, $\Delta t=t_{\mathrm{i}}-t_{\mathrm{o}}=150$ °C;
(3) $t_{\mathrm{i}}=200$ °C, $t_{\mathrm{o}}=150$ °C, $\Delta t=t_{\mathrm{i}}-t_{\mathrm{o}}=50$ °C;
(4) $t_{\mathrm{i}}=300$ °C, $t_{\mathrm{o}}=0$ °C, $\Delta t=t_{\mathrm{i}}-t_{\mathrm{o}}=300$ °C;
(5) $t_{\mathrm{i}}=300$ °C, $t_{\mathrm{o}}=50$ °C, $\Delta t=t_{\mathrm{i}}-t_{\mathrm{o}}=250$ °C;
(6) $t_{\mathrm{i}}=300$ °C, $t_{\mathrm{o}}=150$ °C, $\Delta t=t_{\mathrm{i}}-t_{\mathrm{o}}=150$ °C.

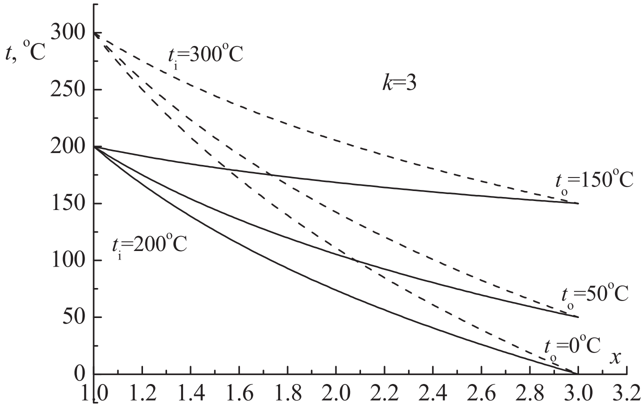

FIG. 7.6 – The distributions of wall temperature of internal heating for $k=3$.

The distribution of wall temperature in the case of external heating for $k=3$ is shown in figure 7.7, where the solid curves are for $t_{\mathrm{i}}=0$ °C, the dash curves are for $t_{\mathrm{i}}=-30$ °C, and the parameters are as follows:

(7) $t_\mathrm{i} = 0$ °C, $t_\mathrm{o} = 200$ °C, $\Delta t = t_\mathrm{i} - t_\mathrm{o} = -200$ °C;
(8) $t_\mathrm{i} = 0$ °C, $t_\mathrm{o} = 150$ °C, $\Delta t = t_\mathrm{i} - t_\mathrm{o} = -150$ °C;
(9) $t_\mathrm{i} = 0$ °C, $t_\mathrm{o} = 50$ °C, $\Delta t = t_\mathrm{i} - t_\mathrm{o} = -50$ °C;
(10) $t_\mathrm{i} = -30$ °C, $t_\mathrm{o} = 270$ °C, $\Delta t = t_\mathrm{i} - t_\mathrm{o} = -300$ °C;
(11) $t_\mathrm{i} = -30$ °C, $t_\mathrm{o} = 220$ °C, $\Delta t = t_\mathrm{i} - t_\mathrm{o} = -250$ °C;
(12) $t_\mathrm{i} = -30$ °C, $t_\mathrm{o} = 120$ °C, $\Delta t = t_\mathrm{i} - t_\mathrm{o} = -150$ °C.

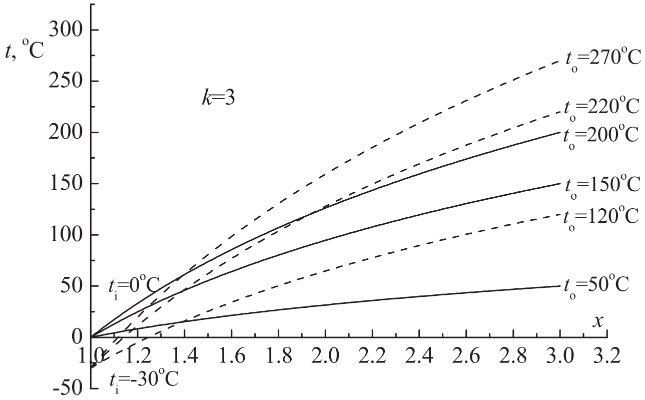

FIG. 7.7 – The distributions of wall temperature of external heating for $k = 3$.

In addition, by integrating equation (7.13) from any radius r to the outside wall, or

$$Q \int_r^{r_\mathrm{o}} \frac{\mathrm{d}r}{r} = -2\pi l \lambda \int_t^{t_\mathrm{o}} \mathrm{d}t$$

we obtain

$$Q = 2\pi l \lambda \frac{t - t_\mathrm{o}}{\ln \frac{r_\mathrm{o}}{r}} \tag{7.17}$$

Letting equation (7.17) equal equation (7.14) also gives equation (7.16), which expresses temperature at any location r under steady heat transfer. Equation (7.16) indicates that the temperature of any point in the wall of the cylindrical pressure vessel depends on the temperatures of the inside and outside walls (t_i and t_o) and the internal and external radii (r_i and r_o), and has nothing to do with material natures. When $t_\mathrm{i} > t_\mathrm{o}$, the temperature in the wall of the cylindrical pressure vessel strictly decreases from the internal wall to the external wall, and when $t_\mathrm{i} < t_\mathrm{o}$, it strictly increases from the internal wall to the external wall. However, from equation (7.14), the quantity of heat conducted through a wall of a cylinder is associated with the thermal conductivity of the material, the temperature difference, and the radii of the inside and outside walls. Thus, it can be seen that once the inside and outside wall temperatures and the inside and outside radii are determined, heat transfer is related to the physical properties of the materials; and once the material is determined, heat transfer is related to the boundary temperatures and sizes, and has no relation to the specific distribution of temperature field.

Substituting equation (7.16) into equation (7.7) obtains the thermal stresses for elastic materials at any location.

$$\sigma_{\mathrm{r}}^{\mathrm{t}} = p_{\mathrm{t}}\left(-\frac{\ln(k/x)}{\ln k} + \frac{k^2 - x^2}{x^2 k^2 - x^2}\right) \tag{7.18}$$

$$\sigma_{\theta}^{\mathrm{t}} = p_{\mathrm{t}}\left(\frac{1 - \ln(k/x)}{\ln k} - \frac{k^2 + x^2}{x^2 k^2 - x^2}\right) \tag{7.19}$$

$$\sigma_{\mathrm{z}}^{\mathrm{t}} = p_{\mathrm{t}}\left(\frac{1 - 2\ln(k/x)}{\ln k} - \frac{2}{k^2 - 1}\right) \tag{7.20}$$

where p_{t} is called thermal loading, $p_{\mathrm{t}} = \frac{E\alpha\Delta t}{2(1-\mu)}$. Equations (7.18) and (7.19) are consistent with the results in refs.[5–7].

Therefore, the thermal stresses when $t_{\mathrm{i}} > t_{\mathrm{o}}$ are equal to minus thermal stresses when $t_{\mathrm{i}} < t_{\mathrm{o}}$; the greater Δt is, the greater the absolute value of thermal stresses are.

From equations (7.18)–(7.20) and equation (7.11), it is seen that at any location,

$$\sigma_{\mathrm{z}}^{\mathrm{t}} = \sigma_{\mathrm{r}}^{\mathrm{t}} + \sigma_{\theta}^{\mathrm{t}} \tag{7.21}$$

In addition, the thermal stresses increase as elastic modules and the thermal expansion coefficient of the materials increase. The mechanism of this phenomenon is analyzed as follows:

The elastic modulus indicates the ability of the connected atoms of the material to resist separation or atomic bonding interactions, thus, greater stresses are needed for the materials with greater elastic modulus to obtain a certain strain. The thermal expansion coefficient is a parameter to characterize the expansion or contraction of the material with temperature changing. For the materials with great thermal expansion coefficients, the free strains are great when heated or cooled. However, if the materials are subject to constraints, the stresses caused because of the constraints must be great.

The thermal stresses decrease as the Poisson ratio of the material increases, but the Poisson ratios of most materials are similar. Therefore, adjusting the Poisson ratio to adjust the thermal stresses is not quite effective.

7.3 The Characteristics of the Thermal Stresses

Thermal stresses in the material are the foundation of material selection and design of apparatuses. Therefore, it is necessary to study the principal features of thermal stresses. From equations (7.18)–(7.20), at the inside and outside walls, the thermal stresses are as follows,

$$\sigma^{\mathrm{t}}_{\mathrm{ri}} = \sigma^{\mathrm{t}}_{\mathrm{ro}} = 0 \tag{7.22}$$

$$\sigma^{\mathrm{t}}_{\theta\mathrm{i}} = \sigma^{\mathrm{t}}_{\mathrm{zi}} = -p_{\mathrm{t}}\left(\frac{2k^2}{k^2-1} - \frac{1}{\ln k}\right) \text{ or } \frac{\sigma^{\mathrm{t}}_{\theta\mathrm{i}}}{p_{\mathrm{t}}} = \frac{\sigma^{\mathrm{t}}_{\mathrm{zi}}}{p_{\mathrm{t}}} = -\left(\frac{2k^2}{k^2-1} - \frac{1}{\ln k}\right) \tag{7.23}$$

$$\sigma^{\mathrm{t}}_{\theta\mathrm{o}} = \sigma^{\mathrm{t}}_{\mathrm{zo}} = p_{\mathrm{t}}\left(\frac{1}{\ln k} - \frac{2}{k^2-1}\right) \text{ or } \frac{\sigma^{\mathrm{t}}_{\theta\mathrm{o}}}{p_{\mathrm{t}}} = \frac{\sigma^{\mathrm{t}}_{\mathrm{zo}}}{p_{\mathrm{t}}} = \frac{1}{\ln k} - \frac{2}{k^2-1} \tag{7.24}$$

where the subscripts i and o represent the quantity at the inside and outside wall respectively. According to equations (7.22)–(7.24), when $r = r_{\mathrm{o}}$, $\sigma^{\mathrm{t}}_{\mathrm{ro}} = \sigma^{\mathrm{t}}_{\mathrm{zo}} - \sigma^{\mathrm{t}}_{\theta\mathrm{o}} = 0$, or $\sigma^{\mathrm{t}}_{\mathrm{zo}} = \sigma^{\mathrm{t}}_{\theta\mathrm{o}}$; when $r = r_{\mathrm{i}}$, $\sigma^{\mathrm{t}}_{\mathrm{ri}} = \sigma^{\mathrm{t}}_{\mathrm{zi}} - \sigma^{\mathrm{t}}_{\theta\mathrm{i}} = 0$, or $\sigma^{\mathrm{t}}_{\mathrm{zi}} = \sigma^{\mathrm{t}}_{\theta\mathrm{i}}$. This means that at the inside and outside wall, the circumferential thermal stress and axial thermal stress are equal to each other. Within $1 \leq x \leq k$, $\sigma^{\mathrm{t}}_{\mathrm{r}} = \sigma^{\mathrm{t}}_{\mathrm{z}} - \sigma^{\mathrm{t}}_{\theta} = 0$ has a sole extreme point at $x = \sqrt{\frac{k^2 \ln k^2}{k^2-1}}$[8], and if $\Delta t > 0$ (called internal heating), $\frac{\mathrm{d}\sigma^{\mathrm{t}}_{\mathrm{r}}}{\mathrm{d}x^2} > 0$; if $\Delta t < 0$ (called external heating), $\frac{\mathrm{d}\sigma^{\mathrm{t}}_{\mathrm{r}}}{\mathrm{d}x^2} < 0$. Therefore, if $\Delta t > 0$, $\sigma^{\mathrm{t}}_{\mathrm{r}}$ gets minimum at $x = \sqrt{\frac{k^2 \ln k^2}{k^2-1}}$, which means $\sigma^{\mathrm{t}}_{\mathrm{r}} = \sigma^{\mathrm{t}}_{\mathrm{z}} - \sigma^{\mathrm{t}}_{\theta} < 0$, or $\sigma^{\mathrm{t}}_{\mathrm{z}} < \sigma^{\mathrm{t}}_{\theta}$ within $1 \leq x \leq k$; if $\Delta t < 0$, $\sigma^{\mathrm{t}}_{\mathrm{r}}$ gets maximum at $x = \sqrt{\frac{k^2 \ln k^2}{k^2-1}}$, which means $\sigma^{\mathrm{t}}_{\mathrm{r}} = \sigma^{\mathrm{t}}_{\mathrm{z}} - \sigma^{\mathrm{t}}_{\theta} > 0$, or $\sigma^{\mathrm{t}}_{\mathrm{z}} > \sigma^{\mathrm{t}}_{\theta}$ within $1 \leq x \leq k$.

Since $\ln k^2 > 2\frac{k^2-1}{k^2+1}$, it is easy to obtain $\frac{1}{\ln k} < \frac{2k^2}{k^2-1}$, then $\sigma^{\mathrm{t}}_{\mathrm{zi}} = \sigma^{\mathrm{t}}_{\theta\mathrm{i}} < 0$; since $k^2 - 1 > \ln k^2$, it is easy to obtain $\frac{1}{\ln k} > \frac{2}{k^2-1}$, then $\sigma^{\mathrm{t}}_{\mathrm{zo}} = \sigma^{\mathrm{t}}_{\theta\mathrm{o}} > 0$. Within $1 \leq x \leq k$, neither $\sigma^{\mathrm{t}}_{\theta}$ nor $\sigma^{\mathrm{t}}_{\mathrm{z}}$ has pole. Therefore, when $\Delta t > 0$, $\sigma^{\mathrm{t}}_{\theta}$ and $\sigma^{\mathrm{t}}_{\mathrm{z}}$ increase monotonically from compressive stress on inside wall to tension on the outside wall; when $\Delta t < 0$, $\sigma^{\mathrm{t}}_{\theta}$ and $\sigma^{\mathrm{t}}_{\mathrm{z}}$ decrease monotonically from tension on the inside wall to compressive stress on the outside wall. Thus, it can be seen that the variation tendency of thermal stresses and the variation tendency of wall temperature difference are opposite. The reason is that deformation of the material is great where the temperature is high, when the material is constrained, the thermal stresses are certainly great; and *vice versa*. See figure 7.6 with figures 7.8 and 7.7 with figures 7.9, respectively.

From equations (7.22)–(7.24), we know that

$$\begin{aligned}|\sigma^{\mathrm{t}}_{\theta\mathrm{i}}| - |\sigma^{\mathrm{t}}_{\theta\mathrm{o}}| = |\sigma^{\mathrm{t}}_{\mathrm{zi}}| - |\sigma^{\mathrm{t}}_{\mathrm{zo}}| &= |p_{\mathrm{t}}|\left[\left(\frac{2k^2}{k^2-1} - \frac{1}{\ln k}\right) - \left(\frac{1}{\ln k} - \frac{2}{k^2-1}\right)\right] \\ &= 2|p_{\mathrm{t}}|\left(\frac{k^2+1}{k^2-1} - \frac{1}{\ln k}\right).\end{aligned}$$

Since $\ln k^2 = 2\ln k > 2\frac{k^2-1}{k^2+1}$[8], or $\frac{1}{\ln k} < \frac{k^2+1}{k^2-1}$, $|\sigma^{\mathrm{t}}_{\theta\mathrm{i}}| > |\sigma^{\mathrm{t}}_{\theta\mathrm{o}}|$ and $|\sigma^{\mathrm{t}}_{\mathrm{zi}}| > |\sigma^{\mathrm{t}}_{\mathrm{zo}}|$, which means that whether $\Delta t > 0$ or $\Delta t < 0$, the absolute values of thermal stresses on the inside wall are greater than those on an outside wall. Since $\sigma^{\mathrm{t}}_{\mathrm{ri}} = 0$, as $|\Delta t|$ increases, the inside wall becomes yielded always earlier than the outside wall. If $\Delta t > 0$, when equivalent thermal stress $\sigma_{\mathrm{ei}}^{\mathrm{ts}} = \sigma_{\mathrm{ri}}^{\mathrm{t}} - \sigma_{\mathrm{zi}}^{\mathrm{t}} = \sigma_{\mathrm{y}}$ (Tresca criterion), inside wall initiates yielding, where σ_{y} is yield limit of the materials. From equation (7.22)–(7.24),

$$\sigma_{\mathrm{ei}}^{\mathrm{ts}} = \sigma_{\mathrm{ri}}^{\mathrm{t}} - \sigma_{\mathrm{zi}}^{\mathrm{t}} = -\sigma_{\mathrm{zi}}^{\mathrm{t}} = -\sigma_{\theta\mathrm{i}}^{\mathrm{t}} = p_{\mathrm{t}}\left(\frac{2k^2}{k^2-1} - \frac{1}{\ln k}\right) = p_{\mathrm{t}}\frac{k^2\ln k^2 - k^2 + 1}{(k^2-1)\ln k} \tag{7.25}$$

For equation (7.25), letting $\sigma_{\mathrm{ei}}^{\mathrm{ts}} = \sigma_{\mathrm{y}}$ obtains the temperature difference at which the inside wall initiates yielding, called critical temperature difference, Δt_{c}:

$$\Delta t_{\mathrm{c}} = \frac{(k^2-1)\ln k}{k^2\ln k^2 - k^2 + 1}\frac{2(1-\mu)\sigma_{\mathrm{y}}}{E\alpha} = mf \tag{7.26}$$

where $m = \frac{(k^2-1)\ln k}{k^2\ln k^2 - k^2+1}$, reflecting the size of a cylindrical pressure vessel; $f = \frac{2(1-\mu)\sigma_{\mathrm{y}}}{E\alpha}$, reflecting material properties. Equation (7.26) shows that the thicker the cylindrical pressure vessel is, the smaller the $|\Delta t_{\mathrm{c}}|$ is, this means that the thicker the cylindrical pressure vessel is, the more easily it becomes yielded under the thermal stresses. When $k \to 1$, $m = 1$, $\Delta t_{\mathrm{c}} = \frac{2(1-\mu)\sigma_{\mathrm{y}}}{E\alpha}$; and $k \to \infty$, $m = 0.5$, $\Delta t_{\mathrm{c}} = \frac{(1-\mu)\sigma_{\mathrm{y}}}{E\alpha}$. The critical temperature difference at $\Delta t < 0$ is equal to minus equation (7.26).

When $\Delta t > 0$, the thermal loading at which the inside wall initiates yielding, called critical thermal loading p_{tc}, is obtained from equation (7.26):

$$p_{\mathrm{tc}} = \frac{(k^2-1)\ln k}{k^2\ln k^2 - k^2 + 1}\sigma_{\mathrm{y}} \quad \text{or} \quad \frac{p_{\mathrm{tc}}}{\sigma_{\mathrm{y}}} = \frac{(k^2-1)\ln k}{k^2\ln k^2 - k^2 + 1} = m \tag{7.27}$$

Equation (7.27) is plotted in figure 7.8 to show the variation tendency of m with the increase of k.

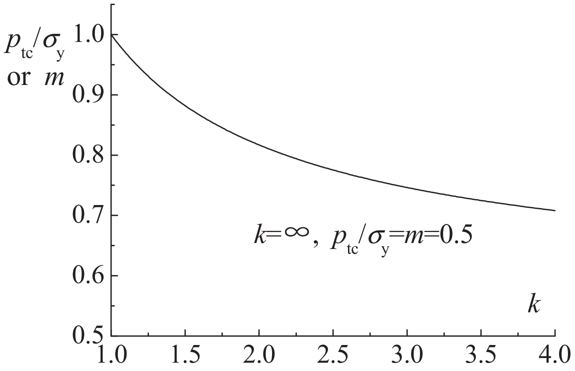

FIG. 7.8 – The variation tendency of m with the increase of k.

An example: commonly, elastic modules of steel materials $E = 1.95 \times 10^5$ MPa, Poisson ratio $\mu = 0.3$, thermal expansion coefficient of the materials $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, if $\Delta t = 50$ °C, then, $p_{\mathrm{t}} = 83.57143$ MPa. Setting $k = 2.5$, the thermal stresses are shown as the solid curves in figure 7.9. If $\Delta t < 0$, the trend of variations of thermal stresses is opposite to figure 7.9, as is shown in figure 7.10. The meaning of each curve and the main parameters are marked in the figures.

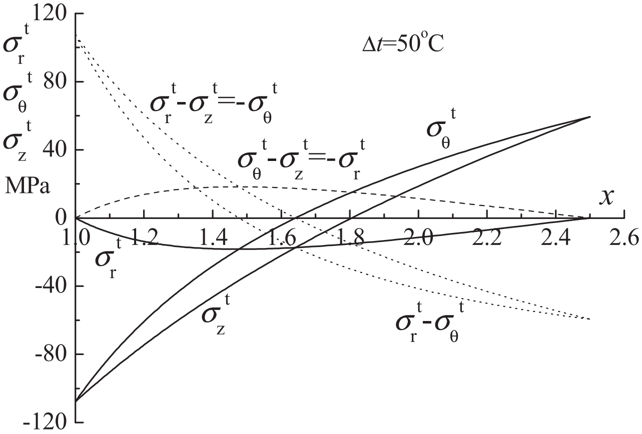

FIG. 7.9 – Distribution of thermal stresses for elastic cylindrical pressure vessel when $\Delta t > 0$.

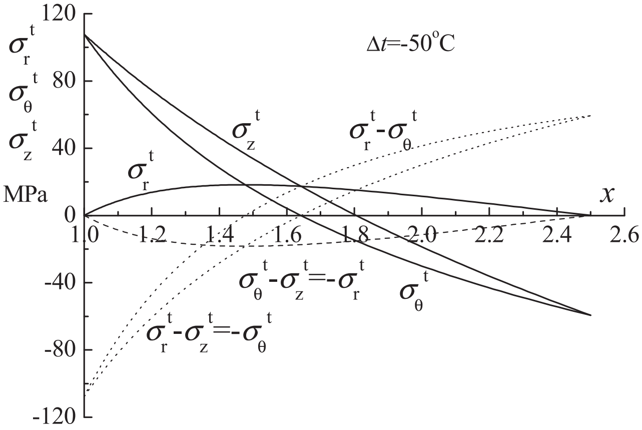

FIG. 7.10 – Distribution of thermal stresses for elastic cylindrical pressure vessel when $\Delta t < 0$.

From equations (7.23) and (7.24), $\frac{\sigma_{\theta i}^t}{p_t} - \frac{\sigma_{\theta o}^t}{p_t} = \frac{\sigma_{zi}^t}{p_t} - \frac{\sigma_{zo}^t}{p_t} = -2$; with $k \to \infty$, $\frac{\sigma_{\theta i}^t}{p_t}$ and $\frac{\sigma_{zi}^t}{p_t} \to -2$, $\frac{\sigma_{\theta o}^t}{p_t}$ and $\frac{\sigma_{zo}^t}{p_t} \to 0$.

The variation tendencies of $\frac{\sigma_{\theta i}^t}{p_t}$ and $\frac{\sigma_{zi}^t}{p_t}$ as well as $\frac{\sigma_{\theta o}^t}{p_t}$ and $\frac{\sigma_{zo}^t}{p_t}$ with the increase of k are shown in figure 7.11.

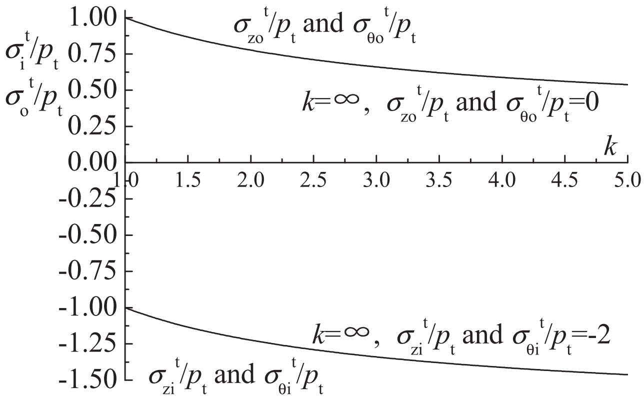

FIG. 7.11 – The variation tendencies of $\frac{\sigma_{\theta i}^{t}}{p_{t}}$ and $\frac{\sigma_{zi}^{t}}{p_{t}}$ as well as $\frac{\sigma_{\theta o}^{t}}{p_{t}}$ and $\frac{\sigma_{zo}^{t}}{p_{t}}$ with the increase of k when $\Delta t > 0$.

$\frac{\mathrm{d}\left(\frac{\sigma_{\theta i}^{t}}{p_{t}}\right)}{\mathrm{d}k}=\frac{\mathrm{d}\left(\frac{\sigma_{zi}^{t}}{p_{t}}\right)}{\mathrm{d}k}=\frac{\mathrm{d}\left(\frac{\sigma_{\theta o}^{t}}{p_{t}}\right)}{\mathrm{d}k}=\frac{\mathrm{d}\left(\frac{\sigma_{zo}^{t}}{p_{t}}\right)}{\mathrm{d}k}=\frac{(2k\ln k+k^2-1)(2k\ln k-k^2+1)}{k[(k^2-1)\ln k]^2}$. When $k=1$, $2k\ln k-k^2+1=0$, while $\frac{\mathrm{d}(2k\ln k-k^2+1)}{\mathrm{d}k}=2[\ln k-(k-1)]<0$ because $k-1>\ln k$. Therefore, $2k\ln k-k^2+1<0$, while $2k\ln k+k^2-1>0$, $k[(k^2-1)\ln k]^2>0$, accordingly, $\frac{\mathrm{d}\left(\frac{\sigma_{\theta i}^{t}}{p_{t}}\right)}{\mathrm{d}k}=\frac{\mathrm{d}\left(\frac{\sigma_{zi}^{t}}{p_{t}}\right)}{\mathrm{d}k}=\frac{\mathrm{d}\left(\frac{\sigma_{\theta o}^{t}}{p_{t}}\right)}{\mathrm{d}k}=\frac{\mathrm{d}\left(\frac{\sigma_{zo}^{t}}{p_{t}}\right)}{\mathrm{d}k}<0$. Besides, $-\left(\frac{2k^2}{k^2-1}-\frac{1}{\ln k}\right)<0$ and $\frac{1}{\ln k}-\frac{2}{k^2-1}>0$. Thus, the larger k is, the larger $\left|\frac{\sigma_{\theta i}^{t}}{p_{t}}\right|$ and $\left|\frac{\sigma_{zi}^{t}}{p_{t}}\right|$ are; the larger k is, the smaller $\left|\frac{\sigma_{\theta o}^{t}}{p_{t}}\right|$ and $\left|\frac{\sigma_{zo}^{t}}{p_{t}}\right|$ are, as is seen from figure 7.11.

Figure 7.11 includes $p_t>0$ (or $\Delta t>0$, and $\sigma_{zi}^{t}<0$, $\sigma_{\theta i}^{t}<0$) and $p_t<0$ (or $\Delta t<0$, and $\sigma_{zi}^{t}>0$, $\sigma_{\theta i}^{t}>0$). If figure 7.11 represents the case of $p_t>0$ ($\Delta t>0$), when $p_t<0$ ($\Delta t<0$), the variation tendencies of $\frac{\sigma_{\theta i}^{t}}{p_{t}}$ and $\frac{\sigma_{zi}^{t}}{p_{t}}$ as well as $\frac{\sigma_{\theta o}^{t}}{p_{t}}$ and $\frac{\sigma_{zo}^{t}}{p_{t}}$ with the increase of k are shown in figure 7.12. When $p_t<0$ ($\Delta t<0$), as k tends to ∞, $\frac{\sigma_{\theta i}^{t}}{p_{t}}$ and $\frac{\sigma_{zi}^{t}}{p_{t}}$ tend to 2, $\frac{\sigma_{\theta o}^{t}}{p_{t}}$ and $\frac{\sigma_{zo}^{t}}{p_{t}}$ tend to 0. In a word, regardless of $p_t>0$ ($\Delta t>0$) or $p_t<0$ ($\Delta t<0$), as the k increases, $\frac{\sigma_{\theta o}^{t}}{p_{t}}$ and $\frac{\sigma_{zo}^{t}}{p_{t}}$ get closer to the k axis (horizontal axis), and $\frac{\sigma_{\theta i}^{t}}{p_{t}}$ and $\frac{\sigma_{zi}^{t}}{p_{t}}$ gradually leave the horizontal axis (k axis) and are stabilized at $\frac{\sigma_{\theta i}^{t}}{p_{t}}=\frac{\sigma_{zi}^{t}}{p_{t}}=2$.

When $\Delta t>0$ ($p_t>0$), σ_r^t reaches the minimum at $x=\sqrt{\frac{k^2\ln k^2}{k^2-1}}$, the minimum is

$$\frac{\sigma_{\mathrm{rmin}}^{t}}{p_{t}}=\frac{\ln\sqrt{\frac{\ln k^2}{k^2-1}}}{\ln k}+\frac{1}{\ln k^2}-\frac{1}{k^2-1} \tag{7.28}$$

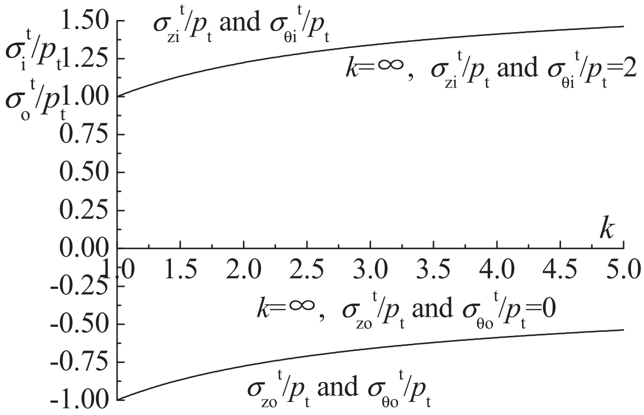

FIG. 7.12 – The variation tendencies of $\frac{\sigma_{\theta i}^{t}}{p_t}$ and $\frac{\sigma_{zi}^{t}}{p_t}$ as well as $\frac{\sigma_{\theta o}^{t}}{p_t}$ and $\frac{\sigma_{zo}^{t}}{p_t}$ with the increase of k when $\Delta t < 0$.

When $\Delta t < 0$ ($p_t < 0$), σ_r^t reaches the maximum at $x = \sqrt{\frac{k^2 \ln k^2}{k^2-1}}$, the maximum is

$$\frac{\sigma_{\mathrm{rmax}}^{t}}{p_t} = -\frac{\sigma_{\mathrm{rmin}}^{t}}{p_t} = \frac{1}{k^2-1} - \frac{1}{\ln k^2} - \frac{\ln\sqrt{\frac{\ln k^2}{k^2-1}}}{\ln k} \tag{7.29}$$

Figure 7.13 shows the variation tendencies of $\frac{\sigma_{\mathrm{rmin}}^{t}}{p_t}$ and $\frac{\sigma_{\mathrm{rmax}}^{t}}{p_t}$ with the increase of k. From figure 7.13, the larger k is, the larger $\left|\frac{\sigma_{\mathrm{rmin}}^{t}}{p_t}\right|$ and $\left|\frac{\sigma_{\mathrm{rmax}}^{t}}{p_t}\right|$ are.

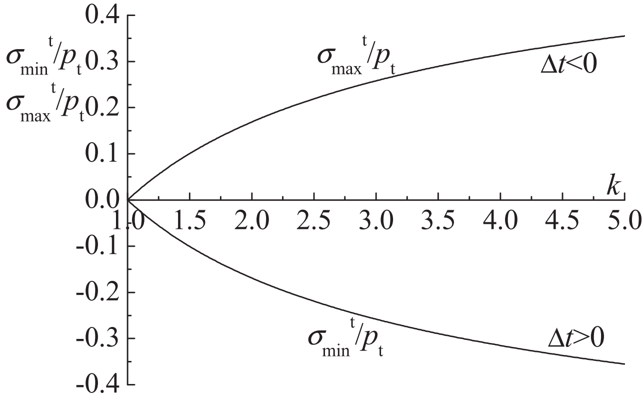

FIG. 7.13 – The variation tendencies of $\frac{\sigma_{\mathrm{rmin}}^{t}}{p_t}$ and $\frac{\sigma_{\mathrm{rmax}}^{t}}{p_t}$ with the increase of k.

In order to show the above conclusions, figure 7.14 illustrates the distributions of $\frac{\sigma_z^t}{p_t}$ and $\frac{\sigma_\theta^t}{p_t}$ along x for $k = 1.5$, 2, 3 and 4, and figure 7.15 illustrates the distributions of $\frac{\sigma_r^t}{p_t}$ along x for $k = 1.5$, 2, 3 and 4.

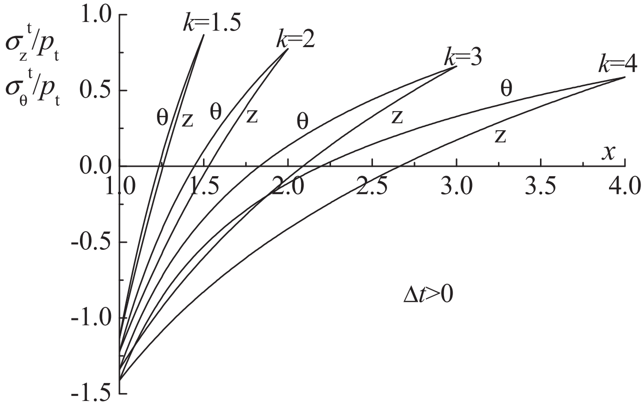

FIG. 7.14 – The distributions of $\frac{\sigma_z^t}{p_t}$ and $\frac{\sigma_\theta^t}{p_t}$ along x with $k = 1.5$, 2, 3 and 4.

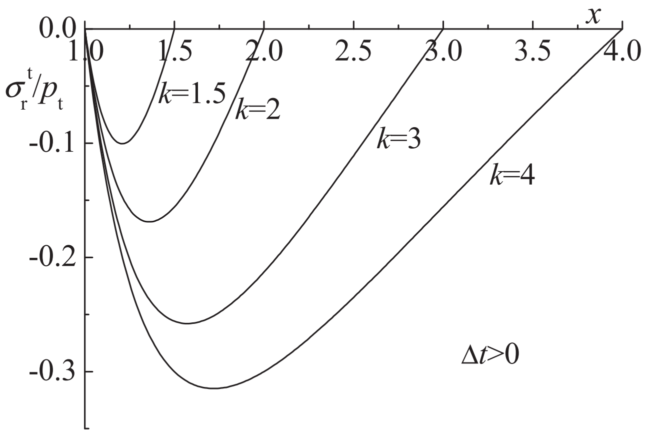

FIG. 7.15 – The distributions of $\frac{\sigma_r^t}{p_t}$ along x with $k = 1.5$, 2, 3 and 4.

Figures 7.16 and 7.17 is a combination of figures 7.14 and 7.15 for $k = 2$ and 4, respectively.

When $\Delta t < 0$ ($p_t < 0$), the thermal stresses are negative, those in figures 7.14–7.17, or they are symmetric about x-axis.

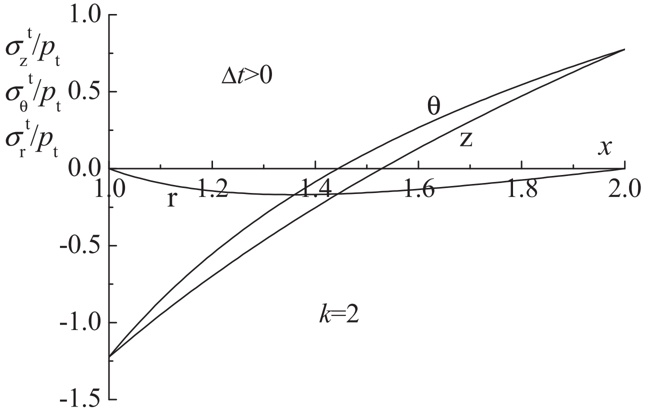

FIG. 7.16 – The distributions of $\frac{\sigma_z^t}{p_t}$, $\frac{\sigma_\theta^t}{p_t}$ and $\frac{\sigma_r^t}{p_t}$ along x with $k = 2$.

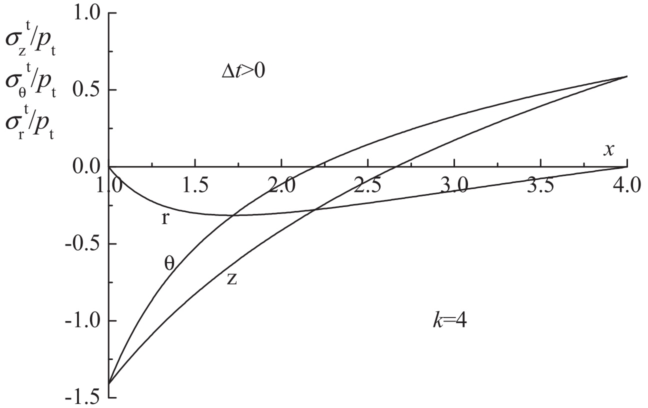

FIG. 7.17 – The distributions of $\frac{\sigma_z^t}{p_t}$, $\frac{\sigma_\theta^t}{p_t}$ and $\frac{\sigma_r^t}{p_t}$ along x with $k = 4$.

If the temperature difference $|\Delta t|$ exceeds critical temperature difference $|\Delta t_c|$, the plastic zone (yielded zone) develops further, thus the wall material becomes elastoplastic, part of the material is elastic, and part of the material is plastic. After the temperature difference is released, the plastic part of the material cannot be recovered, and the elastic part is trying to recover, then, the residual stresses (thermal stresses) thereupon remained. These residual stresses (thermal stresses) can be controlled in order to obtain a favorable state of stresses for the cylindrical pressure vessel.

In an elastic state, the most dangerous stress under internal pressure operation is tension at the inside wall, and when $\Delta t > 0$, the compressive thermal stresses at the internal wall are relatively large. Thus, the thermal stresses when $\Delta t > 0$ are beneficial to decreasing dangerous stress under internal pressure operation, then, this chapter studies the internal heating and internal pressure operation problems.

7.4 The Analysis of Total Stresses and Investigation of Optimum Operation Conditions

The stresses caused by internal operation pressure (p) contained in a pressure vessel are called mechanical stresses, *i.e.*, equation (1.1), thermal stresses plus mechanical stresses are called total stresses. When a pressure vessel is subjected to temperature difference ($\Delta t > 0$) and internal operation pressure at the same time, the total stresses are as follows,

$$\sigma_{\mathrm{r}}^{\mathrm{T}} = \sigma_{\mathrm{r}}^{\mathrm{t}} + \sigma_{\mathrm{r}}^{\mathrm{p}} = p_{\mathrm{t}}\left(-\frac{\ln(k/x)}{\ln k} + \frac{k^2 - x^2}{x^2 k^2 - x^2}\right) + \frac{p}{k^2 - 1}\left(1 - \frac{k^2}{x^2}\right) \tag{7.30}$$

$$\sigma_{\theta}^{\mathrm{T}} = \sigma_{\theta}^{\mathrm{t}} + \sigma_{\theta}^{\mathrm{p}} = p_{\mathrm{t}}\left(\frac{1 - \ln(k/x)}{\ln k} - \frac{k^2 + x^2}{x^2 k^2 - x^2}\right) + \frac{p}{k^2 - 1}\left(1 + \frac{k^2}{x^2}\right) \tag{7.31}$$

$$\sigma_{\mathrm{z}}^{\mathrm{T}} = \sigma_{\mathrm{z}}^{\mathrm{t}} + \sigma_{\mathrm{z}}^{\mathrm{p}} = p_{\mathrm{t}}\left(\frac{1 - 2\ln(k/x)}{\ln k} - \frac{2}{k^2 - 1}\right) + \frac{p}{k^2 - 1} \tag{7.32}$$

where $\sigma_{\mathrm{r}}^{\mathrm{T}}$, $\sigma_{\theta}^{\mathrm{T}}$ and $\sigma_{\mathrm{z}}^{\mathrm{T}}$ are total radial stress, total circumferential stress, and total axial stress, respectively, $\sigma_{\mathrm{r}}^{\mathrm{p}}$, $\sigma_{\theta}^{\mathrm{p}}$ and $\sigma_{\mathrm{z}}^{\mathrm{p}}$ are mechanical radial stress, mechanical circumferential stress, and mechanical axial stress, respectively.

It is easy to prove that the curve for $\sigma_{\theta}^{\mathrm{T}}$ is parallel to the curve for $\sigma_{\mathrm{z}}^{\mathrm{T}} - \sigma_{\mathrm{r}}^{\mathrm{T}}$, or $\sigma_{\theta}^{\mathrm{T}} - (\sigma_{\mathrm{z}}^{\mathrm{T}} - \sigma_{\mathrm{r}}^{\mathrm{T}})$ is a constant:

$$\begin{aligned}
\sigma_{\theta}^{\mathrm{T}} - (\sigma_{\mathrm{z}}^{\mathrm{T}} - \sigma_{\mathrm{r}}^{\mathrm{T}}) &= (\sigma_{\theta}^{\mathrm{t}} + \sigma_{\theta}^{\mathrm{p}}) - [(\sigma_{\mathrm{z}}^{\mathrm{t}} + \sigma_{\mathrm{z}}^{\mathrm{p}}) - (\sigma_{\mathrm{r}}^{\mathrm{t}} + \sigma_{\mathrm{r}}^{\mathrm{p}})] \\
&= (\sigma_{\theta}^{\mathrm{t}} - \sigma_{\mathrm{z}}^{\mathrm{t}} + \sigma_{\mathrm{r}}^{\mathrm{t}}) + (\sigma_{\theta}^{\mathrm{p}} + \sigma_{\mathrm{r}}^{\mathrm{p}}) - \sigma_{\mathrm{z}}^{\mathrm{p}} \\
&= 0 + 2\sigma_{\mathrm{z}}^{\mathrm{p}} - \sigma_{\mathrm{z}}^{\mathrm{p}} = \sigma_{\mathrm{z}}^{\mathrm{p}} \\
&= \frac{p}{k^2 - 1} = \text{a constant} > 0.
\end{aligned}$$

Therefore, $\sigma_{\mathrm{z}}^{\mathrm{T}} - \sigma_{\mathrm{r}}^{\mathrm{T}} < \sigma_{\theta}$.

$$\sigma_{\theta}^{\mathrm{T}} - \sigma_{\mathrm{r}}^{\mathrm{T}} = p_{\mathrm{t}}\left(\frac{1}{\ln k} - \frac{2k^2}{x^2 k^2 - x^2}\right) + \frac{2k^2 p}{x^2 k^2 - x^2} = \frac{2k^2}{x^2 k^2 - x^2}(p - p_{\mathrm{t}}) + \frac{p_{\mathrm{t}}}{\ln k} \tag{7.33}$$

$$\sigma_{\theta}^{\mathrm{T}} - \sigma_{\mathrm{r}}^{\mathrm{T}} = (\sigma_{\theta}^{\mathrm{T}} - \sigma_{\mathrm{z}}^{\mathrm{T}}) + (\sigma_{\mathrm{z}}^{\mathrm{T}} - \sigma_{\mathrm{r}}^{\mathrm{T}})$$

$\sigma_{\mathrm{ro}}^{\mathrm{T}} = 0$, so if $\sigma_{\theta\mathrm{o}}^{\mathrm{T}} - \sigma_{\mathrm{ro}}^{\mathrm{T}} \leq \sigma_{\mathrm{y}}$, then $\sigma_{\theta\mathrm{o}}^{\mathrm{T}} \leq \sigma_{\mathrm{y}}$. $\sigma_{\theta\mathrm{i}}^{\mathrm{T}} = -p < 0$, so if $\sigma_{\theta\mathrm{i}}^{\mathrm{T}} - \sigma_{\mathrm{ri}}^{\mathrm{T}} \leq \sigma_{\mathrm{y}}$, then $\sigma_{\theta\mathrm{i}}^{\mathrm{T}} \leq \sigma_{\mathrm{y}}$.

It is known by further analysis that $\sigma_\theta^{\mathrm{T}} - \sigma_{\mathrm{r}}^{\mathrm{T}}$ is generally the most dangerous stress in engineering.

$\frac{\mathrm{d}(\sigma_\theta - \sigma_{\mathrm{r}})^{\mathrm{T}}}{\mathrm{d}x} = -\frac{4k^2(p-p_{\mathrm{t}})}{x^3(k^2-1)}$, it is thus clear that $\sigma_\theta^{\mathrm{T}} - \sigma_{\mathrm{r}}^{\mathrm{T}}$ has no stationary point within $-\infty < x < +\infty$, except $p = p_{\mathrm{t}}$. If $p < p_{\mathrm{t}}$, $\sigma_\theta^{\mathrm{T}} - \sigma_{\mathrm{r}}^{\mathrm{T}}$ increase monotonically; if $p > p_{\mathrm{t}}$, $\sigma_\theta^{\mathrm{T}} - \sigma_{\mathrm{r}}^{\mathrm{T}}$ decrease monotonically; if $p = p_{\mathrm{t}}$, $\sigma_\theta^{\mathrm{T}} - \sigma_{\mathrm{r}}^{\mathrm{T}}$ is constant: $\frac{p_{\mathrm{t}}}{\ln k}$.

Therefore, $\sigma_\theta^{\mathrm{T}} - \sigma_{\mathrm{r}}^{\mathrm{T}}$ at the inside and outside walls is the key to ensuring the safety of a pressure vessel.

At the inside wall, $x = 1$, then

$$\sigma_{\theta\mathrm{i}}^{\mathrm{T}} - \sigma_{\mathrm{ri}}^{\mathrm{T}} = p_{\mathrm{t}}\left(\frac{1}{\ln k} - \frac{2k^2}{k^2-1}\right) + \frac{2k^2 p}{k^2-1} = \frac{2k^2}{k^2-1}(p - p_{\mathrm{t}}) + \frac{p_{\mathrm{t}}}{\ln k} \tag{7.34}$$

If $\frac{p}{\sigma_{\mathrm{y}}} \geq \frac{k^2\ln k^2 - k^2 + 1}{k^2 \ln k^2}\frac{p_{\mathrm{t}}}{\sigma_{\mathrm{y}}} - \frac{k^2-1}{2k^2}$, $\sigma_{\theta\mathrm{i}}^{\mathrm{T}} - \sigma_{\mathrm{ri}}^{\mathrm{T}} \geq -\sigma_{\mathrm{y}}$. Substituting equation (7.26) into equation (7.34) shows that even if $p_{\mathrm{t}} = p_{\mathrm{tc}}$, $\sigma_{\theta\mathrm{i}}^{\mathrm{T}} - \sigma_{\mathrm{ri}}^{\mathrm{T}} \geq -\sigma_{\mathrm{y}}$ provided $p > 0$. So $\sigma_{\theta\mathrm{i}}^{\mathrm{T}} - \sigma_{\mathrm{ri}}^{\mathrm{T}} \geq -\sigma_{\mathrm{y}}$ in engineering.

At the outside wall, $x = k$, then

$$\sigma_{\theta\mathrm{o}}^{\mathrm{T}} - \sigma_{\mathrm{ro}}^{\mathrm{T}} = p_{\mathrm{t}}\left(\frac{1}{\ln k} - \frac{2}{k^2-1}\right) + \frac{2p}{k^2-1} = \frac{2}{k^2-1}(p - p_{\mathrm{t}}) + \frac{p_{\mathrm{t}}}{\ln k} > 0 \tag{7.35}$$

Setting $\sigma_{\theta\mathrm{i}}^{\mathrm{T}} - \sigma_{\mathrm{ri}}^{\mathrm{T}} \leq \sigma_{\mathrm{y}}$ obtains load-bearing capacity or allowable loading, p/σ_{y}:

$$\frac{p}{\sigma_{\mathrm{y}}} \leq \frac{k^2-1}{2k^2} + \frac{p_{\mathrm{t}}}{\sigma_{\mathrm{y}}}\frac{k^2\ln k^2 - k^2 + 1}{k^2\ln k^2} = \frac{k^2-1}{2k^2} + \frac{E\alpha}{2(1-\mu)\sigma_{\mathrm{y}}}\frac{k^2\ln k^2 - k^2 + 1}{k^2\ln k^2}\Delta t = \frac{p_1}{\sigma_{\mathrm{y}}} \tag{7.36}$$

$$\text{or } \Delta t \geq \left(k^2\frac{p}{\sigma_{\mathrm{y}}} - \frac{k^2-1}{2}\right)\frac{\ln k^2}{k^2\ln k^2 - k^2 + 1}\frac{2(1-\mu)\sigma_{\mathrm{y}}}{E\alpha} = \Delta t_1 \tag{7.37}$$

The greater Δt is, the greater p_1 is. When $\frac{p}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{2k^2} = \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$, $\Delta t_1 = 0$.

Setting $\sigma_{\theta\mathrm{o}}^{\mathrm{T}} - \sigma_{\mathrm{ro}}^{\mathrm{T}} \leq \sigma_{\mathrm{y}}$ obtains another load-bearing capacity or allowable loading, p/σ_{y}:

$$\frac{p}{\sigma_{\mathrm{y}}} \leq \frac{k^2-1}{2} - \frac{p_{\mathrm{t}}}{\sigma_{\mathrm{y}}}\frac{k^2-1-\ln k^2}{\ln k^2} = \frac{k^2-1}{2} - \frac{E\alpha}{2(1-\mu)\sigma_{\mathrm{y}}}\frac{k^2-1-\ln k^2}{\ln k^2}\Delta t = \frac{p_2}{\sigma_{\mathrm{y}}} \tag{7.38}$$

$$\text{or } \quad \Delta t \leq \left(\frac{k^2-1}{2} - \frac{p}{\sigma_{\mathrm{y}}}\right)\frac{\ln k^2}{k^2-1-\ln k^2}\frac{2(1-\mu)\sigma_{\mathrm{y}}}{E\alpha} = \Delta t_2 \tag{7.39}$$

The greater Δt is, the smaller p_2 is. When $\frac{p}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{2} = k^2\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$, $\Delta t_2 = 0$.

Therefore, to ensure that $\sigma_{\theta\mathrm{i}}^{\mathrm{T}} - \sigma_{\mathrm{ri}}^{\mathrm{T}} \leq \sigma_{\mathrm{y}}$ and $\sigma_{\theta\mathrm{o}}^{\mathrm{T}} - \sigma_{\mathrm{ro}}^{\mathrm{T}} \leq \sigma_{\mathrm{y}}$ at the same time, there should be $\Delta t_1 \leq \Delta t \leq \Delta t_2$.

Setting $\Delta t_1 = \Delta t_2$ results in

$$\frac{p}{\sigma_\mathrm{y}} = \ln k = \frac{p_\mathrm{y}}{\sigma_\mathrm{y}} \tag{7.40}$$

Equation (7.40) is just the mechanical entire yield pressure.

If $\frac{p}{\sigma_\mathrm{y}} < \ln k$, $\Delta t_1 < \Delta t_2$; if $\frac{p}{\sigma_\mathrm{y}} > \ln k$, $\Delta t_1 > \Delta t_2$. This is not allowed in engineering. It happens that there is a similar case that in mechanical autofrettage technology, the load-bearing capacity or allowable loading is also always smaller than the entire yield loading in any case.

At the inside wall, $x = 1$, then,

$$\sigma_\mathrm{ri}^\mathrm{T} = -p,\ \sigma_{\theta\mathrm{i}}^\mathrm{T} = p_\mathrm{t}\left(\frac{1}{\ln k} - \frac{2k^2}{k^2-1}\right) + \frac{k^2+1}{k^2-1}p \tag{7.41}$$

If $\frac{p}{\sigma_\mathrm{y}} \geq \frac{k^2\ln k^2 - k^2 + 1}{(k^2+1)\ln k}\frac{p_\mathrm{t}}{\sigma_\mathrm{y}} - \frac{k^2-1}{k^2+1}$, $\sigma_{\theta\mathrm{i}}^\mathrm{T} \geq -\sigma_\mathrm{y}$. Substituting equation (7.26) into equation (7.41) shows that even if $p_\mathrm{t} = p_\mathrm{tc}$, $\sigma_{\theta\mathrm{i}}^\mathrm{T} \geq -\sigma_\mathrm{y}$ provided $p > 0$. So $\sigma_{\theta\mathrm{i}}^\mathrm{T} \geq -\sigma_\mathrm{y}$ in engineering.

At the outside wall, $x = k$, then,

$$\sigma_\mathrm{ro}^\mathrm{T} = 0,\ \sigma_{\theta\mathrm{o}}^\mathrm{T} = \sigma_{\theta\mathrm{o}}^\mathrm{T} - \sigma_\mathrm{ro}^\mathrm{T} = p_\mathrm{t}\left(\frac{1}{\ln k} - \frac{2}{k^2-1}\right) + \frac{2p}{k^2-1} = \frac{2}{k^2-1}(p - p_\mathrm{t}) + \frac{p_\mathrm{t}}{\ln k} > 0 \tag{7.42}$$

If $\Delta t = \Delta t_\mathrm{c}$, substituting equation (7.26) into equation (7.36) obtains:

$$\frac{p}{\sigma_\mathrm{y}} \leq \frac{k^2-1}{k^2} = \frac{p_3}{\sigma_\mathrm{y}} = 2\frac{p_\mathrm{e}}{\sigma_\mathrm{y}} \quad \text{or} \quad k = \sqrt{\frac{\sigma_\mathrm{y}}{\sigma_\mathrm{y} - p}} \tag{7.43}$$

where p_e is the maximum mechanical elastic load-bearing capability of an unautofrettaged cylindrical pressure vessel or initial mechanical yield pressure, $\frac{p_\mathrm{e}}{\sigma_\mathrm{y}} = \frac{k^2-1}{2k^2}$.

Substituting equation (7.26) into equation (7.38) obtains:

$$\frac{p}{\sigma_\mathrm{y}} \leq \frac{(k^4-1)\ln k - (k^2-1)^2}{k^2\ln k^2 - k^2 + 1} = \frac{p_4}{\sigma_\mathrm{y}} \tag{7.44}$$

In fact, setting $\Delta t_1 \leq \Delta t_\mathrm{c}$ obtains $\frac{p}{\sigma_\mathrm{y}} \leq \frac{k^2-1}{k^2} = 2\frac{p_\mathrm{e}}{\sigma_\mathrm{y}}$; setting $\Delta t_2 \leq \Delta t_\mathrm{c}$ results in $p \geq p_4$. $\frac{p_3}{\sigma_\mathrm{y}}$, $\frac{p_4}{\sigma_\mathrm{y}}$ and $\frac{p_\mathrm{y}}{\sigma_\mathrm{y}}$ are plotted in figure 7.18. The meaning of each curve and the main parameters are marked in the figures.

Figure 7.18 shows that the three curves for $\frac{p_3}{\sigma_\mathrm{y}}$, $\frac{p_4}{\sigma_\mathrm{y}}$ and $\frac{p_\mathrm{y}}{\sigma_\mathrm{y}}$ have a common intersection point a, whose coordinates are a (k_c, 0.796812…).

As mentioned above, when $\Delta t = \Delta t_\mathrm{c}$, if $\frac{p}{\sigma_\mathrm{y}} \leq \frac{k^2-1}{k^2} = \frac{p_3}{\sigma_\mathrm{y}} = 2\frac{p_\mathrm{e}}{\sigma_\mathrm{y}}$, $\sigma_{\theta\mathrm{i}}^\mathrm{T} - \sigma_\mathrm{ri}^\mathrm{T} \leq \sigma_\mathrm{y}$ can be guaranteed; if $\frac{p}{\sigma_\mathrm{y}} \leq \frac{(k^4-1)\ln k - (k^2-1)^2}{k^2\ln k^2 - k^2 + 1} = \frac{p_4}{\sigma_\mathrm{y}}$, $\sigma_{\theta\mathrm{o}}^\mathrm{T} - \sigma_\mathrm{ro}^\mathrm{T} = \sigma_\mathrm{y}$ can be guaranteed. From figure 7.18, when $k < k_\mathrm{c}$, $\frac{p_3}{\sigma_\mathrm{y}} > \frac{p_4}{\sigma_\mathrm{y}}$, therefore, when $k < k_\mathrm{c}$ and $\Delta t = \Delta t_\mathrm{c}$, only if $p \leq p_4$, $\sigma_{\theta\mathrm{i}}^\mathrm{T} - \sigma_\mathrm{ri}^\mathrm{T} \leq \sigma_\mathrm{y}$ and $\sigma_{\theta\mathrm{o}}^\mathrm{T} - \sigma_\mathrm{ro}^\mathrm{T} \leq \sigma_\mathrm{y}$ can be guaranteed simultaneously. However, p_4 is

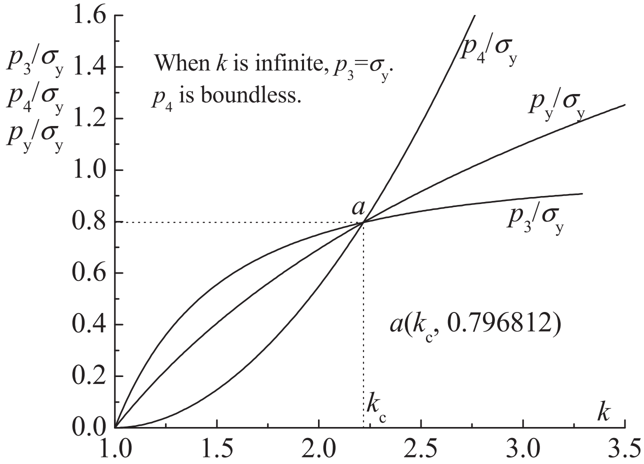

FIG. 7.18 – The graphs of $\frac{p_3}{\sigma_y}$, $\frac{p_4}{\sigma_y}$ and $\frac{p_y}{\sigma_y}$.

too low. Thus, when $k < k_c$, if $\Delta t = \Delta t_c$ can not be changed, p can only take the value of p_4; if Δt can be changed within $0 < \Delta t < \Delta t_c$, the allowable load p may be raised. The specific method to raise the allowable load p will be presented later. From figure 7.18 again, when $k > k_c$, $\frac{p_3}{\sigma_y} < \frac{p_4}{\sigma_y}$, therefore, when $k > k_c$ and $\Delta t = \Delta t_c$, if $p \leq p_3$, $\sigma_{\theta i}^T - \sigma_{ri}^T \leq \sigma_y$ and $\sigma_{\theta o}^T - \sigma_{ro}^T \leq \sigma_y$ can be guaranteed simultaneously.

Substituting Δt_1 into equation (7.34) must result in $\sigma_{\theta i}^T - \sigma_{ri}^T = \sigma_y$, substituting Δt_2 into equation (7.35) must result in $\sigma_{\theta o}^T - \sigma_{ro}^T = \sigma_y$; Substituting Δt_1 into equation (7.35) and setting $\sigma_{\theta o}^T - \sigma_{ro}^T \leq \sigma_y$, and substituting Δt_2 into equation (7.34) and setting $\sigma_{\theta i}^T - \sigma_{ri}^T \leq \sigma_y$ both result in $\frac{p}{\sigma_y} < \ln k$. This result is equivalent to $\Delta t_1 < \Delta t_2$. This means that as long as $\Delta t_1 < \Delta t_2$, or $\frac{p}{\sigma_y} < \ln k$, it is possible to assure $\sigma_{\theta i}^T - \sigma_{ri}^T \leq \sigma_y$ and $\sigma_{\theta o}^T - \sigma_{ro}^T \leq \sigma_y$ in the meantime. Setting $p_1 = p_2$ obtains:

$$\frac{E\alpha\Delta t_x}{(1-\mu)\sigma_y} = \ln k^2 \quad \text{or} \quad \Delta t_x = \frac{(1-\mu)\sigma_y \ln k^2}{E\alpha} = n\frac{2(1-\mu)\sigma_y}{E\alpha} = \frac{k^2 \ln k^2 - k^2 + 1}{k^2 - 1}\Delta t_c \tag{7.45}$$

where $n = \ln k$, Δt_x is called the ideal temperature difference. The greater k is, the greater Δt_x is. When $\Delta t \leq \Delta t_x$, $p_1 \leq p_2$; when $\Delta t \geq \Delta t_x$, $p_1 \geq p_2$.

Substituting Δt_x into equations (7.36) and (7.38) both obtains

$$\frac{p}{\sigma_y} = \ln k \quad \text{or} \quad k = e^{p/\sigma_y} \tag{7.46}$$

Setting $\Delta t_1 \leq \Delta t_x$ results in $\frac{p}{\sigma_y} \leq \ln k$, and setting $\Delta t_2 \leq \Delta t_x$ results in $\frac{p}{\sigma_y} \geq \ln k$. In fact, setting $\Delta t = \Delta t_x$ in equations (7.36) and (7.38) obtains the entire yield pressure $\frac{p_1}{\sigma_y} = \frac{p_2}{\sigma_y} = \frac{p}{\sigma_y} = \ln k$.

Setting $\Delta t_x = \Delta t_c$ or setting $p_3 = p_4$ obtains

$$\frac{k^2 \ln k}{k^2 - 1} = 1 \quad \text{or} \quad \ln k = \frac{k^2 - 1}{k^2} \tag{7.47}$$

Equation (7.47) is just equation (2.32), or equation (3.28), etc. The solution for equation (7.47) is $k = k_c = 2.218\ 457\ 489\ 916\ 7\ldots$, where k_c is just the critical radius ratio. k_c has been shown in figure 7.18.

In fact, $\frac{k^2 \ln k}{k^2-1} = 1$ is equivalent to $k^2 \ln k = k^2 - 1$, substituting $k^2 \ln k = k^2 - 1$ into equation (7.26) and equation (7.45) both gives $\Delta t_x = \Delta t_c$.

n and m reflect magnitudes of Δt_x and Δt_c, respectively. n and m are shown in figure 7.19. Letting $n = m$ just results in equation (7.47). Then, the coordinate of point b in figure 7.19 is b (k_c, 0.796812).

When $k < k_c$, $n < m$ or $\Delta t_x < \Delta t_c$; when $k = k_c$, $n = m$ or $\Delta t_x = \Delta t_c$, the load-bearing capacity of a pressure vessel reaches the maximum: $\frac{p_1}{\sigma_y} = \frac{p_2}{\sigma_y} = \frac{p}{\sigma_y} = \ln k$. If $\Delta t_x < \Delta t < \Delta t_c$, $\frac{p}{\sigma_y}$ should be calculated by equation (7.38), the result is $\frac{p}{\sigma_y} < \ln k$; if $\Delta t < \Delta t_x$ $\frac{p}{\sigma_y}$ should be calculated by equation (7.36), the result is also $\frac{p}{\sigma_y} < \ln k$.

When $k > k_c$, $n > m$, or $\Delta t_x > \Delta t_c$. Under the case of $k > k_c$, if $\Delta t = \Delta t_x$, there still is $p_1 = p_2 = \sigma_y \ln k$, but when $\Delta t > \Delta t_c$, the pressure vessel has been yielded, thermal stresses are not the elastic stresses expressed by equations (7.18)–(7.20). So, when $k > k_c$, the maximum temperature difference is Δt_c. When $k > k_c$, $\Delta t_c < \Delta t_x$, so, when $k > k_c$, $\frac{p}{\sigma_y}$ is controlled by equation (7.36). When $\Delta t = \Delta t_c$, the load-bearing capacity of a pressure vessel when $k > k_c$ reaches the maximum. When $k > k_c$ and $\Delta t = \Delta t_c$, equation (7.36) becomes equation (7.43): $\frac{p}{\sigma_y} = \frac{k^2-1}{k^2} < \ln k$.

In brief, if $k < k_c$, the maximum load-bearing capacity is $\frac{p}{\sigma_y} = \ln k$; if $k > k_c$, the maximum load-bearing capacity is $\frac{p}{\sigma_y} = \frac{k^2-1}{k^2}$. Letting $\frac{p}{\sigma_y} = \frac{k^2-1}{k^2} = \ln k$ just results in equation (7.47). It is interesting and wonderful that in mechanical autofrettage, if

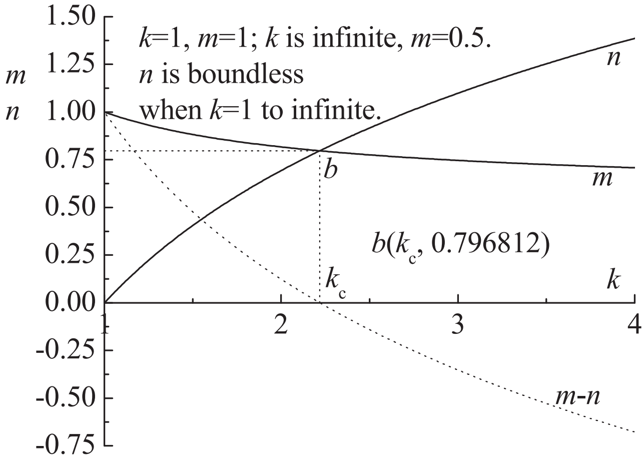

FIG. 7.19 – The variation tendency of m and n.

the plastic region covers the whole wall of a pressure vessel, for compressive yield not to occur after removing autofrettage pressure p_a, the expression for ultimate k is also equation (7.47). This fact deserves to be studied further.

Setting $p_3 = p_4$, we obtain equation (7.47); setting $\frac{p}{\sigma_y} = \frac{k^2-1}{k^2} = \ln k$ we also obtain equation (7.47); setting $m = n$, we still obtain equation (7.47). $\frac{k^2 \ln k}{k^2-1} = 1$ appeared many times and played an important role in mechanical and thermal autofrettage technology.

When $k < k_c$, the larger k is, the smaller $\Delta t_c - \Delta t_x$ is; When $k = k_c$, $\Delta t_c = \Delta t_x$. When $k > k_c$, the larger k is, the larger $\Delta t_x - \Delta t_c$ is.

If p_1 and p_2, Δt_1 and Δt_2 are illustrated by taking some typical parameters as examples, the above results will be clearer and more intuitive. The following figures can also be used as a reference for engineering design.

For Elastic modulus $E = 1.95 \times 10^5$ MPa, Poisson ratio $\mu = 0.3$, thermal expansion coefficient of the materials $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, strength limit $\sigma_y = 350$ MPa and radius ratio $k = 1.5$, $k = 2$, $k = k_c$, $k = 3$ and $k = 4$, Δt_1 and Δt_2 are plotted in figures 7.20–7.24, respectively. For other parameters, the changing trends and patterns of Δt_1 and Δt_2 are similar to those of figures 7.20–7.24, because for steel materials, the change of parameters $\frac{E\alpha}{2(1-\mu)}$ is small, popularly $\frac{E\alpha}{2(1-\mu)} \approx 1.5 \sim 1.8$. For example, here $\frac{E\alpha}{2(1-\mu)} = 1.671429\ldots$

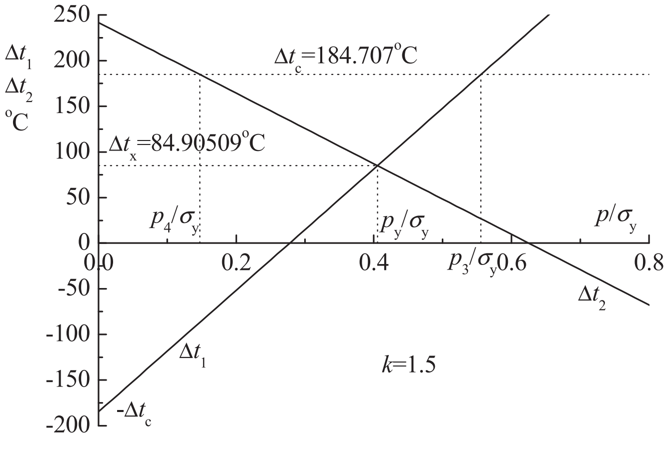

FIG. 7.20 – The graphs of Δt_1 and Δt_2 for $k = 1.5$.

In figure 7.20, the abscissa of the intersection point of straight line Δt_1 and straight line Δt_2 is $\frac{p_y}{\sigma_y}$. When $k = 1.5$, $\Delta t_c = 184.707$ °C, $\frac{p_3}{\sigma_y} = 0.555556$, $\frac{p_4}{\sigma_y} = 0.147412$, $\frac{p_y}{\sigma_y} = \ln k = 0.405465$. When $p = p_3$, $\Delta t_1 = \Delta t_c$, $\Delta t_2 < \Delta t_c$; When $p = p_4$, $\Delta t_2 = \Delta t_c$, $\Delta t_1 < \Delta t_c$ $(\Delta t_1 < 0)$; At the intersection point of straight line Δt_1 and straight line Δt_2, or when $\frac{p}{\sigma_y} = \frac{p_3}{\sigma_y} = \frac{p_4}{\sigma_y} = \frac{p_y}{\sigma_y}$, $\Delta t_1 = \Delta t_2 = \Delta t_x = 84.90509$ °C $< \Delta t_c$. When $\frac{p}{\sigma_y} = \frac{k^2-1}{2k^2} = \frac{p_e}{\sigma_y} = 0.277778$, $\Delta t_1 = 0$; When $\frac{p}{\sigma_y} = \frac{k^2-1}{2} = k^2 \frac{p_e}{\sigma_y} = 0.625$, $\Delta t_2 = 0$. See figure 7.20.

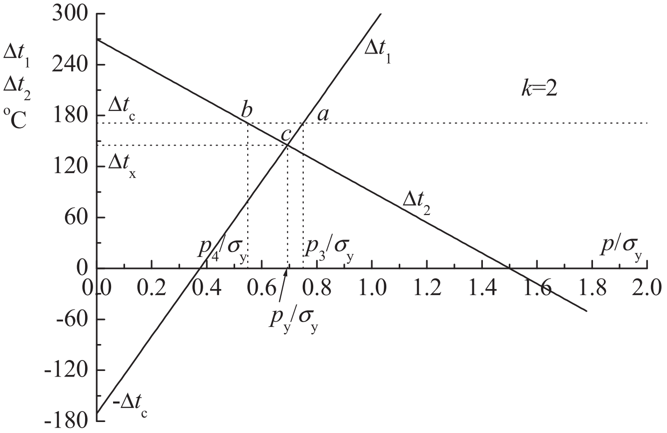

Fig. 7.21 – The graphs of Δt_1 and Δt_2 for $k = 2$.

In figure 7.21, the abscissa of the intersection point of straight line Δt_1 and straight line Δt_2 is $\frac{p_\mathrm{y}}{\sigma_\mathrm{y}}$. When $k = 2$, $\Delta t_\mathrm{c} = 171.0838$ °C, $\frac{p_3}{\sigma_\mathrm{y}} = 0.75$, $\frac{p_4}{\sigma_\mathrm{y}} = 0.548963$, $\frac{p_\mathrm{y}}{\sigma_\mathrm{y}} = \ln k = 0.693147$. When $p = p_3$, $\Delta t_1 = \Delta t_\mathrm{c}$, $\Delta t_2 < \Delta t_\mathrm{c}$; When $p = p_4$, $\Delta t_2 = \Delta t_\mathrm{c}$, $\Delta t_1 < \Delta t_\mathrm{c}$. At the intersection point of straight line Δt_1 and straight line Δt_2, or when $\frac{p}{\sigma_\mathrm{y}} = \frac{p_3}{\sigma_\mathrm{y}} = \frac{p_4}{\sigma_\mathrm{y}} = \frac{p_\mathrm{y}}{\sigma_\mathrm{y}}$, $\Delta t_1 = \Delta t_2 = \Delta t_\mathrm{x} = 145.1462$ °C $< \Delta t_\mathrm{c}$. When $\frac{p}{\sigma_\mathrm{y}} = \frac{k^2-1}{2k^2} = \frac{p_\mathrm{e}}{\sigma_\mathrm{y}} = 0.375$, $\Delta t_1 = 0$; When $\frac{p}{\sigma_\mathrm{y}} = \frac{k^2-1}{2} = k^2\frac{p_\mathrm{e}}{\sigma_\mathrm{y}} = 1.5$, $\Delta t_2 = 0$. See figure 7.21.

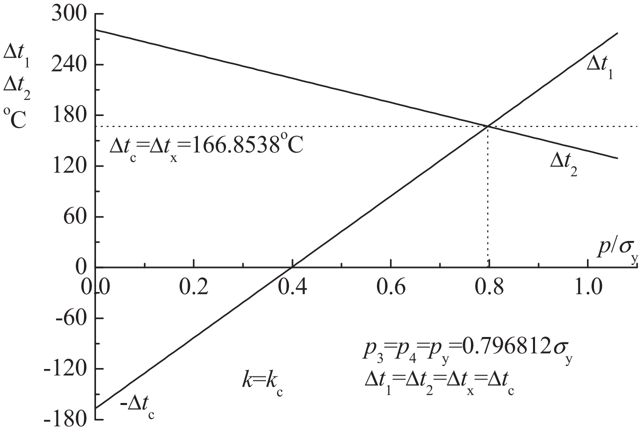

Fig. 7.22 – The graphs of Δt_1 and Δt_2 for $k = k_\mathrm{c}$.

In figure 7.22, the abscissa of the intersection point of straight line Δt_1 and straight line Δt_2 is $\frac{p_\mathrm{y}}{\sigma_\mathrm{y}}$. When $k = k_\mathrm{c}$, $\Delta t_\mathrm{c} = 166.8538$ °C, $\frac{p_3}{\sigma_\mathrm{y}} = \frac{p_4}{\sigma_\mathrm{y}} = \frac{p_\mathrm{y}}{\sigma_\mathrm{y}} = \ln k = 0.796812$. At the intersection point of straight line Δt_1 and straight line Δt_2, or when

$\frac{p}{\sigma_y} = \frac{p_3}{\sigma_y} = \frac{p_4}{\sigma_y} = \frac{p_y}{\sigma_y}$, $\Delta t_1 = \Delta t_2 = \Delta t_x = \Delta t_c$. When $\frac{p}{\sigma_y} = \frac{k^2-1}{2k^2} = \frac{p_e}{\sigma_y} = 0.398406$, $\Delta t_1 = 0$; When $\frac{p}{\sigma_y} = \frac{k^2-1}{2} = k^2 \frac{p_e}{\sigma_y} = 1.960777$, $\Delta t_2 = 0$. See figure 7.22.

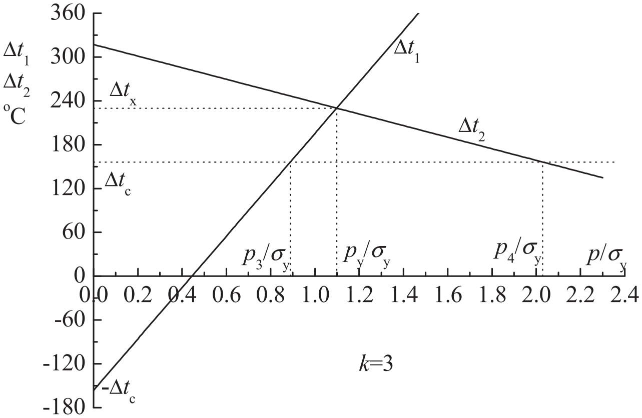

FIG. 7.23 – The graphs of Δt_1 and Δt_2 for $k = 3$.

In figure 7.23, the abscissa of the intersection point of straight line Δt_1 and straight line Δt_2 is $\frac{p_y}{\sigma_y}$. When $k = 3$, $\Delta t_c = 156.2978$ °C, $\frac{p_3}{\sigma_y} = 0.888889$, $\frac{p_4}{\sigma_y} = 2.028785$, $\frac{p_y}{\sigma_y} = \ln k = 1.098612$. When $p = p_3$, $\Delta t_1 = \Delta t_c$, $\Delta t_2 > \Delta t_c$; When $p = p_4$, $\Delta t_2 = \Delta t_c$, $\Delta t_1 > \Delta t_c$. At the intersection point of straight line Δt_1 and straight line Δt_2, or when $\frac{p}{\sigma_y} = \frac{p_3}{\sigma_y} = \frac{p_4}{\sigma_y} = \frac{p_y}{\sigma_y}$, $\Delta t_1 = \Delta t_2 = \Delta t_x = 230.0513$ °C $> \Delta t_c$. When $\frac{p}{\sigma_y} = \frac{k^2-1}{2k^2} = \frac{p_e}{\sigma_y} = 0.444444$, $\Delta t_1 = 0$; When $\frac{p}{\sigma_y} = \frac{k^2-1}{2} = k^2 \frac{p_e}{\sigma_y} = 4$, $\Delta t_2 = 0$. See figure 7.23.

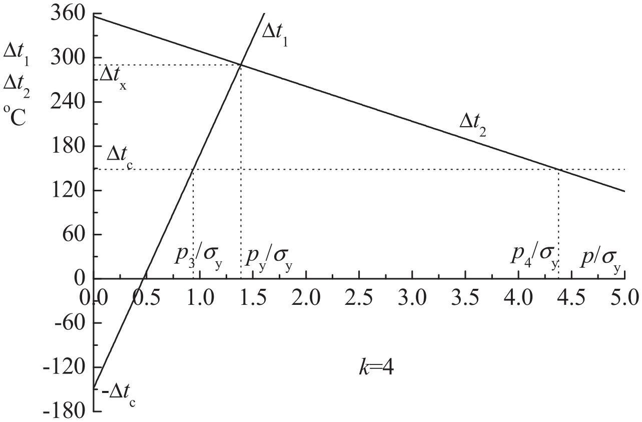

FIG. 7.24 – The graphs of Δt_1 and Δt_2 for $k = 4$.

In figure 7.24, the abscissa of the intersection point of straight line Δt_1 and straight line Δt_2 is $\frac{p_y}{\sigma_y}$. When $k = 4$, $\Delta t_c = 148.303$ °C, $\frac{p_3}{\sigma_y} = 0.9375$, $\frac{p_4}{\sigma_y} = 4.376664$, $\frac{p_y}{\sigma_y} = \ln k = 1.386294$. When $p = p_3$, $\Delta t_1 = \Delta t_c$, $\Delta t_2 > \Delta t_c$; When $p = p_4$, $\Delta t_2 = \Delta t_c$, $\Delta t_1 > \Delta t_c$. At the intersection point of straight line Δt_1 and straight line Δt_2, or when $\frac{p}{\sigma_y} = \frac{p_3}{\sigma_y} = \frac{p_4}{\sigma_y} = \frac{p_y}{\sigma_y}$, $\Delta t_1 = \Delta t_2 = \Delta t_x = 290.2924$ °C $> \Delta t_c$. When $\frac{p}{\sigma_y} = \frac{k^2-1}{2k^2} = \frac{p_e}{\sigma_y} = 0.46875$, $\Delta t_1 = 0$; When $\frac{p}{\sigma_y} = \frac{k^2-1}{2} = k^2 \frac{p_e}{\sigma_y} = 7.5$, $\Delta t_2 = 0$. See figure 7.24.

When $k < k_c$, $p_4 < p_y < p_3$, the smaller k is, the smaller p_4 is; and the smaller k is, the greater $p_3 - p_4$ is. When $k = k_c$, $p_3 = p_4 = p_y$. When $k > k_c$, $p_3 < p_y < p_4$, the greater k is, the greater p_4 is, and the greater k is, the greater $p_4 - p_3$ is. When $k > k_c$, at the intersection point of straight line Δt_1 and straight line Δt_2, where $\Delta t_1 = \Delta t_2$, there is $\Delta t_x > \Delta t_c$. Therefore, when $k > k_c$, $\frac{p_y}{\sigma_y} = \ln k$ cannot be taken as the allowable load, the optimum allowable load is equation (7.43), or $\frac{p}{\sigma_y} = \frac{k^2-1}{k^2} = \frac{p_3}{\sigma_y} = 2\frac{p_e}{\sigma_y}$.

For $E = 1.95 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, $\sigma_y = 350$ MPa and $k = 1.5$, $k = 2$, $k = k_c$, $k = 3$ and $k = 4$, $\frac{p_1}{\sigma_y}$ and $\frac{p_2}{\sigma_y}$ are plotted in figures 7.25–7.29, respectively. For other parameters, the changing trends and patterns of $\frac{p_1}{\sigma_y}$ and $\frac{p_2}{\sigma_y}$ are similar to those of figures 7.25–7.29.

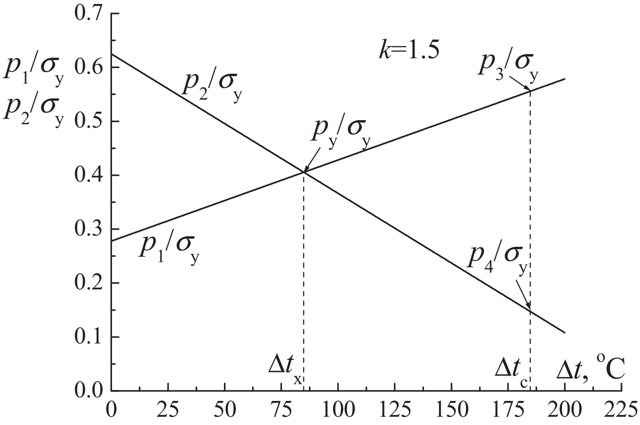

FIG. 7.25 – The graphs of $\frac{p_1}{\sigma_y}$ and $\frac{p_2}{\sigma_y}$ for $k = 1.5$.

In figure 7.25, the abscissa of the intersection point of straight line $\frac{p_1}{\sigma_y}$ and straight line $\frac{p_2}{\sigma_y}$ is Δt_x ($<\Delta t_c$). When $\Delta t < \Delta t_x$, $p_1 < p_2$; When $\Delta t > \Delta t_x$, $p_1 > p_2$. When $k = 1.5$, $\Delta t_c = 184.707$ °C, $\Delta t_x = 84.90509$ °C $< \Delta t_c$, $\frac{p_3}{\sigma_y} = 0.555556$, $\frac{p_4}{\sigma_y} = 0.147412$, $\frac{p_y}{\sigma_y} = \ln k = 0.405465$. When $\Delta t = \Delta t_c$, $\frac{p_1}{\sigma_y} = \frac{p_3}{\sigma_y} > \frac{p_2}{\sigma_y} = \frac{p_4}{\sigma_y}$ and $\frac{p_3}{\sigma_y} > \frac{p_y}{\sigma_y}$, $\frac{p_4}{\sigma_y} < \frac{p_y}{\sigma_y}$. When $\Delta t = \Delta t_x$, $\frac{p_1}{\sigma_y} = \frac{p_2}{\sigma_y} = \frac{p_3}{\sigma_y} = \frac{p_4}{\sigma_y} = \frac{p_y}{\sigma_y}$. See figure 7.25.

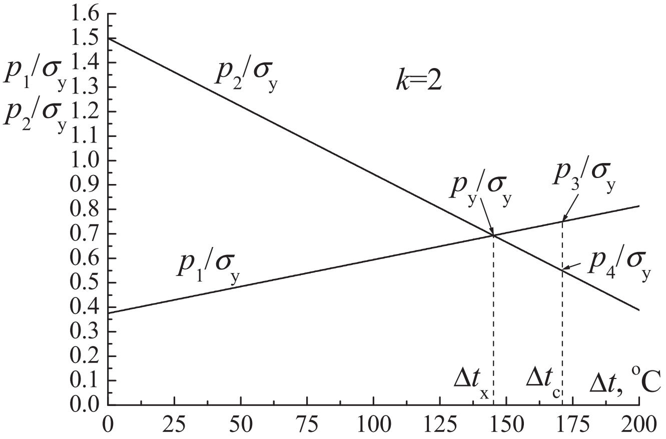

FIG. 7.26 – The graphs of $\frac{p_1}{\sigma_y}$ and $\frac{p_2}{\sigma_y}$ for $k = 2$.

In figure 7.26, the abscissa of the intersection point of straight line $\frac{p_1}{\sigma_y}$ and straight line $\frac{p_2}{\sigma_y}$ is Δt_x ($<\Delta t_c$). When $\Delta t < \Delta t_x$, $p_1 < p_2$; When $\Delta t > \Delta t_x$, $p_1 > p_2$. When $k = 2$, $\Delta t_c = 171.0838$ °C, $\Delta t_x = 145.1462$ °C $< \Delta t_c$, $\frac{p_3}{\sigma_y} = 0.75$, $\frac{p_4}{\sigma_y} = 0.548963$, $\frac{p_y}{\sigma_y} = \ln k = 0.693147$. When $\Delta t = \Delta t_c$, $\frac{p_1}{\sigma_y} = \frac{p_3}{\sigma_y} > \frac{p_2}{\sigma_y} = \frac{p_4}{\sigma_y}$ and $\frac{p_3}{\sigma_y} > \frac{p_y}{\sigma_y}$, $\frac{p_4}{\sigma_y} < \frac{p_y}{\sigma_y}$. When $\Delta t = \Delta t_x$, $\frac{p_1}{\sigma_y} = \frac{p_2}{\sigma_y} = \frac{p_3}{\sigma_y} = \frac{p_4}{\sigma_y} = \frac{p_y}{\sigma_y}$. See figure 7.26.

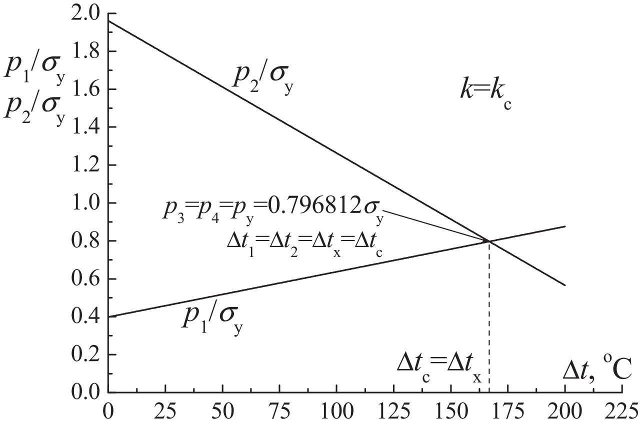

FIG. 7.27 – The graphs of p_1/σ_y and p_2/σ_y for $k = k_c$.

In figure 7.27, the abscissa of the intersection point of straight line $\frac{p_1}{\sigma_y}$ and straight line $\frac{p_2}{\sigma_y}$ is Δt_x ($=\Delta t_c$). When $\Delta t < \Delta t_x$, $p_1 < p_2$; When $\Delta t > \Delta t_x$, $p_1 > p_2$. When $k = k_c$, $\Delta t_c = \Delta t_x = 166.8538$ °C, $\frac{p_3}{\sigma_y} = \frac{p_4}{\sigma_y} = \frac{p_y}{\sigma_y} = \ln k = 0.796812$. When $\Delta t = \Delta t_x = \Delta t_c$, $\frac{p_1}{\sigma_y} = \frac{p_2}{\sigma_y} = \frac{p_3}{\sigma_y} = \frac{p_4}{\sigma_y} = \frac{p_y}{\sigma_y}$. See figure 7.27.

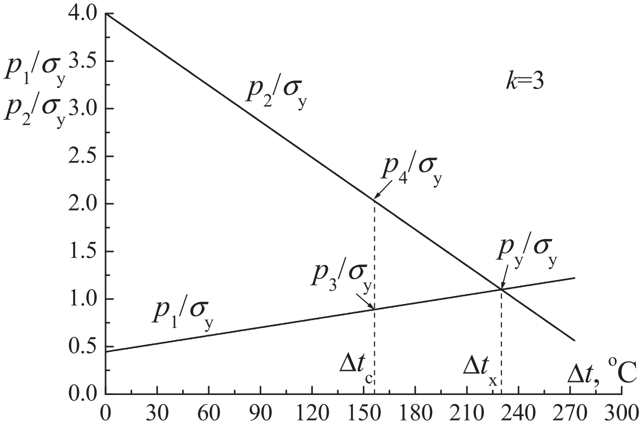

FIG. 7.28 – The graphs of p_1/σ_y and p_2/σ_y for $k = 3$.

In figure 7.28, the abscissa of the intersection point of straight line $\frac{p_1}{\sigma_y}$ and straight line $\frac{p_2}{\sigma_y}$ is Δt_x ($<\Delta t_c$). When $\Delta t < \Delta t_x$, $p_1 < p_2$; When $\Delta t > \Delta t_x$, $p_1 > p_2$. When $k = 3$, $\Delta t_c = 156.2978$ °C, $\Delta t_x = 230.0513$ °C $> \Delta t_c$, $\frac{p_3}{\sigma_y} = 0.888889$, $\frac{p_4}{\sigma_y} = 2.028785$, $\frac{p_y}{\sigma_y} = \ln k = 1.098612$. When $\Delta t = \Delta t_c$, $\frac{p_1}{\sigma_y} = \frac{p_3}{\sigma_y} < \frac{p_2}{\sigma_y} = \frac{p_4}{\sigma_y}$ and $\frac{p_3}{\sigma_y} < \frac{p_y}{\sigma_y}$, $\frac{p_4}{\sigma_y} > \frac{p_y}{\sigma_y}$. When $\Delta t = \Delta t_x$, $\frac{p_1}{\sigma_y} = \frac{p_2}{\sigma_y} = \frac{p_3}{\sigma_y} = \frac{p_4}{\sigma_y} = \frac{p_y}{\sigma_y}$. See figure 7.28.

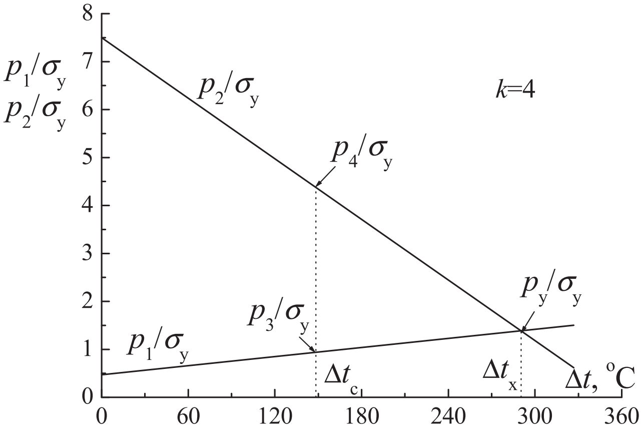

FIG. 7.29 – The graphs of p_1/σ_y and p_2/σ_y for $k = 4$.

If Elastic modulus E, Poisson ratio μ, the thermal expansion coefficient of the materials α, strength limit σ_y can be changed, the design projects will be more flexible.

7.5 Examples

Example 1. $k = 2$, $E = 1.95 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, $\sigma_{\mathrm{y}} = 300$ MPa.

This is the case of $k < k_{\mathrm{c}}$. If the temperature difference is not exerted, the load-bearing capacity of the pressure vessel with $k = 2$ is only $\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{2k^2} = 0.375$.

(1) From equation (7.45), $\Delta t_{\mathrm{x}} = 124.411$ °C $< \Delta t_{\mathrm{c}} = 146.6433$ °C obtained from equation (7.26). When $\Delta t = \Delta t_{\mathrm{x}} = 124.411$ °C, equations (7.36) and (7.38) both lead to $\frac{p}{\sigma_{\mathrm{y}}} = 0.693147391 = \ln k = \ln 2 > 0.375$. This is the optimum design project; the following projects are available for reference.

(2) Taking $\Delta t = 130$ °C $> \Delta t_{\mathrm{x}}$, from equation (7.36) $\frac{p}{\sigma_{\mathrm{y}}} = 0.707439 > \ln 2$, from equation (7.38) $\frac{p}{\sigma_{\mathrm{y}}} = 0.656900602 < \ln 2$. Substituting $\frac{p}{\sigma_{\mathrm{y}}} = 0.707439$ into equations (7.34) and (7.35) obtains $\sigma^{\mathrm{T}}_{\theta\mathrm{i}} - \sigma^{\mathrm{T}}_{\mathrm{ri}} = 300$ MPa and $\sigma^{\mathrm{T}}_{\theta\mathrm{o}} - \sigma^{\mathrm{T}}_{\mathrm{ro}} = 310$ MPa $> \sigma_{\mathrm{y}}$, respectively. Substituting $\frac{p}{\sigma_{\mathrm{y}}} = 0.656900602$ into equations (7.34) and (7.35) obtains $\sigma_{\theta\mathrm{i}} - \sigma_{\mathrm{ri}} = 260$ MPa, $\sigma^{\mathrm{T}}_{\theta\mathrm{o}} - \sigma^{\mathrm{T}}_{\mathrm{ro}} = 300$ MPa, respectively. Therefore, when $\Delta t = 130$ °C, $\frac{p}{\sigma_{\mathrm{y}}} = 0.656900602 < \ln 2$.

(3) Taking $\Delta t = 110$ °C $< \Delta t_{\mathrm{x}}$, from equation (7.36) $\frac{p}{\sigma_{\mathrm{y}}} = 0.656295$, from equation (7.38) $p/\sigma_{\mathrm{y}} = 0.786608202$. Substituting $\frac{p}{\sigma_{\mathrm{y}}} = 0.656295$ into equations (7.36) and (7.38) obtains $\sigma^{\mathrm{T}}_{\theta\mathrm{i}} - \sigma^{\mathrm{T}}_{\mathrm{ri}} = 300$ MPa and $\sigma^{\mathrm{T}}_{\theta\mathrm{o}} - \sigma^{\mathrm{T}}_{\mathrm{ro}} = 274$ MPa, respectively. Substituting $\frac{p}{\sigma_{\mathrm{y}}} = 0.786608202$ into equations (7.36) and (7.38) obtains $\sigma^{\mathrm{T}}_{\theta\mathrm{i}} - \sigma^{\mathrm{T}}_{\mathrm{ri}} = 404$ MPa $> \sigma_{\mathrm{y}}$ and $\sigma^{\mathrm{T}}_{\theta\mathrm{o}} - \sigma^{\mathrm{T}}_{\mathrm{ro}} = 300$ MPa, respectively. Therefore, when $\Delta t = 110$ °C, $\frac{p}{\sigma_{\mathrm{y}}} = 0.656295 < \ln 2$.

(4) Substituting $\Delta t = \Delta t_{\mathrm{c}} = 146.6433$ °C into equation (7.36) or substituting $k = 2$ into equation (7.43) obtains $\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_3}{\sigma_{\mathrm{y}}} = 0.75 > \ln k$. However, substituting $\frac{p}{\sigma_{\mathrm{y}}} = 0.75$ into equations (7.34) and (7.35) respectively obtains $\sigma^{\mathrm{T}}_{\theta\mathrm{i}} - \sigma^{\mathrm{T}}_{\mathrm{ri}} = 300$ MPa and $\sigma^{\mathrm{T}}_{\theta\mathrm{o}} - \sigma^{\mathrm{T}}_{\mathrm{ro}} = 340$ MPa $> \sigma_{\mathrm{y}}$. Substituting $\Delta t = \Delta t_{\mathrm{c}} = 146.6433$ °C into equation (7.38) or substituting $k = 2$ into equation (7.44) obtains $\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_4}{\sigma_{\mathrm{y}}} = 0.548962477 < \ln k$.

Therefore, for a pressure vessel with $k = 2$, the optimum operation parameters are $\Delta t = \Delta t_{\mathrm{x}} = 124.411$ °C, $\frac{p}{\sigma_{\mathrm{y}}} = 0.693147391 = \ln k$. Under these parameters, the distributions of thermal stresses and total stresses are shown in figures 7.30 and 7.31, respectively. The meaning of each curve and the main parameters are marked in the figures.

Example 2. $k = 2.5$, $E = 1.95 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, $\sigma_{\mathrm{y}} = 300$ MPa.

This is the case of $k > k_{\mathrm{c}}$. If there is no temperature difference, the load-bearing capacity of the pressure vessel with $k = 2.5$ is only $\frac{p}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{2k^2} = 0.42$.

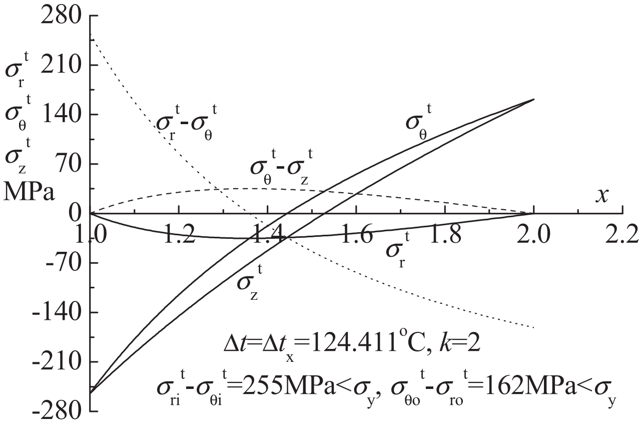

FIG. 7.30 – Distribution of thermal stresses for $k = 2$ under optimum operation parameters: $\Delta t = \Delta t_\mathrm{x}$.

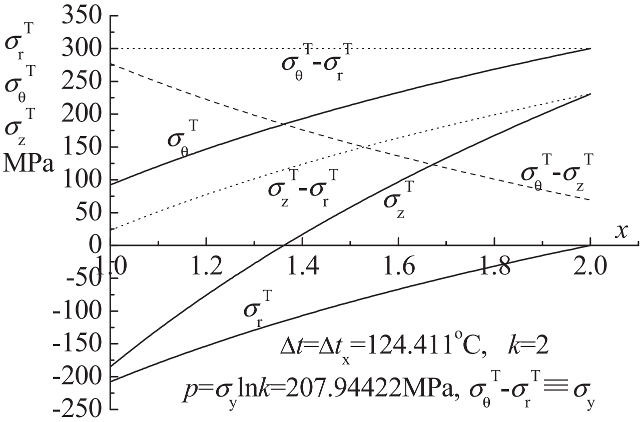

FIG. 7.31 – Distribution of total stresses for $k = 2$ under optimum operation parameters: $\Delta t = \Delta t_\mathrm{x}$ and $p = \sigma_\mathrm{y} \ln k$.

(1) From equation (7.26) $\Delta t_\mathrm{c} = 139.181$ °C; from equation (7.45) $\Delta t_\mathrm{x} = 164.4624$ °C $> \Delta t_\mathrm{c}$. Taking $\Delta t = \Delta t_\mathrm{c} = 139.181$ °C, from equations (7.36) or (7.43) $\frac{p}{\sigma_\mathrm{y}} = \frac{p_3}{\sigma_\mathrm{y}} = \frac{k^2-1}{k^2} = 2\frac{p_\mathrm{e}}{\sigma_\mathrm{y}} = 2 \times 0.42 = 0.84$. This is the optimum design project; the following projects are available for reference.

(2) If taking $\Delta t = 130$ °C $< \Delta t_\mathrm{c}$, from equation (7.36) $\frac{p}{\sigma_\mathrm{y}} = 0.812295$, from equation (7.38) $\frac{p}{\sigma_\mathrm{y}} = 1.274343849$. Substituting $\frac{p}{\sigma_\mathrm{y}} = 0.812295$ and $\Delta t = 130$ °C into equations (7.34) and (7.35) obtains $\sigma_{\theta\mathrm{i}}^\mathrm{T} - \sigma_{\mathrm{ri}}^\mathrm{T} = 300$ MPa and $\sigma_{\theta\mathrm{o}}^\mathrm{T} - \sigma_{\mathrm{ro}}^\mathrm{T} = 247.2$ MPa, respectively; Substituting $p/\sigma_\mathrm{y} = 1.274343849$ and $\Delta t = 130$ °C into equations (7.34) and (7.35) obtains $\sigma_{\theta\mathrm{i}}^\mathrm{T} - \sigma_{\mathrm{ri}}^\mathrm{T} = 630$ MPa $> \sigma_\mathrm{y}$ and

$\sigma_{\theta o}^{T} - \sigma_{ro}^{T} = 300$ MPa, respectively. Then, when $\Delta t = 130$ °C, $\frac{p}{\sigma_y} = 0.812295 < 0.84$.

Therefore, for a pressure vessel with $k = 2.5$, the optimum operation parameters are $\Delta t = \Delta t_c = 139.181$ °C, $p = 2p_e$. Under these parameters, the distributions of thermal stresses and total stresses are shown in figures 7.32 and 7.33, respectively. The meaning of each curve and the main parameters are marked in the figures.

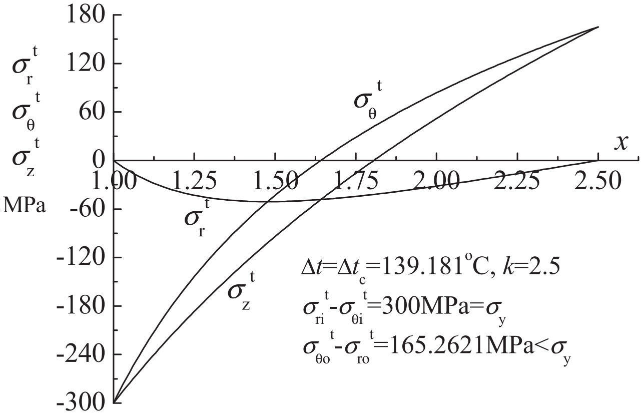

FIG. 7.32 – Distribution of thermal stresses for $k = 2.5$ under optimum operation parameters: $\Delta t = \Delta t_c$.

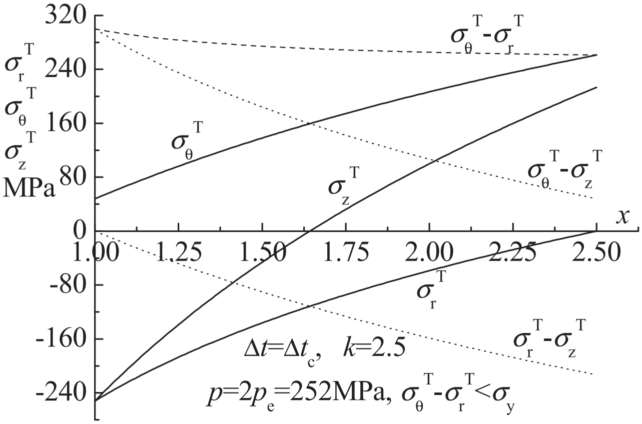

FIG. 7.33 – Distribution of total stresses for $k = 2.5$ under optimum operation parameters: $\Delta t = \Delta t_c$ and $p = 2p_e$.

If temperature difference can be chosen within the range below Δt_c, then, for a pressure vessel with some k, take the smaller of the values determined by equations (7.36) and (7.38) as allowable loading, this has been shown in examples 1 and 2.

For a certain Δt and p, determine radius ratio k by equations (7.36) and (7.38) or by equations (7.43) and (7.46), and take the greater radius ratio as the real radius ratio. By equations (7.36) and (7.38), we can weigh and adjust the parameters k, p, properties of materials (E, α, μ, and σ_y) and Δt to construct a pressure vessel that fits practical requirements.

Example 3. A pressure vessel is to be subjected to a pressure 225 MPa, determine radius ratio ($E = 1.95 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, $\sigma_y = 350$ MPa).

If there is no temperature difference, under an elastic state, even if k is infinite, the load-bearing capacity of the pressure vessel $p = 0.5\sigma_y$ (based on the maximum shear stress theory) or $\frac{p}{\sigma_y} = \frac{1}{\sqrt{3}} = 0.577$ (based on the maximum distortion strain energy theory). However, in this example, $\frac{p}{\sigma_y} = \frac{225}{350} = 0.642857 > 0.577$, so even if the wall is infinite, the pressure vessel cannot bear a pressure 225 MPa.

(1) If wall is required to be minimum, then $k = \sqrt{\frac{\sigma_y}{\sigma_y - p}} = 1.67332$ from equation (7.43) or $k = e^{p/\sigma_y} = 1.901907$ from equation (7.46). Take the greater $k = 1.901907$ as the real radius ratio. When $k = 1.901907$, from equation (7.45) $\Delta t_x = 134.6154$ °C, from equation (7.26) $\Delta t_c = 173.259$ °C. According to the above conclusions, take $\Delta t = \Delta t_x = 134.6154$ °C. Substituting $\frac{p}{\sigma_y} = 0.642857$ and $\Delta t = \Delta t_x = 134.6154$ °C into equations (7.34) and (7.35) both obtains $\sigma_{\theta i}^T - \sigma_{ri}^T = \sigma_{\theta o}^T - \sigma_{ro}^T = 350\text{ MPa} = \sigma_y$. The distributions of thermal stresses and total stresses are shown in figures 7.34 and 7.35, respectively. The meaning of each curve and the main parameters are marked in the figures.

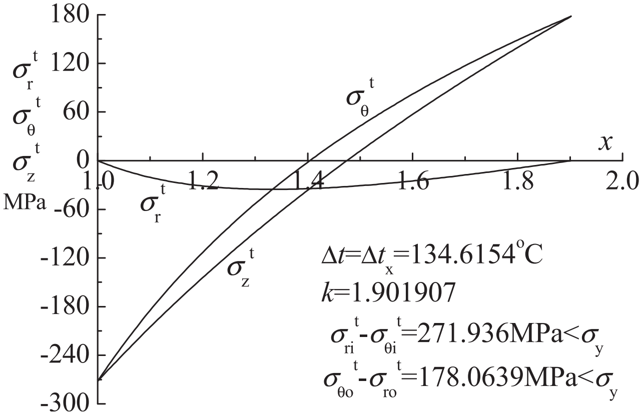

Fig. 7.34 – Distribution of thermal stresses for $k = 1.901907$ under optimum operation parameters: $\Delta t = \Delta t_c$.

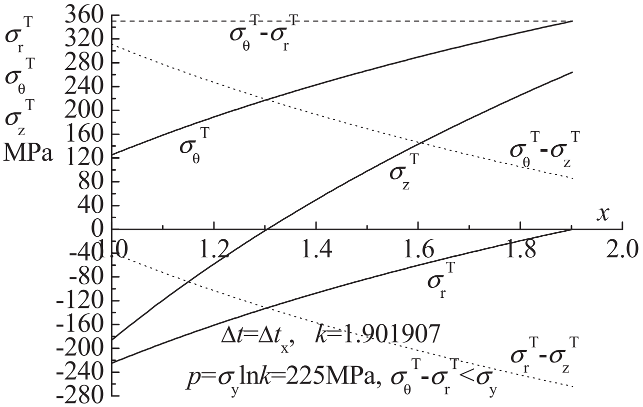

FIG. 7.35 – Distribution of total stresses for $k = 1.901907$ under optimum operation parameters: $\Delta t = \Delta t_c$ and $p = \sigma_y \ln k = 225$ MPa.

If the wall is not required to be minimum, then choose a suitable k on the basis of practical requirements.

(2) Saying $k = 2.2$, then, from equation (7.45) $\Delta t_x = 165.1043$ °C, from equation (7.26) $\Delta t_c = 167.1825$ °C, from equation (7.37) $\Delta t_1 = 103.7427$ °C, from equation (7.39) $\Delta t_2 = 186.349\ °\mathrm{C} > \Delta t_c$, which is of no significance. Substituting $\frac{p}{\sigma_y} = 0.642857$ and $\Delta t = \Delta t_1$ into equations (7.34) and (7.35) obtains $\sigma_{\theta i}^T - \sigma_{ri}^T = 350\ \mathrm{MPa} = \sigma_y$ and $\sigma_{\theta o}^T - \sigma_{ro}^T = 246.8\ \mathrm{MPa} < \sigma_y$, respectively. Substituting $\frac{p}{\sigma_y} = 0.642857$ and $\Delta t = \Delta t_c$ into equations (7.34) and (7.35) obtains $\sigma_{\theta i}^T - \sigma_{ri}^T = 217.2\ \mathrm{MPa} < \sigma_y$ and $\sigma_{\theta o}^T - \sigma_{ro}^T = 326.1\ \mathrm{MPa} < \sigma_y$, respectively.

When $\Delta t = \Delta t_1 = 103.7427$ °C, $k = 2.2$, the distributions of thermal stresses and total stresses are shown in figures 7.36 and 7.37, respectively. The meaning of each curve and the main parameters are marked in the figures.

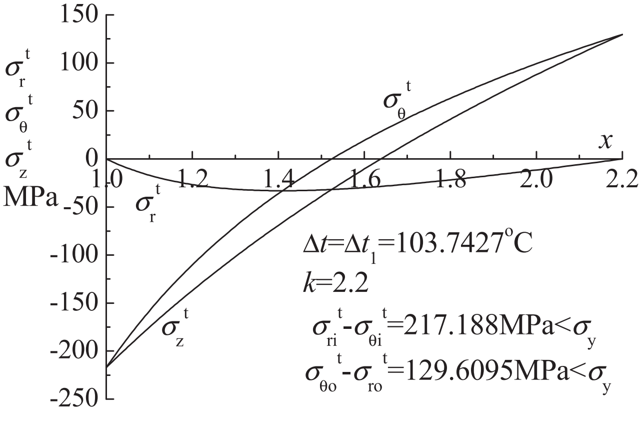

FIG. 7.36 – Distribution of thermal stresses for $k = 2.2$, $\Delta t = \Delta t_1 = 103.7427$ °C.

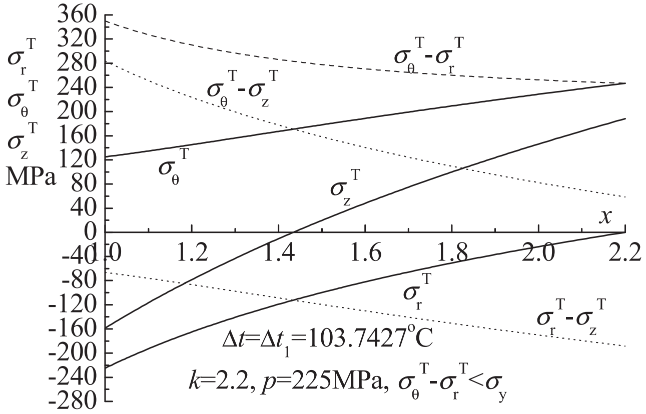

FIG. 7.37 – Distribution of total stresses for $k = 2.2$, $\Delta t = \Delta t_1 = 103.7427$ °C, and $p = 225$ MPa.

When $\Delta t = \Delta t_c = 167.1825$ °C, $k = 2.2$, $p = 225$ MPa, the distributions of thermal stresses and total stresses are shown in figures 7.38 and 7.39, respectively. The meaning of each curve and the main parameters are marked in the figures.

(3) Saying $k = 3$, from equation (7.26) $\Delta t_c = 156.2978$ °C, from equation (7.45) $\Delta t_x = 230.0513$ °C $> \Delta t_c$, Δt_x is of no significance. From equation (7.37) $\Delta t_1 = 69.77582$ °C, from equation (7.39) $\Delta t_2 = 266.1882$ °C $> \Delta t_c$. Δt_2 is of no significance. Substituting $\frac{p}{\sigma_y} = 0.642857$ and $\Delta t = \Delta t_1$ into equations (7.34) and (7.35) obtains $\sigma_{\theta i}^T - \sigma_{ri}^T = 350$ MPa $= \sigma_y$ and $\sigma_{\theta o}^T - \sigma_{ro}^T = 133.3$ MPa $< \sigma_y$, respectively. Substituting $\frac{p}{\sigma_y} = 0.642857$ and $\Delta t = \Delta t_c$ into equations (7.34) and (7.35) obtains $\sigma_{\theta i}^T - \sigma_{ri}^T = 156.3$ MPa $< \sigma_y$ and $\sigma_{\theta o}^T - \sigma_{ro}^T = 228.7$ MPa $< \sigma_y$, respectively.

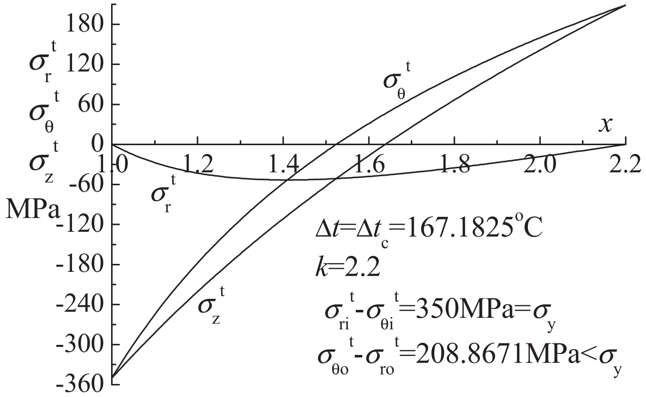

FIG. 7.38 – Distribution of thermal stresses for $k = 2.2$, $\Delta t = \Delta t_c = 167.1825$ °C.

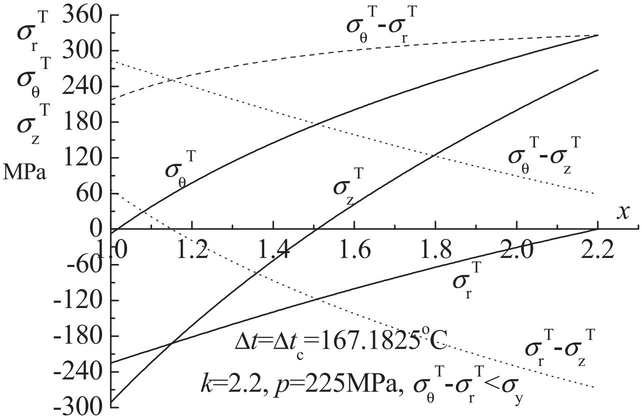

FIG. 7.39 – Distribution of total stresses for $k = 2.2$, $\Delta t = \Delta t_c = 167.1825$ °C, and $p = 225$ MPa.

When $\Delta t = \Delta t_1 = 69.77582$ °C, $k = 3$, the distributions of thermal stresses and total stresses are shown in figures 7.40 and 7.41, respectively. The meaning of each curve and the main parameters are marked in the figures.

When $\Delta t = \Delta t_c = 156.2978$ °C, $k = 3$, $p = 225$ MPa, the distributions of thermal stresses and total stresses are shown in figures 7.42 and 7.43, respectively. The meaning of each curve and the main parameters are marked in the figures.

Example 4. A process can supply a temperature difference $\Delta t = 120$ °C for a pressure vessel, study the size and the load-bearing capacity of the pressure vessel, $E = 1.95 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, $\sigma_y = 350$ MPa.

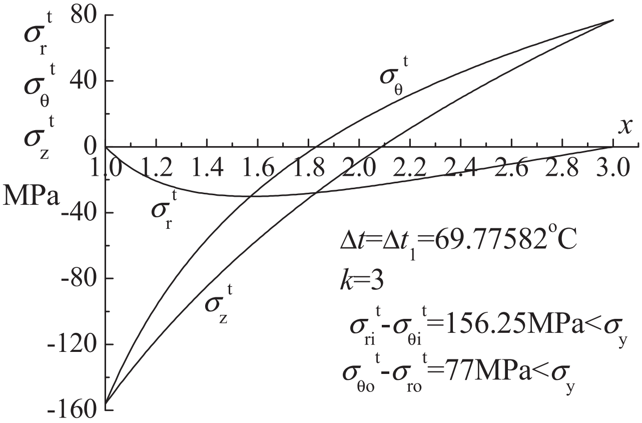

FIG. 7.40 – Distribution of thermal stresses for $k = 3$, $\Delta t = \Delta t_1 = 69.77582$ °C.

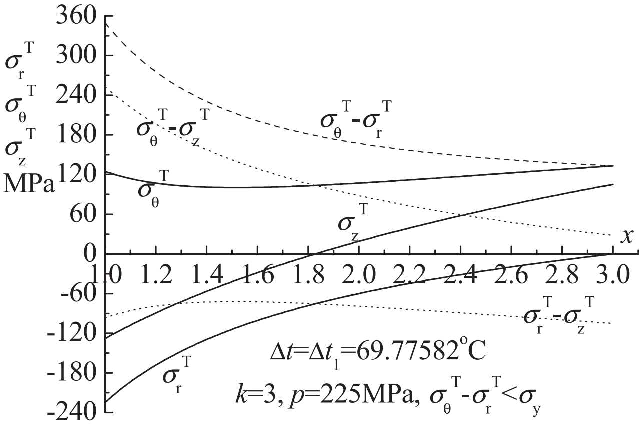

FIG. 7.41 – Distribution of total stresses for $k = 3$, $\Delta t = \Delta t_1 = 69.77582$ °C and $p = 225$ MPa.

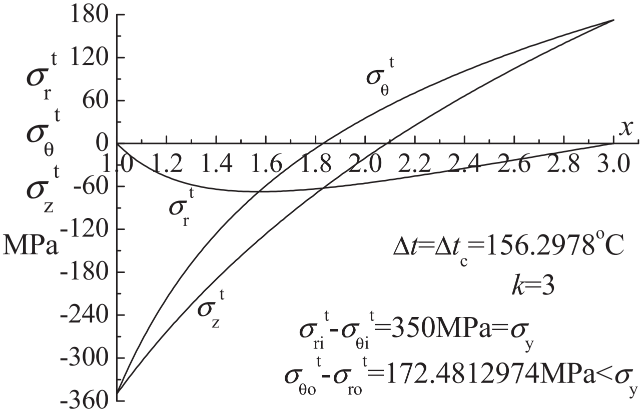

FIG. 7.42 – Distribution of thermal stresses for $k = 3$, $\Delta t = \Delta t_c = 156.2978$ °C.

(1) If the load-bearing capacity is not required, there are many schemes. Such as,

I. taking $k = 2.5$, from equation (7.26), $\Delta t_c = 162.3777958 > \Delta t$. From equation (7.36) $\frac{p}{\sigma_y} = 0.730387265$. From equation (7.37) $\frac{p}{\sigma_y} = 1.556349$. Take $\frac{p}{\sigma_y} = 0.730387265$, or $p = 255.63554$ MPa as real loading.

When $\Delta t = 120$ °C, $k = 2.5$, $p = 255.63554$ MPa, the distributions of thermal stresses and total stresses are shown in figures 7.44 and 7.45, respectively. The meaning of each curve and the main parameters are marked in the figures.

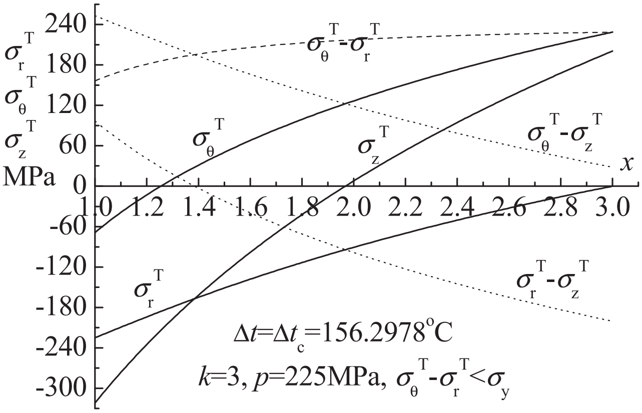

FIG. 7.43 – Distribution of total stresses for $k = 3$, $\Delta t = \Delta t_c = 156.2978$ °C and $p = 225$ MPa.

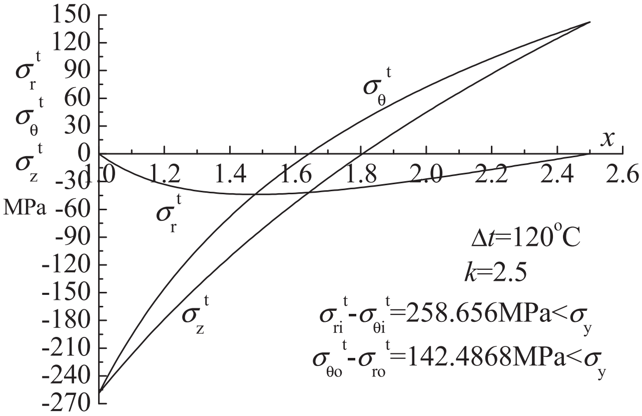

FIG. 7.44 – Distribution of thermal stresses for $k = 2.5$, $\Delta t = 120$ °C.

II. taking $k = 1.6$, from equation (7.26), $\Delta t_c = 181.3931336 > \Delta t$. From equation (7.36) $\frac{p}{\sigma_y} = 0.506252$, From equation (7.34) $\frac{p}{\sigma_y} = 0.402030791$. Take $\frac{p}{\sigma_y} = 0.402030791$ or $p = 140.71078$ MPa as real loading.

When $\Delta t = 120$ °C, $k = 1.6$, $p = 140.71078$ MPa, the distributions of thermal stresses and total stresses are shown in figures 7.46 and 7.47, respectively. The meaning of each curve and the main parameters are marked in the figures.

(2) Supposing $\Delta t = 120$ °C $= \Delta t_x$, then, from equation (7.45) $k = 1.773688 < k_c$ and $\frac{p}{\sigma_y} = \ln k = 0.573060794$, or $p = 200.57128$ MPa.

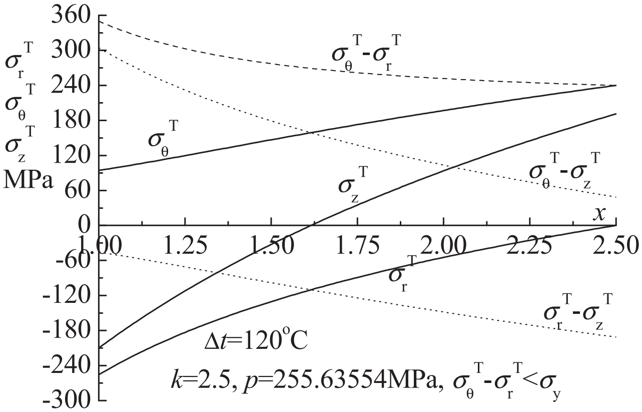

FIG. 7.45 – Distribution of total stresses for $k = 2.5$, $\Delta t = 120$ °C and $p = 255.63554$ MPa.

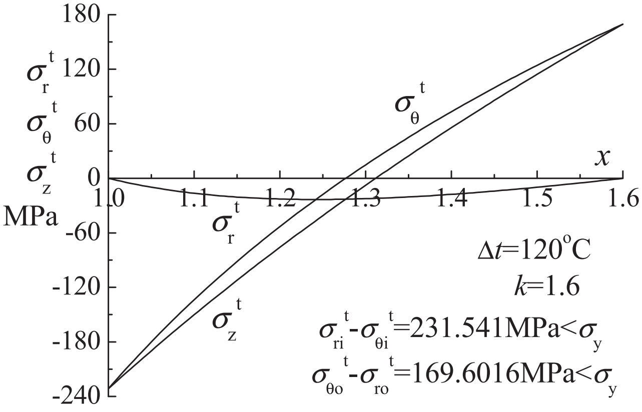

FIG. 7.46 – Distribution of thermal stresses for $k = 1.6$, $\Delta t = 120$ °C.

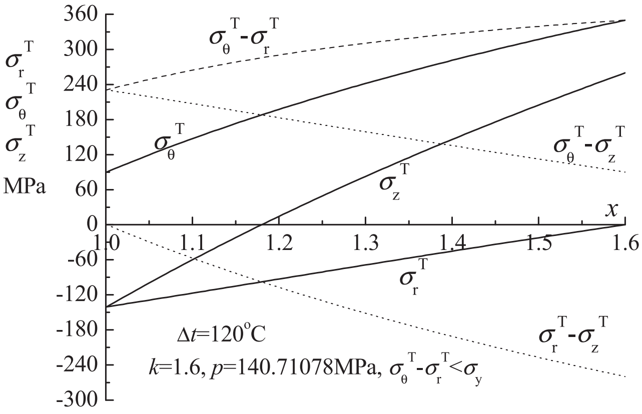

FIG. 7.47 – Distribution of total stresses for $k = 1.6$, $\Delta t = 120$ °C and $p = 140.71078$ MPa.

When $\Delta t = \Delta t_x$=120 °C, $k = 1.773688$, $p = 200.57128$ MPa, the distributions of thermal stresses and total stresses are shown in figures 7.48 and 7.49, respectively. The meaning of each curve and the main parameters are marked in the figures.

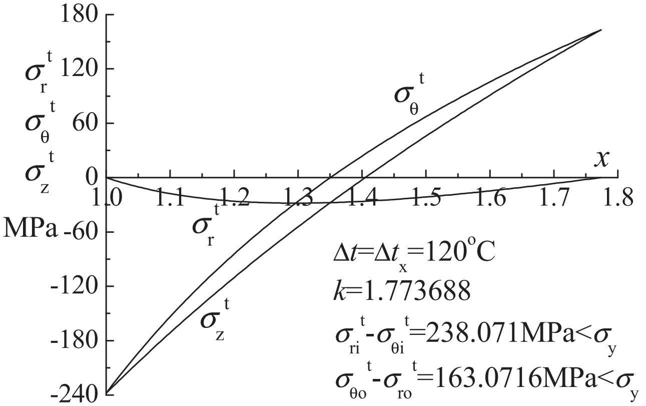

FIG. 7.48 – Distribution of thermal stresses for $k = 1.773688$, $\Delta t = \Delta t_x = 120$ °C.

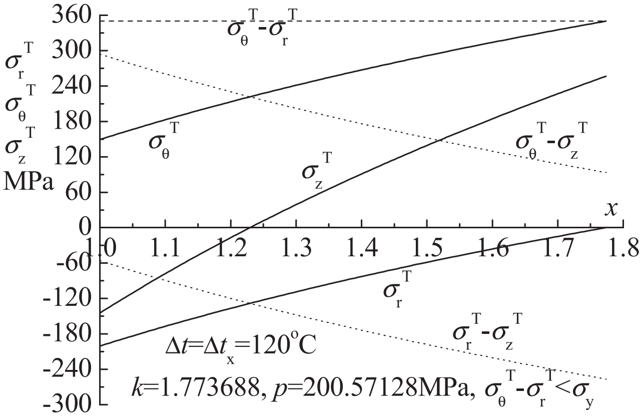

FIG. 7.49 – Distribution of total stresses for $k = 1.773688$, $\Delta t = \Delta t_x = 120$ °C and $p = 200.57128$ MPa.

(3) Supposing $\Delta t = 120$ °C $= \Delta t_c$, then, from equation (7.26) $k = 49.872$, it is unrealistic.

(4) I. Supposing the pressure vessel is required to contain 200 MPa, or $\frac{p}{\sigma_y} = \frac{200}{350} = 0.571429$.

From equation (7.36) by approximation, $k = 1.76884$; from equation (7.37) by approximation, $k = 1.772141$. Take $k = 1.772141$.

When $\Delta t = 120$ °C, $k = 1.772141$, $p = 200$ MPa the distributions of thermal stresses and total stresses are similar to figures 7.48 and 7.49, respectively.

II. Supposing the pressure vessel is required to contain 300 MPa or $\frac{p}{\sigma_y} = \frac{300}{350} = 0.857143$.

From equation (7.36) by approximation, $k = 4.219641$; from equation (7.37) by approximation, $k = 2.019384$. Take $k = 4.219641$.

When $\Delta t = 120$ °C, $k = 4.219641$, $p = 300$ MPa, the distributions of thermal stresses and total stresses are shown in figures 7.50 and 7.51, respectively. The meaning of each curve and the main parameters are marked in the figures.

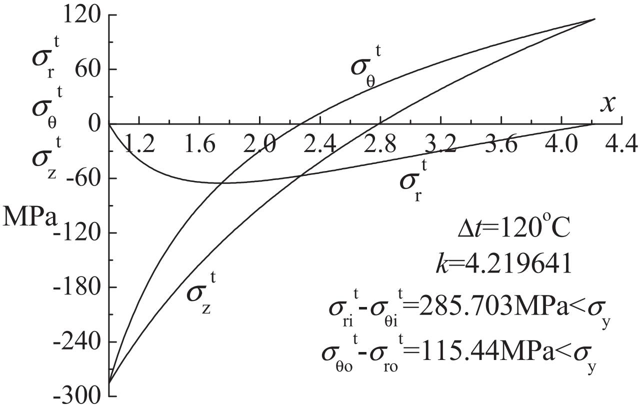

FIG. 7.50 – Distribution of thermal stresses for $k = 4.219641$, $\Delta t = 120$ °C.

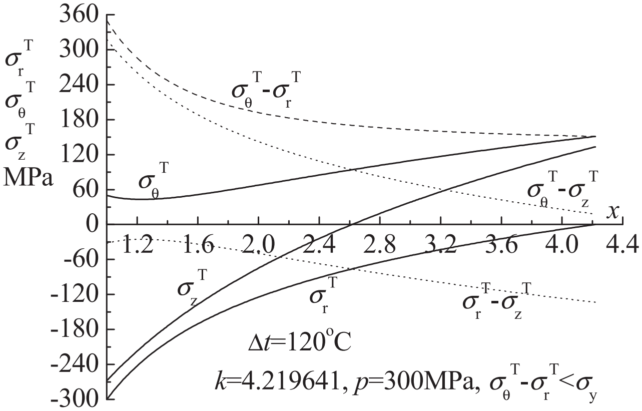

FIG. 7.51 – Distribution of total stresses for $k = 4.219641$, $\Delta t = 120$ °C and $p = 300$ MPa.

7.6 The Total Stresses Under Optimum Operation Conditions

When $k \leq k_c$, if $\Delta t = \Delta t_x$, then $p = p_y = \sigma_y \ln k$ and $p_t = \sigma_y \ln k = p_y = p$, both are entire yield pressure. Therefore, when $k \leq k_c$ and $\Delta t = \Delta t_x$, the total stresses are

$$\sigma_r^T = \sigma_r^t + \sigma_r^p = -p_y \frac{\ln(k/x)}{\ln k} = -\sigma_y \ln \frac{k}{x} \tag{7.48}$$

$$\sigma_\theta^T = \sigma_\theta^t + \sigma_\theta^p = p_y \frac{1 - \ln(k/x)}{\ln k} = \sigma_y \left(1 - \ln \frac{k}{x}\right) \tag{7.49}$$

$$\sigma_z^T = \sigma_z^t + \sigma_z^p = p_y \left(\frac{1 - 2\ln(k/x)}{\ln k} - \frac{1}{k^2 - 1}\right) = \sigma_y \left(1 - \ln \frac{k^2}{x^2} - \frac{\ln k}{k^2 - 1}\right) \tag{7.50}$$

In this case, $p - p_t = 0$, $\frac{p_t}{\ln k} = \sigma_y$, then, $\sigma_{\theta i}^T - \sigma_{ri}^T = \sigma_{\theta o}^T - \sigma_{ro}^T = \sigma_y$. In fact, it is seen from equation (7.48) to equation (7.48) that, within the whole wall $\sigma_\theta^T - \sigma_r^T \equiv \sigma_y$, which can be also referred to in example 1 and figure 7.29.

According to equation (7.48) to equation (7.48),

$$\sigma_\theta^T - \sigma_z = p_y \left(\frac{\ln(k/x)}{\ln k} + \frac{1}{k^2 - 1}\right) = \sigma_y \left(\ln \frac{k}{x} + \frac{\ln k}{k^2 - 1}\right)$$

$$\sigma_z^T - \sigma_r = p_y \left(\frac{1 - \ln(k/x)}{\ln k} - \frac{1}{k^2 - 1}\right) = \sigma_y \left(1 - \ln \frac{k}{x} - \frac{\ln k}{k^2 - 1}\right)$$

As x increases, $\sigma_\theta^T - \sigma_z^T$ decreases. When $x = 1$, $\sigma_{\theta i}^T - \sigma_{zi}^T = \frac{k^2 p_y}{k^2 - 1} = \frac{k^2 \ln k}{k^2 - 1} \sigma_y \leq \sigma_y$; when $x = k$, $\sigma_{\theta o}^T - \sigma_{zo}^T = \frac{\sigma_y \ln k}{k^2 - 1} < \sigma_y$.

As x increases, $\sigma_z^T - \sigma_r^T$ increases.

When $x = 1$, $-\sigma_y < \sigma_{zi}^T - \sigma_{ri}^T = p_y \left(\frac{1 - \ln k}{\ln k} - \frac{1}{k^2 - 1}\right) = \sigma_y \left(1 - \frac{k^2 \ln k}{k^2 - 1}\right) < \sigma_y$;

When $x = k$, $-\sigma_y < \sigma_{zo}^T - \sigma_{ro}^T = p_y \left(\frac{1}{\ln k} - \frac{1}{k^2 - 1}\right) = \sigma_y \left(1 - \frac{\ln k}{k^2 - 1}\right) < \sigma_y$.

$\sigma_z^T - \sigma_r^T$ is parallel with σ_θ^T and smaller than σ_θ^T.

The total stresses when $k = 2$ and $\Delta t = \Delta t_x$ have been shown in example 1, see figure 7.31.

When $k \leq k_c$, if $\Delta t = \Delta t_x$, $p_t = \sigma_y \ln k = p_y$, the equations of thermal stresses or equation (7.18)–(7.20) become

$$\frac{\sigma_r^t}{\sigma_y} = \ln x + \frac{k^2 \ln k}{k^2 - 1} \frac{1}{x^2} - \frac{k^2 \ln k}{k^2 - 1} \tag{7.51}$$

$$\frac{\sigma_\theta^t}{\sigma_y} = 1 + \ln x - \frac{k^2 \ln k}{k^2 - 1} \frac{1}{x^2} - \frac{k^2 \ln k}{k^2 - 1} \tag{7.52}$$

$$\frac{\sigma_z^t}{\sigma_y} = 1 + \ln x^2 - \frac{k^2 \ln k^2}{k^2 - 1} \tag{7.53}$$

Equation (7.48) to equations (7.48), (7.51)–(7.53) tell us that the total stresses and thermal stresses are not related to elastic modulus E, Poisson ratio μ and thermal expansion coefficient of the materials α under the case that $k \leq k_c$, $\Delta t = \Delta t_x$ and $p_t = \sigma_y \ln k = p_y$.

By taking $k = 1.5$, $k = 2$ and $k = k_c$ as examples, the thermal stresses and the total stresses are illustrated in figures 7.52–7.57, respectively. The meaning of each curve is marked in the figures.

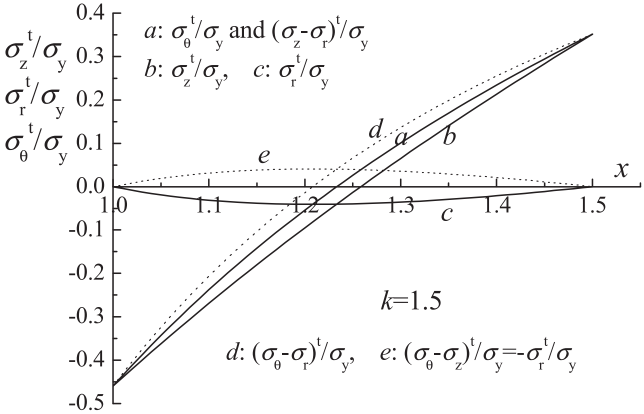

FIG. 7.52 – The thermal stresses with $\Delta t = \Delta t_x$, $p_t = p_y$ and $k = 1.5$.

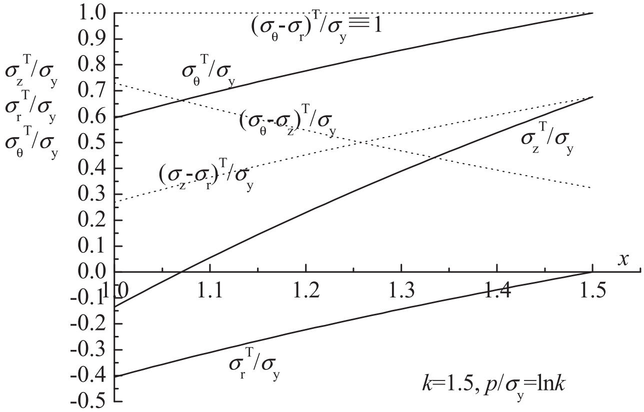

FIG. 7.53 – The total stresses with $\Delta t = \Delta t_x$, $p_t = p = p_y$ and $k = 1.5$.

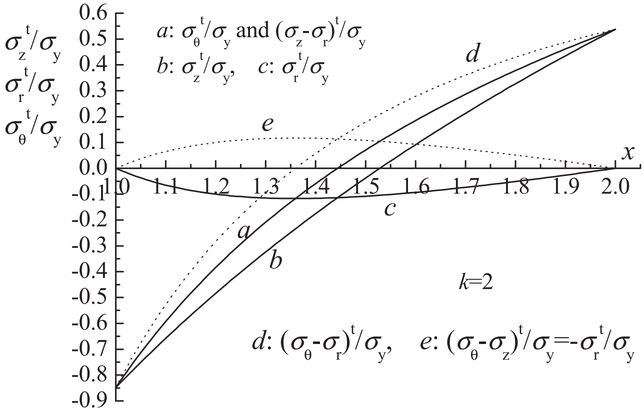

FIG. 7.54 – The thermal stresses with $\Delta t = \Delta t_x$, $p_t = p_y$ and $k = 2$.

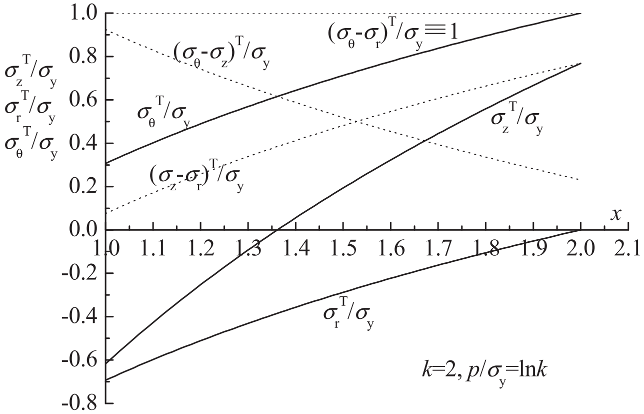

FIG. 7.55 – The total stresses with $\Delta t = \Delta t_x$, $p_t = p = p_y$ and $k = 2$.

Under the operation condition that $\Delta t = \Delta t_x$ and $p = p_t = \sigma_y \ln k = p_y$, the total stresses are cut down and evened greatly, especially the maximum circumferential tension on the inside surface under internal pressure is cut down considerably.

When $k \geq k_c$, if $\Delta t = \Delta t_c$, then $\frac{p}{\sigma_y} = \frac{k^2-1}{k^2} = \frac{p_3}{\sigma_y} = 2\frac{p_e}{\sigma_y}$, thermal loading is critical thermal loading p_{tc}, or equation (7.27). Therefore, the equations of thermal stresses or equations (7.18)–(7.20) become

$$\frac{\sigma_r^t}{\sigma_y} = -\frac{(k^2-1)\ln\frac{k}{x}}{k^2\ln k^2 - k^2 + 1} + \frac{(k^2-x^2)\ln k}{x^2(k^2\ln k^2 - k^2 + 1)} \tag{7.54}$$

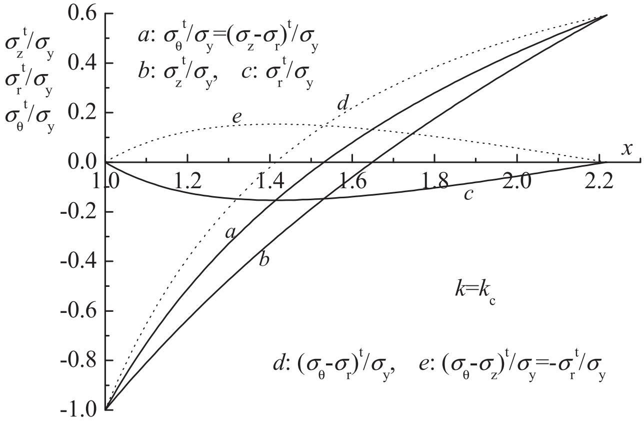

FIG. 7.56 – The thermal stresses with $\Delta t = \Delta t_x$, $p_t = p_y$ and $k = k_c$.

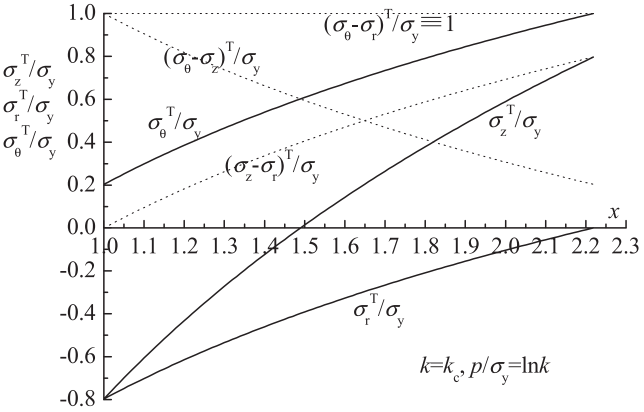

FIG. 7.57 – The total stresses with $\Delta t = \Delta t_x$, $p_t = p = p_y$ and $k = k_c$.

$$\frac{\sigma_\theta^t}{\sigma_y} = \frac{(k^2-1)(1-\ln\frac{k}{x})}{k^2\ln k^2 - k^2 + 1} - \frac{(k^2+x^2)\ln k}{x^2(k^2\ln k^2 - k^2 + 1)} \tag{7.55}$$

$$\frac{\sigma_z^t}{\sigma_y} = \frac{(k^2-1)(1-2\ln\frac{k}{x})}{k^2\ln k^2 - k^2 + 1} - \frac{\ln k^2}{k^2\ln k^2 - k^2 + 1} = \frac{(k^2-1)(\ln x^2+1) - k^2\ln k^2}{k^2\ln k^2 - k^2 + 1} \tag{7.56}$$

The thermal stresses are irrelevant to the properties of the materials (E, α, μ and σ_y etc.) and are only related to the sizes of pressure vessels under the case that $k \geq k_c$, $\Delta t = \Delta t_c$ and $\frac{p}{\sigma_y} = \frac{k^2-1}{k^2} = \frac{p_3}{\sigma_y} = 2\frac{p_e}{\sigma_y}$.

When $k = k_c$, $\frac{k^2\ln k}{k^2-1} = 1$, equations (7.54)–(7.56) accordingly become equations (7.51)–(7.53).

For $k = 3$ and $k = 4$, the thermal stresses are plotted in figures 7.58 and 7.59, respectively.

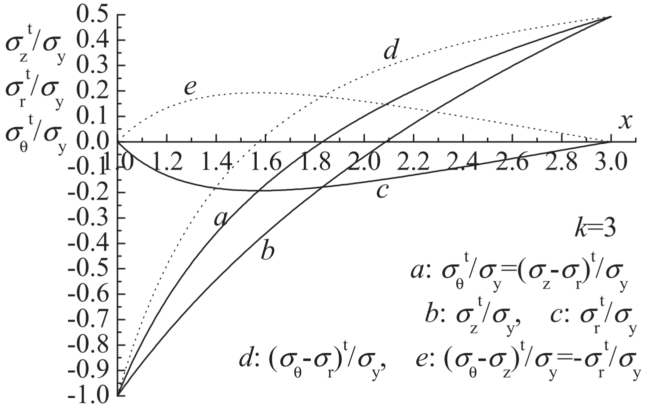

FIG. 7.58 – The thermal stresses for $\Delta t = \Delta t_c$, $k = 3$.

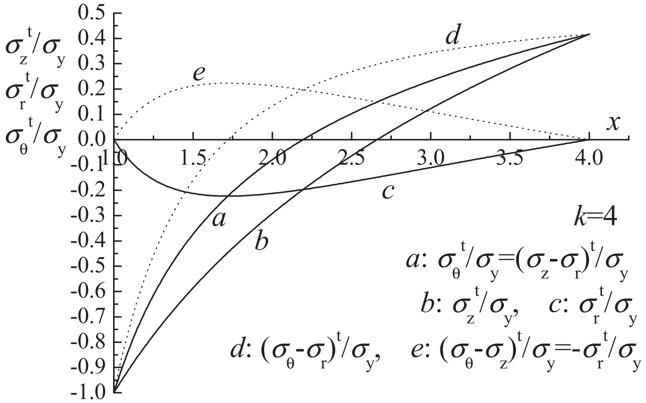

FIG. 7.59 – The thermal stresses for $\Delta t = \Delta t_c$, $k = 4$.

The total stress is the superposition of thermal stress and operating stress, when $k \geq k_c$ and $\Delta t = \Delta t_c$, the total stresses are

$$\frac{\sigma_r^T}{\sigma_y} = \frac{\sigma_r^t}{\sigma_y} + \frac{\sigma_r^p}{\sigma_y} = -\frac{(k^2-1)\ln\frac{k}{x}}{k^2\ln k^2 - k^2 + 1} + \frac{(k^2-x^2)\ln k}{x^2(k^2\ln k^2 - k^2 + 1)} - \frac{k^2-x^2}{x^2k^2} \tag{7.57}$$

$$\frac{\sigma_\theta^T}{\sigma_y} = \frac{\sigma_\theta^t}{\sigma_y} + \frac{\sigma_\theta^p}{\sigma_y} = \frac{(k^2-1)(1-\ln\frac{k}{x})}{k^2\ln k^2 - k^2 + 1} - \frac{(k^2+x^2)\ln k}{x^2(k^2\ln k^2 - k^2 + 1)} + \frac{k^2+x^2}{x^2k^2} \tag{7.58}$$

$$\frac{\sigma_z^T}{\sigma_y}=\frac{\sigma_z^t}{\sigma_y}+\frac{\sigma_z^p}{\sigma_y}=\frac{(k^2-1)(1-2\ln\frac{k}{x})}{k^2\ln k^2-k^2+1}-\frac{\ln k^2}{k^2\ln k^2-k^2+1}+\frac{1}{k^2} \tag{7.59}$$

$\frac{\sigma_\theta^T}{\sigma_y}-\frac{\sigma_r^T}{\sigma_y}=\frac{k^2\ln k^2+(k^2-1)(x^2-2)}{x^2(k^2\ln k^2-k^2+1)}$. It is easy to prove that, when $k \geq k_c$, within the whole wall there is $\sigma_\theta^T-\sigma_r^T\leq\sigma_y$; when $x=1$, or at inside wall, $\sigma_{\theta i}^T-\sigma_{ri}^T=\sigma_y$; when $x=k$, or at outside wall, $\frac{\sigma_{\theta o}^T}{\sigma_y}-\frac{\sigma_{ro}^T}{\sigma_y}=\frac{k^2\ln k^2+(k^2-1)(k^2-2)}{k^2(k^2\ln k^2-k^2+1)}$, when $k=k_c$, $\frac{k^2\ln k}{k^2-1}=1$, then $\sigma_{\theta o}^T-\sigma_{ro}^T=\sigma_y$.

The thermal stresses have nothing to do with modulus of elasticity E, coefficient of thermal expansion α, Poisson's ratio μ under the case that $k \geq k_c$, $\Delta t=\Delta t_c$ and $\frac{p}{\sigma_y}=\frac{k^2-1}{k^2}=\frac{p_3}{\sigma_y}=2\frac{p_e}{\sigma_y}$.

For $k=3$ and $k=4$, the total stresses are plotted in figures 7.60 and 7.61, respectively.

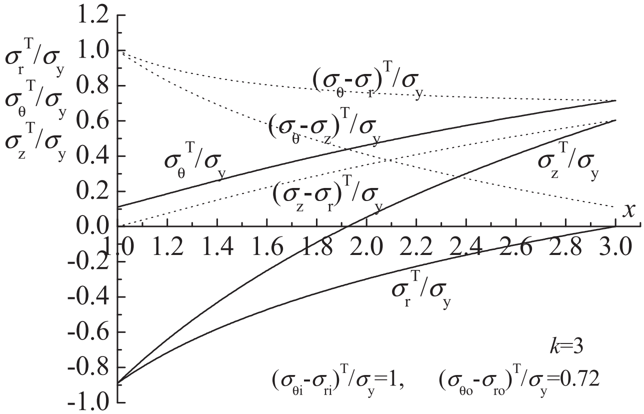

FIG. 7.60 – The total stresses with $\Delta t=\Delta t_c$, $k=3$ and $\frac{p}{\sigma_y}=\frac{k^2-1}{k^2}$.

When $k=3$, $\Delta t=\Delta t_c=133.9696$ °C ($\sigma_y=300$ MPa), $\frac{p}{\sigma_y}=\frac{k^2-1}{k^2}=0.888889$, $\frac{p_t}{\sigma_y}=0.746402$.

When $k=4$, $\Delta t=\Delta t_c=127.1168$ °C ($\sigma_y=300$ MPa), $\frac{p}{\sigma_y}=\frac{k^2-1}{k^2}=0.9375$, $\frac{p_t}{\sigma_y}=0.755437$.

Under the operation condition that $\Delta t=\Delta t_c$ and $\frac{p}{\sigma_y}=\frac{k^2-1}{k^2}=\frac{p_3}{\sigma_y}=2\frac{p_e}{\sigma_y}$, the total stresses are reduced and evened markedly, especially the maximum circumferential tension on the inner surface under internal pressure is reduced enormously.

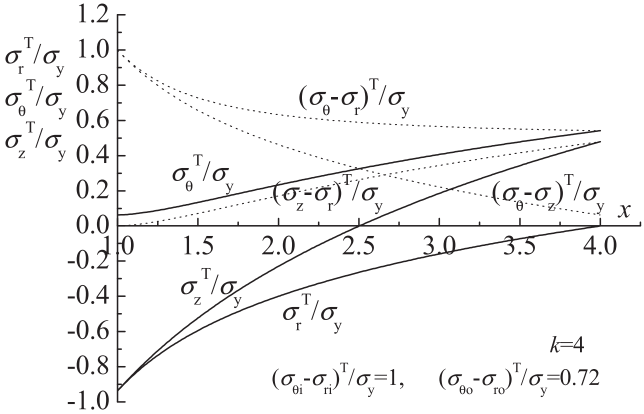

FIG. 7.61 – The total stresses with $\Delta t = \Delta t_c$, $k = 4$ and $\frac{p}{\sigma_y} = \frac{k^2-1}{k^2}$.

7.7 Chapter Summary

1. Thermal stresses of materials are related to not only the sizes of the materials but also the physical properties of the materials. The materials with a great modulus of elasticity and/or thermal expansion coefficient will have a great thermal stresses, and the materials with great Poisson ratio will have small thermal stresses.
2. The quantity of heat conducted through a wall of a cylindrical pressure vessel in unit time is related to the thermal conductivity of the material, but the distribution of the temperature in the wall of the pressure vessel is irrelevant to the physical properties of the materials, it depends on the temperatures of inside and outside walls and radii of inside and outside walls. When $\Delta t > 0$, the wall temperature in the wall of the pressure vessel gradually drops from the inside surface to the outside surface; when $\Delta t < 0$, the temperature in the wall of the pressure vessel gradually rises from the inside surface to the outside surface.

   3.1 When $\Delta t > 0$, σ_θ^t and σ_z^t monotonically increase from compressive stress on the inside surface to tension on the outside surface; when $\Delta t < 0$, σ_θ^t and σ_z^t monotonically decrease from tension on the inside surface to compressive stress on the outside surface; that is to say, the material is compressed where the temperature is high, and tensioned where the temperature is low.

   3.2 No matter $\Delta t > 0$ or $\Delta t < 0$, the absolute values of stresses on the inside surface are greater than those on the outside surface. The larger k is, the larger $\left|\frac{\sigma_{\theta i}^t}{p_t}\right|$ and $\left|\frac{\sigma_{zi}^t}{p_t}\right|$ as well as $\left|\frac{\sigma_{r\min}^t}{p_t}\right|$ and $\left|\frac{\sigma_{r\max}^t}{p_t}\right|$ are; the larger k is, the smaller $\left|\frac{\sigma_{\theta o}^t}{p_t}\right|$ and $\left|\frac{\sigma_{zo}^t}{p_t}\right|$ are; as $|\Delta t|$ increases, the inside surface gets yielded always earlier than the outside surface.

3.3 The thicker the pressure vessel is, the smaller the critical temperature difference $|\Delta t_c|$ is.

4. When engineers design thick-wall pressure apparatuses, the parameters k, p, properties of materials (E, α, μ, σ_y and r_i, r_o) and Δt can be adjusted to construct a variety of technique proposals for a pressure vessel to fit practical requirements and to draw on the advantages and avoid disadvantages of thermal stresses.
5. Using thermal stresses as pre-stresses in cylindrical high-pressure apparatuses is a potential and promising autofrettage technology for the apparatuses. In the elastic stage, $\Delta t > 0$ is beneficial to decreasing dangerous stresses in the cylindrical pressure vessel under internal pressure operation, and $\Delta t < 0$ is beneficial to decreasing dangerous stresses in the cylindrical pressure vessel under external pressure operation. If the temperature difference is increased above the critical temperature difference $|\Delta t_c|$, the materials will enter the elastic-plastic phase, at this time, the residual thermal stresses can be obtained after eliminating the temperature difference. This is another way of autofrettage that adopts residual thermal stresses.
6. For the thermal stress, $\sigma_z^t = \sigma_r^t + \sigma_\theta^t$ at any location.
7. At the inside surface and outside surface, the thermal circumferential stress and thermal axial stress are equal to each other.
8. If $\Delta t > 0$, σ_r^t gets minimum at $x = \sqrt{\frac{k^2 \ln k^2}{k^2-1}}$, which means $\sigma_r^t = \sigma_z^t - \sigma_\theta^t < 0$, or $\sigma_z^t < \sigma_\theta^t$ within $1 \leq x \leq k$.
9. For the total stress, $\sigma_\theta^T - (\sigma_z^T - \sigma_r^T) = \sigma_z^p = \frac{p}{k^2-1} = \text{a constant} > 0$, thus $\sigma_z^T - \sigma_r^T \leq \sigma_\theta$.
10. When $k \leq k_c$, $\Delta t_x \leq \Delta t_c$; when $k \geq k_c$, $\Delta t_x \geq \Delta t_c$. When $k < k_c$, the optimum operation conditions are $\Delta t = \Delta t_x$, and the load-bearing capacity of a pressure vessel is $p = \sigma_y \ln k = p_t$, $\sigma_\theta - \sigma_r \equiv \sigma_y$. When $k > k_c$, the optimum operation conditions are $\Delta t = \Delta t_c$, and the load-bearing capacity of a pressure vessel is $\frac{p}{\sigma_y} = \frac{k^2-1}{k^2} = 2\frac{p_e}{\sigma_y}$; when $k = k_c$, $\frac{p}{\sigma_y} = \frac{k^2-1}{k^2} = \ln k$. Unexpectedly, this result is the same as that obtained in mechanical autofrettage except that Δt_x and Δt_c are not concerned with mechanical autofrettage.
11. Usually, $\sigma_{\theta i}^T \geq -\sigma_y$ in engineering. If $\sigma_{\theta o}^T - \sigma_{ro}^T \leq \sigma_y$, $\sigma_{\theta o}^T \leq \sigma_y$. If $\sigma_{\theta i}^T - \sigma_{ri}^T \leq \sigma_y$, $\sigma_{\theta i}^T \leq \sigma_y$. If $p < p_t$, $\sigma_\theta^T - \sigma_r^T$ increase monotonically; if $p > p_t$, $\sigma_\theta^T - \sigma_r^T$ decrease monotonically; if $p = p_t$, $\sigma_\theta^T - \sigma_r^T$ is a constant: $\frac{p_t}{\ln k}$.

The main equations and conclusion are listed in table 7.1.

TAB. 7.1 – The main equations and conclusion of this chapter.

Thermal stresses

$$\sigma_{\mathrm{r}}^{\mathrm{t}} = p_{\mathrm{t}}\left(-\frac{\ln(k/x)}{\ln k} + \frac{k^2 - x^2}{x^2 k^2 - x^2}\right)$$

$$\sigma_{\theta}^{\mathrm{t}} = p_{\mathrm{t}}\left(\frac{1 - \ln(k/x)}{\ln k} - \frac{k^2 + x^2}{x^2 k^2 - x^2}\right)$$

$$\sigma_{\mathrm{z}}^{\mathrm{t}} = p_{\mathrm{t}}\left(\frac{1 - 2\ln(k/x)}{\ln k} - \frac{2}{k^2 - 1}\right)$$

$$p_{\mathrm{t}} = \frac{E\alpha\Delta t}{2(1-\mu)}$$

The thermal stresses at the inside and outside walls

$$\sigma_{\mathrm{ri}}^{\mathrm{t}} = \sigma_{\mathrm{ro}}^{\mathrm{t}} = 0$$

$$\sigma_{\theta\mathrm{i}}^{\mathrm{t}} = \sigma_{\mathrm{zi}}^{\mathrm{t}} = -p_{\mathrm{t}}\left(\frac{2k^2}{k^2 - 1} - \frac{1}{\ln k}\right)$$

$$\sigma_{\theta\mathrm{o}}^{\mathrm{t}} = \sigma_{\mathrm{zo}}^{\mathrm{t}} = p_{\mathrm{t}}\left(\frac{1}{\ln k} - \frac{2}{k^2 - 1}\right)$$

The equivalent thermal stress at the inside wall

$$\sigma_{\mathrm{ei}}^{\mathrm{ts}} = \sigma_{\mathrm{ri}}^{\mathrm{t}} - \sigma_{\mathrm{zi}}^{\mathrm{t}} = -\sigma_{\mathrm{zi}}^{\mathrm{t}} = -\sigma_{\theta\mathrm{i}}^{\mathrm{t}} = p_{\mathrm{t}}\left(\frac{2k^2}{k^2 - 1} - \frac{1}{\ln k}\right) = p_{\mathrm{t}}\frac{k^2 \ln k^2 - k^2 + 1}{(k^2 - 1)\ln k}$$

Critical temperature difference, Δt_{c}

$$\Delta t_{\mathrm{c}} = \frac{(k^2 - 1)\ln k}{k^2 \ln k^2 - k^2 + 1}\frac{2(1-\mu)\sigma_{\mathrm{y}}}{E\alpha}$$

Critical thermal loading p_{tc}

$$p_{\mathrm{tc}} = \frac{(k^2 - 1)\ln k}{k^2 \ln k^2 - k^2 + 1}\sigma_{\mathrm{y}}$$

Total stresses (thermal stresses plus operation stresses)

$$\sigma_{\mathrm{r}}^{\mathrm{T}} = \sigma_{\mathrm{r}}^{\mathrm{t}} + \sigma_{\mathrm{r}}^{\mathrm{p}} = p_{\mathrm{t}}\left(-\frac{\ln(k/x)}{\ln k} + \frac{k^2 - x^2}{x^2 k^2 - x^2}\right) + \frac{p}{k^2 - 1}\left(1 - \frac{k^2}{x^2}\right)$$

$$\sigma_{\theta}^{\mathrm{T}} = \sigma_{\theta}^{\mathrm{t}} + \sigma_{\theta}^{\mathrm{p}} = p_{\mathrm{t}}\left(\frac{1 - \ln(k/x)}{\ln k} - \frac{k^2 + x^2}{x^2 k^2 - x^2}\right) + \frac{p}{k^2 - 1}\left(1 + \frac{k^2}{x^2}\right)$$

$$\sigma_{\mathrm{z}}^{\mathrm{T}} = \sigma_{\mathrm{z}}^{\mathrm{t}} + \sigma_{\mathrm{z}}^{\mathrm{p}} = p_{\mathrm{t}}\left(\frac{1 - 2\ln(k/x)}{\ln k} - \frac{2}{k^2 - 1}\right) + \frac{p}{k^2 - 1}$$

For $\sigma_{\theta\mathrm{i}}^{\mathrm{T}} - \sigma_{\mathrm{ri}}^{\mathrm{T}} \le \sigma_{\mathrm{y}}$,

$$\frac{p}{\sigma_{\mathrm{y}}} \le \frac{k^2 - 1}{2k^2} + \frac{p_{\mathrm{t}}}{\sigma_{\mathrm{y}}}\frac{k^2 \ln k^2 - k^2 + 1}{k^2 \ln k^2} = \frac{k^2 - 1}{2k^2} + \frac{E\alpha}{2(1-\mu)\sigma_{\mathrm{y}}}\frac{k^2 \ln k^2 - k^2 + 1}{k^2 \ln k^2}\Delta t = \frac{p_1}{\sigma_{\mathrm{y}}}$$

$$\text{or}\quad \Delta t \ge \left(k^2\frac{p}{\sigma_{\mathrm{y}}} - \frac{k^2 - 1}{2}\right)\frac{\ln k^2}{k^2 \ln k^2 - k^2 + 1}\frac{2(1-\mu)\sigma_{\mathrm{y}}}{E\alpha} = \Delta t_1$$

p_1 increases with Δt increasing

For $\sigma_{\theta\mathrm{o}}^{\mathrm{T}} - \sigma_{\mathrm{ro}}^{\mathrm{T}} \le \sigma_{\mathrm{y}}$,

$$\frac{p}{\sigma_{\mathrm{y}}} \le \frac{k^2 - 1}{2} - \frac{p_{\mathrm{t}}}{\sigma_{\mathrm{y}}}\frac{k^2 - 1 - \ln k^2}{\ln k^2} = \frac{k^2 - 1}{2} - \frac{E\alpha}{2(1-\mu)\sigma_{\mathrm{y}}}\frac{k^2 - 1 - \ln k^2}{\ln k^2}\Delta t = \frac{p_2}{\sigma_{\mathrm{y}}}$$

$$\text{or}\quad \Delta t \le \left(\frac{k^2 - 1}{2} - \frac{p}{\sigma_{\mathrm{y}}}\right)\frac{\ln k^2}{k^2 - 1 - \ln k^2}\frac{2(1-\mu)\sigma_{\mathrm{y}}}{E\alpha} = \Delta t_2$$

TAB. 7.1 – (continued).

p_2 decreases with Δt increasing

Min $\{p_1,\ p_2\}$ is the allowable load

To ensure that $\sigma_{\theta i}^{T} - \sigma_{ri}^{T} \le \sigma_y$ and $\sigma_{\theta o}^{T} - \sigma_{ro}^{T} \le \sigma_y$ meanwhile, there must be

$$\Delta t_1 \le \Delta t \le \Delta t_2,$$

When $\frac{p}{\sigma_y} > \ln k = \frac{p_y}{\sigma_y}$, $\Delta t_1 > \Delta t_2$, so, Δt which is larger than Δt_1 and smaller than Δt_2 does not exist

When $\Delta t_1 = \Delta t_2$

$\frac{p}{\sigma_y} = \ln k = \frac{p_y}{\sigma_y}$ and $\Delta t_1 = \Delta t_2 = \Delta t_x$; If $k \le k_c$, $\Delta t_x \le \Delta t_c$; If $k \ge k_c$, $\Delta t_x \ge \Delta t_c$. So,

$$\frac{p}{\sigma_y} = \ln k = \frac{p_y}{\sigma_y} \text{ is only applicable to the case of } k \le k_c$$

where p_y is just mechanical entire yield pressure

If $\Delta t = \Delta t_c$, p_1 becomes

$$\frac{p}{\sigma_y} \le \frac{k^2-1}{k^2} = \frac{p_3}{\sigma_y} = 2\frac{p_e}{\sigma_y} \quad \text{or} \quad k = \sqrt{\frac{\sigma_y}{\sigma_y - p}}$$

where p_e is the maximum mechanical elastic load-bearing capability of an unautofrettaged cylindrical pressure vessel, or initial mechanical yield pressure, $\frac{p_e}{\sigma_y} = \frac{k^2-1}{2k^2}$

If $\Delta t = \Delta t_c$, p_2 becomes

$$\frac{p}{\sigma_y} \le \frac{(k^4-1)\ln k - (k^2-1)^2}{k^2 \ln k^2 - k^2 + 1} = \frac{p_4}{\sigma_y}$$

Ideal temperature difference

$$\Delta t_x = \frac{(1-\mu)\sigma_y \ln k^2}{E\alpha} = \frac{k^2 \ln k^2 - k^2 + 1}{k^2 - 1}\Delta t_c$$

Δt_x increases as k increases. When $\Delta t \le \Delta t_x$, $p_1 \le p_2$; when $\Delta t \ge \Delta t_x$, $p_1 \ge p_2$; when $\Delta t = \Delta t_x$,

$$p_1 = p_2$$

When

$$\frac{k^2 \ln k}{k^2 - 1} = 1 \quad \text{or} \quad \ln k = \frac{k^2-1}{k^2}$$

or $k = k_c = 2.2184574899167\ldots$, $\Delta t_x = \Delta t_c$

When $k < k_c$, $\Delta t_x < \Delta t_c$, if $\Delta t = \Delta t_x$, the maximum load-bearing capacity is $\frac{p}{\sigma_y} = \ln k$

When $k = k_c$, $\Delta t_x = \Delta t_c$, if $\Delta t = \Delta t_x = \Delta t_c$, the maximum load-bearing capacity is $\frac{p}{\sigma_y} = \ln k = \frac{k^2-1}{k^2}$

When $k > k_c$, $\Delta t_x > \Delta t_c$, if $\Delta t = \Delta t_c$, the maximum load-bearing capacity is $\frac{p}{\sigma_y} = \frac{k^2-1}{k^2}$

The optimum design

When $k \le k_c$, if $\Delta t = \Delta t_x$,

$$p = p_y = \sigma_y \ln k \leftrightarrow k = e^{\frac{p}{\sigma_y}} \qquad p_t = \sigma_y \ln k = p_y = p$$

When $k \le k_c$ and $\Delta t = \Delta t_x$, the thermal stresses are

$$\frac{\sigma_r^t}{\sigma_y} = \ln x + \frac{k^2 \ln k}{k^2-1}\frac{1}{x^2} - \frac{k^2 \ln k}{k^2-1}$$

Tab. 7.1 – (continued).

$$\frac{\sigma_\theta^\mathrm{t}}{\sigma_\mathrm{y}}=1+\ln x-\frac{k^2\ln k}{k^2-1}\frac{1}{x^2}-\frac{k^2\ln k}{k^2-1}$$

$$\frac{\sigma_\mathrm{z}^\mathrm{t}}{\sigma_\mathrm{y}}=1+\ln x^2-\frac{k^2\ln k^2}{k^2-1}$$

When $k \leq k_\mathrm{c}$ and $\Delta t=\Delta t_\mathrm{x}$, the total stresses are

$$\frac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}}=-\ln\frac{k}{x}$$

$$\frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}}=1-\ln\frac{k}{x}$$

$$\frac{\sigma_\mathrm{z}^\mathrm{T}}{\sigma_\mathrm{y}}=1-\ln\frac{k^2}{x^2}-\frac{\ln k}{k^2-1}$$

within the whole wall $\sigma_\theta^\mathrm{T}-\sigma_\mathrm{r}^\mathrm{T}\equiv\sigma_\mathrm{y}$

When $k \geq k_\mathrm{c}$, if $\Delta t=\Delta t_\mathrm{c}$,

$$\frac{p}{\sigma_\mathrm{y}}=\frac{k^2-1}{k^2}\leftrightarrow k=\sqrt{\frac{\sigma_\mathrm{y}}{\sigma_\mathrm{y}-p}}$$

Thermal loading is critical thermal loading p_tc

When $k \geq k_\mathrm{c}$ and $\Delta t=\Delta t_\mathrm{c}$, the total stresses are

$$\frac{\sigma_\mathrm{r}^\mathrm{T}}{\sigma_\mathrm{y}}=\frac{\sigma_\mathrm{r}^\mathrm{t}}{\sigma_\mathrm{y}}+\frac{\sigma_\mathrm{r}^\mathrm{p}}{\sigma_\mathrm{y}}=-\frac{(k^2-1)\ln\frac{k}{x}}{k^2\ln k^2-k^2+1}+\frac{(k^2-x^2)\ln k}{x^2(k^2\ln k^2-k^2+1)}-\frac{k^2-x^2}{x^2k^2}$$

$$\frac{\sigma_\theta^\mathrm{T}}{\sigma_\mathrm{y}}=\frac{\sigma_\theta^\mathrm{t}}{\sigma_\mathrm{y}}+\frac{\sigma_\theta^\mathrm{p}}{\sigma_\mathrm{y}}=\frac{(k^2-1)(1-\ln\frac{k}{x})}{k^2\ln k^2-k^2+1}-\frac{(k^2+x^2)\ln k}{x^2(k^2\ln k^2-k^2+1)}+\frac{k^2+x^2}{x^2k^2}$$

$$\frac{\sigma_\mathrm{z}^\mathrm{T}}{\sigma_\mathrm{y}}=\frac{\sigma_\mathrm{z}^\mathrm{t}}{\sigma_\mathrm{y}}+\frac{\sigma_\mathrm{z}^\mathrm{p}}{\sigma_\mathrm{y}}=\frac{(k^2-1)(1-2\ln\frac{k}{x})}{k^2\ln k^2-k^2+1}-\frac{\ln k^2}{k^2\ln k^2-k^2+1}+\frac{1}{k^2}$$

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Chapter 8

Thermal Autofrettage Technology Based on Mises Yield Criterion

8.1 The Analysis of Thermal Stresses

The thermal stresses in the elastic state are still equations (7.18)–(7.20), or equations (8.1)–(8.3) of this chapter.

$$\sigma_{\mathrm{r}}^{\mathrm{t}} = p_{\mathrm{t}}\left(-\frac{\ln(k/x)}{\ln k} + \frac{k^2 - x^2}{x^2 k^2 - x^2}\right) \tag{8.1}$$

$$\sigma_{\theta}^{\mathrm{t}} = p_{\mathrm{t}}\left(\frac{1 - \ln(k/x)}{\ln k} - \frac{k^2 + x^2}{x^2 k^2 - x^2}\right) \tag{8.2}$$

$$\sigma_{\mathrm{z}}^{\mathrm{t}} = p_{\mathrm{t}}\left(\frac{1 - 2\ln(k/x)}{\ln k} - \frac{2}{k^2 - 1}\right) \tag{8.3}$$

where p_{t} is called thermal loading, MPa, $p_{\mathrm{t}} = \frac{E\alpha\Delta t}{2(1-\mu)}$. $\sigma_{\mathrm{z}}^{\mathrm{t}} = \sigma_{\mathrm{r}}^{\mathrm{t}} + \sigma_{\theta}^{\mathrm{t}}$ at any location.

At the inside and outside walls, the thermal stresses are the same as equations (7.22)–(7.24), or equations (8.4)–(8.6) of this chapter:

$$\sigma_{\mathrm{ri}}^{\mathrm{t}} = \sigma_{\mathrm{ro}}^{\mathrm{t}} = 0 \tag{8.4}$$

$$\sigma_{\theta\mathrm{i}}^{\mathrm{t}} = \sigma_{\mathrm{zi}}^{\mathrm{t}} = -p_{\mathrm{t}}\left(\frac{2k^2}{k^2 - 1} - \frac{1}{\ln k}\right) \tag{8.5}$$

$$\sigma_{\theta\mathrm{o}}^{\mathrm{t}} = \sigma_{\mathrm{zo}}^{\mathrm{t}} = p_{\mathrm{t}}\left(\frac{1}{\ln k} - \frac{2}{k^2 - 1}\right) \tag{8.6}$$

When based on the maximum distortion strain energy theory (Mises's yield criterion)[1], the equivalent thermal stress is

DOI: 10.1051/978-2-7598-3111-1.c008
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$$\sigma_{\mathrm{e}}^{\mathrm{d}}=\sqrt{\frac{1}{2}\left[(\sigma_\theta-\sigma_{\mathrm{z}})^2+(\sigma_{\mathrm{z}}-\sigma_{\mathrm{r}})^2+(\sigma_{\mathrm{r}}-\sigma_\theta)^2\right]} \tag{8.7}$$

On the inner wall, the equivalent thermal stress is

$$\sigma_{\mathrm{ei}}^{\mathrm{td}}=\sqrt{\frac{1}{2}\left[\left(\sigma_{\theta\mathrm{i}}^{\mathrm{t}}-\sigma_{\mathrm{zi}}^{\mathrm{t}}\right)^2+\left(\sigma_{\mathrm{zi}}^{\mathrm{t}}-\sigma_{\mathrm{ri}}^{\mathrm{t}}\right)^2+\left(\sigma_{\mathrm{ri}}^{\mathrm{t}}-\sigma_{\theta\mathrm{i}}^{\mathrm{t}}\right)^2\right]} \tag{8.8}$$

On the outer wall, the equivalent thermal stress is

$$\sigma_{\mathrm{eo}}^{\mathrm{td}}=\sqrt{\frac{1}{2}\left[\left(\sigma_{\theta\mathrm{o}}^{\mathrm{t}}-\sigma_{\mathrm{zo}}^{\mathrm{t}}\right)^2+\left(\sigma_{\mathrm{zo}}^{\mathrm{t}}-\sigma_{\mathrm{ro}}^{\mathrm{t}}\right)^2+\left(\sigma_{\mathrm{ro}}^{\mathrm{t}}-\sigma_{\theta\mathrm{o}}^{\mathrm{t}}\right)^2\right]} \tag{8.9}$$

When $\sigma_{\mathrm{ei}}^{\mathrm{td}}=\sigma_{\mathrm{y}}$, the inner wall initiates yielding; when $\sigma_{\mathrm{eo}}^{\mathrm{td}}=\sigma_{\mathrm{y}}$, the outer wall initiates yielding. From equations (8.4) to (8.6),

$$\sigma_{\mathrm{ei}}^{\mathrm{td}}=-\sigma_{\mathrm{zi}}^{\mathrm{t}}=-\sigma_{\theta\mathrm{i}}^{\mathrm{t}}=p_{\mathrm{t}}\left(\frac{2k^2}{k^2-1}-\frac{1}{\ln k}\right)=p_{\mathrm{t}}\frac{k^2\ln k^2-k^2+1}{(k^2-1)\ln k} \tag{8.10}$$

Equation (8.10) is the same as equation (7.25).

From equations (8.4) to (8.6),

$$\sigma_{\mathrm{eo}}^{\mathrm{td}}=p_{\mathrm{t}}\left(\frac{1}{\ln k}-\frac{2}{k^2-1}-\right)=p_{\mathrm{t}}\frac{k^2-1-\ln k^2}{(k^2-1)\ln k}<\sigma_{\mathrm{ei}}^{\mathrm{td}} \tag{8.11}$$

For equation (8.10), letting $\sigma_{\mathrm{ei}}^{\mathrm{td}}=\sigma_{\mathrm{y}}$ obtains the temperature difference at which the inside wall initiates yielding, called critical temperature difference, Δt_{c}:

$$\Delta t_{\mathrm{c}}=\frac{(k^2-1)\ln k}{k^2\ln k^2-k^2+1}\frac{2(1-\mu)\sigma_{\mathrm{y}}}{E\alpha}=mf \tag{8.12}$$

where $m=\frac{(k^2-1)\ln k}{k^2\ln k^2-k^2+1}$, reflecting the size of a cylindrical pressure vessel; $f=\frac{2(1-\mu)\sigma_{\mathrm{y}}}{E\alpha}$, reflecting material properties. Equation (8.12) is the same as equation (7.26) of the last chapter. The critical temperature difference when $\Delta t<0$ is equal to negative equation (8.12).

When $\Delta t>0$, the thermal loading at which the inside wall initiates yielding, called critical thermal loading p_{tc}, is obtained from equation (8.12):

$$p_{\mathrm{tc}}=\frac{(k^2-1)\ln k}{k^2\ln k^2-k^2+1}\sigma_{\mathrm{y}} \tag{8.13}$$

An example: commonly, elastic modules of steel materials $E=1.95\times10^5$ MPa, Poisson ratio $\mu=0.3$, thermal expansion coefficient of the materials $\alpha=1.2\times10^{-5}\ {}^\circ\mathrm{C}^{-1}$, $k=2.5$, $\sigma_{\mathrm{y}}=300$ Mpa, from equation (8.12), $\Delta t_{\mathrm{c}}=139.1809679\ {}^\circ\mathrm{C}$. Setting $\Delta t=\Delta t_{\mathrm{c}}=139.1809679\ {}^\circ\mathrm{C}$, then, $p_{\mathrm{t}}=232.631$ MPa. The thermal stresses are shown as the solid curves in figure 8.1. If $\Delta t=-139.1809679\ {}^\circ\mathrm{C}$, the trend of variations of thermal stresses is averse to figure 8.1, as is shown in figure 8.2. The meaning of each curve and the main parameters are marked in the figures.

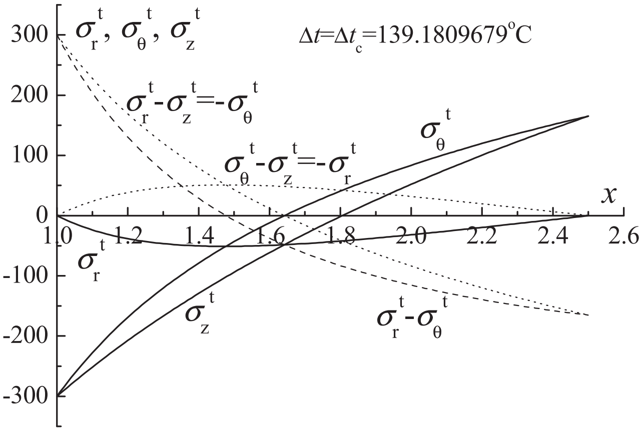


FIG. 8.1 – Distribution of thermal stresses for elastic cylindrical pressure vessel when $\Delta t = \Delta t_c = 139.1809679$ °C.

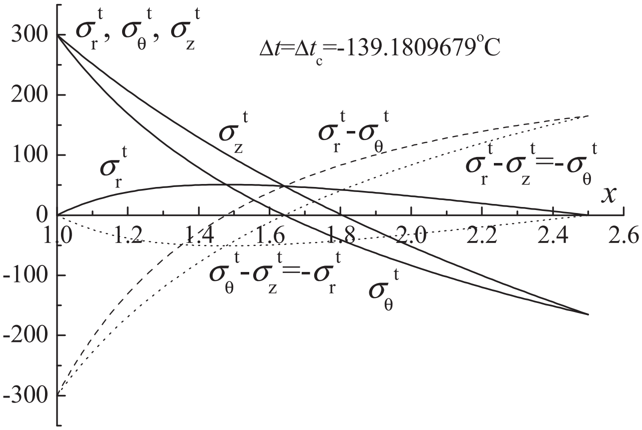


FIG. 8.2 – Distribution of thermal stresses for elastic cylindrical pressure vessel when $\Delta t = \Delta t_c = -139.1809679$ °C.

8.2 The Analysis of Total Stresses and Investigation of Optimum Operation Conditions

The total stresses are the same as equations (7.30)–(7.32) or equations (8.14)–(8.16) of this chapter:

$$\sigma_r^T = \sigma_r^t + \sigma_r^p = p_t\left(-\frac{\ln(k/x)}{\ln k} + \frac{k^2 - x^2}{x^2 k^2 - x^2}\right) + \frac{p}{k^2-1}\left(1 - \frac{k^2}{x^2}\right) \tag{8.14}$$

$$\sigma_\theta^\mathrm{T} = \sigma_\theta^\mathrm{t} + \sigma_\theta^\mathrm{p} = p_\mathrm{t}\left(\frac{1-\ln(k/x)}{\ln k} - \frac{k^2+x^2}{x^2k^2-x^2}\right) + \frac{p}{k^2-1}\left(1+\frac{k^2}{x^2}\right) \tag{8.15}$$

$$\sigma_\mathrm{z}^\mathrm{T} = \sigma_\mathrm{z}^\mathrm{t} + \sigma_\mathrm{z}^\mathrm{p} = p_\mathrm{t}\left(\frac{1-2\ln(k/x)}{\ln k} - \frac{2}{k^2-1}\right) + \frac{p}{k^2-1} \tag{8.16}$$

where $\sigma_\mathrm{r}^\mathrm{T}$, σ_θ^T and $\sigma_\mathrm{z}^\mathrm{T}$ are total radial stress, total circumferential stress, and total axial stress, respectively, $\sigma_\mathrm{r}^\mathrm{p}$, σ_θ^p and $\sigma_\mathrm{z}^\mathrm{p}$ are mechanical radial stress, mechanical circumferential stress, and mechanical axial stress, respectively, which is equation (1.1).

$$\sigma_\theta^\mathrm{T} - \sigma_\mathrm{r}^\mathrm{T} = p_\mathrm{t}\left(\frac{1}{\ln k} - \frac{2k^2}{x^2k^2-x^2}\right) + \frac{2k^2p}{x^2k^2-x^2} = \frac{2k^2}{x^2k^2-x^2}(p-p_\mathrm{t}) + \frac{p_\mathrm{t}}{\ln k} \tag{8.17}$$

$$\sigma_\theta^\mathrm{T} - \sigma_\mathrm{z}^\mathrm{T} = p_\mathrm{t}\left(\frac{\ln(k/x)}{\ln k} - \frac{k^2-x^2}{x^2k^2-x^2}\right) + \frac{k^2p}{x^2k^2-x^2} \tag{8.18}$$

$$\sigma_\mathrm{z}^\mathrm{T} - \sigma_\mathrm{r}^\mathrm{T} = p_\mathrm{t}\left(\frac{1-\ln(k/x)}{\ln k} - \frac{k^2+x^2}{x^2k^2-x^2}\right) + \frac{k^2p}{x^2k^2-x^2} \tag{8.19}$$

According to equation (8.17), if $p \geq \frac{k^2\ln k^2 - x^2(k^2-1)}{k^2\ln k^2}p_\mathrm{t}$, $\sigma_\theta^\mathrm{T} \geq \sigma_\mathrm{r}^\mathrm{T}$. When $x = 1$, $p \geq \left(1 - \frac{k^2-1}{k^2\ln k^2}\right)p_\mathrm{t}$, where $1 - \frac{k^2-1}{k^2\ln k^2} = 1$ monotonically increases as k increases[2]. When $k = 1$, $1 - \frac{k^2-1}{k^2\ln k^2} = 0$; when $k = \infty$, $1 - \frac{k^2-1}{k^2\ln k^2} = 1$; when $k = 4$, $1 - \frac{k^2-1}{k^2\ln k^2} \approx 0.66$. Therefore, in engineering there is generally $\sigma_\theta^\mathrm{T} \geq \sigma_\mathrm{r}^\mathrm{T}$.

According to equation (8.18), if $p \geq \frac{k^2\ln k - x^2(k^2\ln(k/x) + \ln x)}{k^2\ln k}p_\mathrm{t}$, $\sigma_\theta^\mathrm{T} \geq \sigma_\mathrm{z}^\mathrm{T}$. When $x = 1$, $p \geqq 0$. Therefore, in engineering, there is always $\sigma_\theta^\mathrm{T} \geq \sigma_\mathrm{z}^\mathrm{TT}$.

Thus, σ_θ^T is the most dangerous.

On the outer surface, $x = k$, then,

$$\sigma_\mathrm{ro}^\mathrm{T} - \sigma_{\theta\mathrm{o}}^\mathrm{T} = p_\mathrm{t}\left(\frac{1}{\ln k} - \frac{2}{k^2-1}\right) + \frac{2p}{k^2-1} = ap_\mathrm{t} + 2bp \tag{8.20}$$

$$\sigma_{\theta\mathrm{o}}^\mathrm{T} - \sigma_\mathrm{zo}^\mathrm{T} = \frac{p}{k^2-1} = bp \tag{8.21}$$

$$\sigma_\mathrm{zo}^\mathrm{T} - \sigma_\mathrm{ro}^\mathrm{T} = p_\mathrm{t}\left(\frac{1}{\ln k} - \frac{2}{k^2-1}\right) + \frac{p}{k^2-1} = ap_\mathrm{t} + bp \tag{8.22}$$

where $a = \frac{1}{\ln k} - \frac{2}{k^2-1} = \frac{k^2-1-\ln k^2}{(k^2-1)\ln k}$, $b = \frac{1}{k^2-1}$. When k increases, a and b decrease simultaneously; when k decreases, a and b increase simultaneously[2]. When $k \to 1$, $a \to 1$, $b \to \infty$; When $k \to \infty$, $a \to 0$, $b \to 0$. a decreases gently.

Setting $\sigma_{\text{eo}}^{\text{Td}} = \sqrt{\frac{1}{2}\left[\left(\sigma_{\theta\text{o}}^{\text{T}} - \sigma_{\text{zo}}^{\text{T}}\right)^2 + \left(\sigma_{\text{zo}}^{\text{T}} - \sigma_{\text{ro}}^{\text{T}}\right)^2 + \left(\sigma_{\text{ro}}^{\text{T}} - \sigma_{\theta\text{o}}^{\text{T}}\right)^2\right]} \le \sigma_{\text{y}}$ results in

$$3b^2p^2 + 3abp_{\text{t}}p + a^2p_{\text{t}}^2 - \sigma_{\text{y}}^2 \le 0 \tag{8.23}$$

If $p = 0$ or the pressure vessel is subjected only to thermal stresses, from equation (8.23),

$$-\frac{\sigma_{\text{y}}}{a} = -\frac{(k^2-1)\ln k}{k^2-1-\ln k^2}\sigma_{\text{y}} \le p_{\text{t}} \le \frac{\sigma_{\text{y}}}{a} = \frac{(k^2-1)\ln k}{k^2-1-\ln k^2}\sigma_{\text{y}} \tag{8.24}$$

or

$$-\frac{(k^2-1)\ln k}{k^2-1-\ln k^2}\frac{2(1-\mu)}{E\alpha}\sigma_{\text{y}} \le \Delta t \le \frac{(k^2-1)\ln k}{k^2-1-\ln k^2}\frac{2(1-\mu)}{E\alpha}\sigma_{\text{y}} \tag{8.25}$$

Since $\ln k^2 > 2\frac{k^2-1}{k^2+1}$, it is easy to prove that $k^2\ln k^2 - k^2 + 1 > k^2 - 1 - \ln k^2$. Consequently, $\frac{(k^2-1)\ln k}{k^2-1-\ln k^2}\frac{2(1-\mu)}{E\alpha}\sigma_{\text{y}} > \Delta t_{\text{c}}$. This is natural because when $p = 0$ and $\Delta t < \Delta t_{\text{c}}$, the pressure vessel is of course safe.

Finding out p from equation (8.23), we obtain

$$p_7 = \frac{-3ap_{\text{t}} - \sqrt{12\sigma_{\text{y}}^2 - 3a^2p_{\text{t}}^2}}{6b} \le p \le \frac{-3ap_{\text{t}} + \sqrt{12\sigma_{\text{y}}^2 - 3a^2p_{\text{t}}^2}}{6b} = p_5 \tag{8.26}$$

When $\frac{p_{\text{t}}}{\sigma_{\text{y}}} \le \frac{2}{a} = \frac{(k^2-1)\ln k^2}{k^2-1-\ln k^2} = 2\frac{k^2\ln k^2-k^2+1}{k^2-1-\ln k^2}\frac{p_{\text{tc}}}{\sigma_{\text{y}}}$, p_5 is real. Since $\ln k^2 > 2\frac{k^2-1}{k^2+1}$, it is also easy to prove that $2\frac{k^2\ln k^2-k^2+1}{k^2-1-\ln k^2} > 1$. Therefore, as long as $\Delta t \le \Delta t_{\text{c}}$, p_5 is real.

p_7 is totally negative, therefore, it is not taken into account in engineering.

As Δt or p_{t} increases, p_5 decreases.

When $p_{\text{t}} \le \frac{\sigma_{\text{y}}}{a}$ or $p_{\text{t}} \le \frac{(k^2-1)\ln k}{k^2-1-\ln k^2}\sigma_{\text{y}} = p_{\text{t}0}$, $\sqrt{12\sigma_{\text{y}}^2 - 3a^2p_{\text{t}}^2} \ge 3ap_{\text{t}}$ or $p_5 \ge 0$. Since $\ln k^2 > 2\frac{k^2-1}{k^2+1}$, it is easy to prove that $p_{\text{t}0} \ge p_{\text{tc}}$. Therefore, as long as $p_{\text{t}} \le p_{\text{tc}}$, $p_{\text{t}} \le p_{\text{t}0}$, thereby $p_5 \ge 0$. Therefore, generally $p_5 \ge 0$.

Finding out p_{t} from equation (8.23), we obtain

$$p_{\text{t}} \le \frac{-3bp + \sqrt{4\sigma_{\text{y}}^2 - 3b^2p^2}}{2a} = p_{\text{t}3} \tag{8.27}$$

$$\text{or } \Delta t \le \left(-3bp + \sqrt{4\sigma_{\text{y}}^2 - 3b^2p^2}\right)\frac{1-\mu}{E\alpha a} = \Delta t_3 \tag{8.28}$$

$$\text{or } \frac{\Delta t}{\sigma_{\text{y}}} \le \left[-3b\frac{p}{\sigma_{\text{y}}} + \sqrt{4 - 3b^2\left(\frac{p}{\sigma_{\text{y}}}\right)^2}\right]\frac{1-\mu}{E\alpha a} = \frac{\Delta t_3}{\sigma_{\text{y}}} \tag{8.29}$$

When

$$\frac{p}{\sigma_y} \le \frac{2(k^2-1)}{\sqrt{3}},$$

Δt_3 is a real number, and $\frac{2(k^2-1)}{\sqrt{3}} \gg 2\frac{p_e}{\sigma_y}$.

When

$$\frac{p}{\sigma_y} \le \frac{k^2-1}{\sqrt{3}} = \frac{p_f}{\sigma_y},$$

$\sqrt{4\sigma_y^2 - 3b^2p^2} \ge 3bp$, or $\Delta t_3 \geq 0$.

It is easy to prove that $p_5 < \frac{k^2-1}{\sqrt{3}}\sigma_y$. Hence, when $p \leq p_5$, there must be $p \le \frac{k^2-1}{\sqrt{3}}\sigma_y$. Therefore, generally $\Delta t_3 \geq 0$. Besides, $\frac{k^2-1}{\sqrt{3}}\sigma_y > p_e$; when $k \ge \sqrt{2}$, $\frac{k^2-1}{\sqrt{3}}\sigma_y \ge 2p_e$. That is to say, when $k \le \sqrt{2}$, load-bearing capacity or allowable loading cannot reach $2p_e$. In addition, as a or b increases, Δt_3 decreases; as a or b decreases, Δt_3 increases. Accordingly, as k increases, Δt_3 increases; as k decreases, Δt_3 decreases. As p increases, Δt_3 decreases; as p decreases, Δt_3 increases.

When $\Delta t_3 = \Delta t_c$, p_5 is called p_{5c}.

On the inner surface, $x = 1$, then,

$$\sigma_{\theta i}^T - \sigma_{ri}^T = -p_t\left(\frac{2k^2}{k^2-1} - \frac{1}{\ln k}\right) + \frac{2k^2p}{k^2-1} = -cp_t + 2dp \tag{8.30}$$

$$\sigma_{\theta i}^T - \sigma_{zi}^T = \frac{k^2p}{k^2-1} = dp \tag{8.31}$$

$$\sigma_{zi}^T - \sigma_{ri}^T = -p_t\left(\frac{2k^2}{k^2-1} - \frac{1}{\ln k}\right) + \frac{k^2p}{k^2-1} = -cp_t + dp \tag{8.32}$$

where $c = \frac{2k^2}{k^2-1} - \frac{1}{\ln k} = \frac{k^2\ln k^2 - k^2 + 1}{(k^2-1)\ln k} = \frac{2}{k^2-1} - \frac{1}{\ln k} + 2 = 2 - a = \frac{\sigma_y}{p_{tc}}$, $d = \frac{k^2}{k^2-1} = 1 + b$.

$$(ad + bc) = \frac{1}{\ln k} \tag{8.33}$$

When $k \to 1$, $a \to 1$, accordingly $c \to 1$; When $k \to 1$, $b \to \infty$, accordingly $d \to \infty$. When $k \to \infty$, $a \to 0$, accordingly $c \to 2$; When $k \to \infty$, $b \to 0$, accordingly $d \to 1$. a, b, c and d are plotted in figure 8.3.

Setting $\sigma_{ei}^{Td} = \sqrt{\frac{1}{2}\left[\left(\sigma_{\theta i}^T - \sigma_{zi}^T\right)^2 + \left(\sigma_{zi}^T - \sigma_{ri}^T\right)^2 + \left(\sigma_{ri}^T - \sigma_{\theta i}^T\right)^2\right]} \le \sigma_y$ leads to

$$3d^2p^2 - 3cdp_tp + c^2p_t^2 - \sigma_y^2 \le 0 \tag{8.34}$$

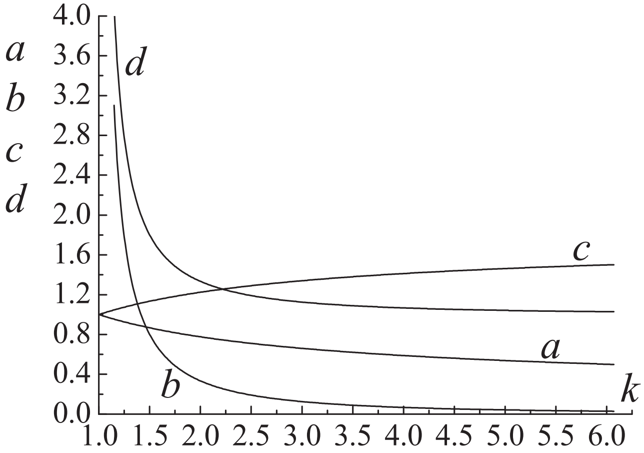


FIG. 8.3 – The graph of a, b, c, and d.

If $p = 0$, or the pressure vessel is subjected only to thermal stresses, from equation (8.30),

$$-\frac{\sigma_y}{c} \leq p_t \leq \frac{\sigma_y}{c} \tag{8.35}$$

or

$$-\Delta t_c \leq \Delta t = \Delta t_c \tag{8.36}$$

This is natural because when $p = 0$ and $\Delta t < \Delta t_c$, the pressure vessel is of course safe.

Finding out p from equation (8.34), we obtain

$$p_{10} = \frac{3cp_t - \sqrt{12\sigma_y^2 - 3c^2p_t^2}}{6d} \leq p \leq \frac{3cp_t + \sqrt{12\sigma_y^2 - 3c^2p_t^2}}{6d} = p_6 \tag{8.37}$$

When $p_t \leq p_{tc}$, or when $\Delta t \leq \Delta t_c$, $p_{10} \leq 0$. Thus, p_{10} may not be taken into consideration in engineering. Then generally equation (3.31) becomes

$$p \leq \frac{3cp_t + \sqrt{12\sigma_y^2 - 3c^2p_t^2}}{6d} = p_6 \tag{8.38}$$

When

$$\frac{p_t}{\sigma_y} \leq \frac{2}{c} = \frac{(k^2-1)\ln k^2}{k^2\ln k^2 - k^2 + 1} = 2\frac{p_{tc}}{\sigma_y}, \text{ or } \Delta t \leq \frac{(k^2-1)\ln k^2}{k^2\ln k^2 - k^2 + 1}\frac{2(1-\mu)\sigma_y}{E\alpha} = 2\Delta t_c = \Delta t_u,$$

p_6 is real. Therefore, generally, p_6 is real in engineering. When $\Delta t = 2\Delta t_{\mathrm{c}} = \Delta t_{\mathrm{u}}$,

$$\frac{p_6}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{k^2} = \sqrt{3}\,\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{p_{6\mathrm{c}}}{\sigma_{\mathrm{y}}}$$

When $\Delta t = 0$

$$\frac{p_6}{\sigma_{\mathrm{y}}} = \frac{1}{\sqrt{3}d} = \frac{k^2-1}{\sqrt{3}k^2} = \frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{s}}}{\sigma_{\mathrm{y}}}$$

When $\Delta t = \Delta t_{\mathrm{c}}$, p_6 is also called $p_{6\mathrm{c}}$, because we will see that $\Delta t = \Delta t_{\mathrm{c}}$ and $\Delta t = 2\Delta t_{\mathrm{c}}$ both result in $\frac{p_6}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{k^2} = \sqrt{3}\,\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{p_{6\mathrm{c}}}{\sigma_{\mathrm{y}}}$.

Finding out p_{t} from equation (8.34), we obtain

$$\frac{3dp - \sqrt{4\sigma_{\mathrm{y}}^2 - 3d^2p^2}}{2c} = p_{\mathrm{t}4} \le p_{\mathrm{t}} \le \frac{3dp + \sqrt{4\sigma_{\mathrm{y}}^2 - 3d^2p^2}}{2c} = p_{\mathrm{t}0} \tag{8.39}$$

$$\text{or } \left[3dp - \sqrt{4\sigma_{\mathrm{y}}^2 - 3d^2p^2}\right]\frac{1-\mu}{E\alpha c} = \Delta t_4 \le \Delta t \le \left[3dp + \sqrt{4\sigma_{\mathrm{y}}^2 - 3d^2p^2}\right]\frac{1-\mu}{E\alpha c} = \Delta t_0 \tag{8.40}$$

$$\text{or } \left[3d\frac{p}{\sigma_{\mathrm{y}}} - \sqrt{4 - 3d^2\left(\frac{p}{\sigma_{\mathrm{y}}}\right)^2}\right]\frac{1-\mu}{E\alpha c} = \frac{\Delta t_4}{\sigma_{\mathrm{y}}} \le \frac{\Delta t}{\sigma_{\mathrm{y}}} \le \left[3d\frac{p}{\sigma_{\mathrm{y}}} + \sqrt{4 - 3d^2\left(\frac{p}{\sigma_{\mathrm{y}}}\right)^2}\right]\frac{1-\mu}{E\alpha c} = \frac{\Delta t_0}{\sigma_{\mathrm{y}}} \tag{8.41}$$

When $p = 0$,

$$\Delta t_4 = -\Delta t_{\mathrm{c}} \text{ and } \Delta t_0 = \Delta t_{\mathrm{c}}.$$

For Δt_0, if we let $\Delta t_0 = \Delta t_{\mathrm{c}}$, we obtain $p = 0$ and $\frac{p}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{k^2} = \sqrt{3}\,\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$, and substituting $\frac{p}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{k^2}$ into Δt_0 obtains $\Delta t_0 = 2\Delta t_{\mathrm{c}}$.

For Δt_4, if we let $\Delta t_4 = \Delta t_{\mathrm{c}}$, we obtain $\frac{p}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{k^2} = \sqrt{3}\,\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}}$, and substituting $\frac{p}{\sigma_{\mathrm{y}}} = \frac{k^2-1}{k^2}$ into Δt_4 just obtains $\Delta t_0 = \Delta t_{\mathrm{c}}$.

When

$$p \ge \frac{k^2-1}{\sqrt{3}k^2}\sigma_{\mathrm{y}} = p_{\mathrm{e}} \text{ (this is the initial mechanical yield pressure)},$$

$\sqrt{4\sigma_{\mathrm{y}}^2 - 3d^2p^2} \le 3dp$, thereupon $\Delta t_4 \ge 0$.

When

$$p \le 2\frac{k^2-1}{\sqrt{3}k^2}\sigma_{\mathrm{y}} = 2p_{\mathrm{e}} = p_{\mathrm{u}} \text{ (two times the initial mechanical yield pressure)},$$

Δt_4 and Δt_0 are real numbers.

With the increase of p, Δt_4 increases; with the decrease of p, Δt_4 decreases.

$\frac{\mathrm{d}\Delta t_0}{\mathrm{d}p} = \frac{3d}{2c}\left(1 - \frac{dp}{\sqrt{4\sigma_{\mathrm{y}}^2 - 3d^2p^2}}\right)\frac{2(1-\mu)}{E\alpha}$. Letting $\frac{\mathrm{d}\Delta t_0}{\mathrm{d}p} \ge 0$ obtains

$$\frac{p}{\sigma_{\mathrm{y}}} \le \frac{1}{d} = \frac{k^2-1}{k^2} = \sqrt{3}\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{p_{6\mathrm{c}}}{\sigma_{\mathrm{y}}}$$

Therefore, if $p < \sqrt{3}p_{\mathrm{e}}$, with the increase of p, Δt_0 increases; with the decrease of p, Δt_0 decreases. If $p > \sqrt{3}p_{\mathrm{e}}$, with the increase of p, Δt_0 decreases; with the decrease of p, Δt_0 increases.

$\frac{\mathrm{d}p_6}{\mathrm{d}p_{\mathrm{t}}} = \frac{c}{2d}\left(1 - \frac{cp_{\mathrm{t}}}{\sqrt{12\sigma_{\mathrm{y}}^2 - 3c^2p_{\mathrm{t}}^2}}\right)$, letting $\frac{\mathrm{d}p_6}{\mathrm{d}p_{\mathrm{t}}} \ge 0$ leads to

$$p_{\mathrm{t}} \le \sqrt{3}\frac{\sigma_{\mathrm{y}}}{c} = \sqrt{3}p_{\mathrm{tc}} \text{ or } \Delta t \le \frac{\sqrt{3}(k^2-1)\ln k^2}{k^2\ln k^2 - k^2 + 1}\frac{(1-\mu)\sigma_{\mathrm{y}}}{E\alpha} = \sqrt{3}\Delta t_{\mathrm{c}} = \Delta t_{\mathrm{m}}.$$

Therefore, generally speaking, as Δt or p_{t} increases, p_5 decreases and p_6 increases; as Δt or p_{t} decreases, p_5 increases and p_6 decreases. When $\Delta t = \sqrt{3}\Delta t_{\mathrm{c}} = \Delta t_{\mathrm{m}}$, p_6 reaches the maximum,

$$\frac{p_{6\mathrm{m}}}{\sigma_{\mathrm{y}}} = 2\frac{k^2-1}{\sqrt{3}k^2} = 2\frac{p_{\mathrm{e}}}{\sigma_{\mathrm{y}}} = \frac{p_{\mathrm{u}}}{\sigma_{\mathrm{y}}}.$$

In practical applications, for a certain k and Δt, we can take the minimum of p_6 and p_5 as allowable load; or for a certain load p and Δt, we can take the maximum of equations (8.26) and (8.38) as the radius ratio k. However, we can also seek a better way to determine operation parameters.

For equations (8.26) and (8.38), setting $p_6 = p_5$ results in

$$3p_{\mathrm{t}}(ad + bc) + b\sqrt{12\sigma_{\mathrm{y}}^2 - 3c^2p_{\mathrm{t}}^2} - d\sqrt{12\sigma_{\mathrm{y}}^2 - 3a^2p_{\mathrm{t}}^2} = 0 \tag{8.42}$$

The temperature difference to make $p_6 = p_5$ is called the optimal temperature difference, Δt_{e}. Δt_{e} can be obtained from equation (8.42). When $\Delta t < \Delta t_{\mathrm{e}}$, $p_5 > p_6$; When $\Delta t > \Delta t_{\mathrm{e}}$, $p_5 < p_6$. When $\Delta t = \Delta t_{\mathrm{e}}$, $p_5 = p_6$, and p_5 and p_6 are written as p_{d}. Δt_{e} must be smaller than Δt_{c}.

Since $(ad + bc) = \frac{1}{\ln k}$, then, equation (8.42) becomes

$$3p_{\mathrm{t}}/\ln k + b\sqrt{12\sigma_{\mathrm{y}}^2 - 3c^2p_{\mathrm{t}}^2} - d\sqrt{12\sigma_{\mathrm{y}}^2 - 3a^2p_{\mathrm{t}}^2} = 0 \tag{8.43}$$

For equation (8.43), setting $p_t = p_{tc}$ leads to

$$\frac{3(k^2-1)}{k^2\ln k^2 - k^2 + 1} + \frac{3}{k^2-1} - \frac{k^2}{k^2-1}\sqrt{12 - 3\frac{\left(k^2-1-\ln k^2\right)^2}{\left(k^2\ln k^2 - k^2+1\right)^2}} = 0 \tag{8.44}$$

From equation (8.44), we solve the boundary radius ratio $k_i = 2.0542956825\ldots$. When $k \leq k_i$, $\Delta t_e \leq \Delta t_c$; when $k \geq k_i$, $\Delta t_e \geq \Delta t_c$. k_i is not related to material characteristics. When $k = k_i$, $\Delta t_e = \Delta t_c = 169.95861$ °C (elastic modules $E = 1.95 \times 10^5$ MPa, Poisson ratio $\mu = 0.3$, thermal expansion coefficient $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, yield strength $\sigma_y = 350$ MPa).

In summary, when $\Delta t = \Delta t_c$, if $k = k_i$, $\frac{p_5}{\sigma_y} = \frac{p_6}{\sigma_y} = \frac{k^2-1}{k^2} = \sqrt{3}\frac{p_e}{\sigma_y} = \frac{p_{6c}}{\sigma_y} = \frac{p_{5c}}{\sigma_y}$; when $k > k_i$, $p_5 > p_6$; when $k < k_i$, $p_5 < p_6$. Δt should be smaller than Δt_c. As Δt decreases, p_6 decreases, and p_5 increases. Therefore, only when $k \leq k_i$, equation (8.42) will be meaningful, or only when $k \leq k_i$, Δt can be reduced to obtain the optimal temperature difference Δt_e that can make $p_6 = p_5$, and $\Delta t_e < \Delta t_c$. When $k \geq k_i$, in the range of $\Delta t \leq \Delta t_c$, there is always $p_6 \leq p_5$, or there is no Δt_e that is smaller than Δt_c and can make $p_6 = p_5$, in other words, it is impossible for p_5 to be equal to p_6. Therefore, when $k \geq k_i$, in the range of $\Delta t \leq \Delta t_c$, the allowable load is $p = p_6$; if $\Delta t = \Delta t_c$ is taken, from equation (8.38), the allowable load is

$$\frac{p}{\sigma_y} = \frac{p_6}{\sigma_y} = \frac{k^2-1}{k^2} = \sqrt{3}\frac{p_e}{\sigma_y} = \frac{p_{6c}}{\sigma_y} \quad \text{or} \quad k = \sqrt{\frac{\sigma_y}{\sigma_y - p}} \tag{8.45}$$

This load is just two times the initial mechanical yield pressure when based on the maximum shear stress theory (Tresca yield criterion). When $k = k_i$, from equation (8.45), $\frac{p}{\sigma_y} = \frac{p_6}{\sigma_y} = 0.763040517$. That is to say, only when the load $\frac{p}{\sigma_y} = \frac{p_6}{\sigma_y} \geq 0.763041$, or $k \geq k_i$, equation (8.45) can be used to determine the radius ratio k.

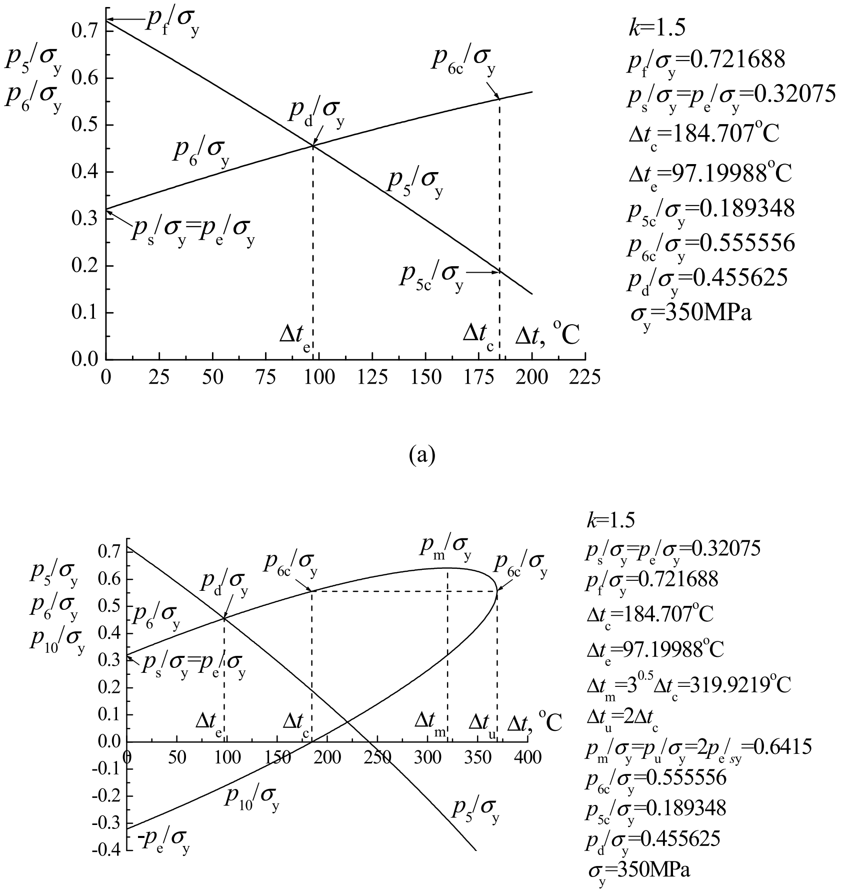
As long as $\Delta t = \Delta t_c$, equation (8.38) becomes equation (8.45).

When $k \leq k_i$, after solving Δt_e by equation (8.42) or equation (8.43), p_6 or p_5 ($p_6 = p_5$) can be found.

As Δt or p_t changes, the curves of $\frac{p_5}{\sigma_y} \sim \Delta t$ and $\frac{p_6}{\sigma_y} \sim \Delta t$ are bound to intersect. The intersection is the optimum design point. As mentioned above, as Δt or p_t increases, p_5 decreases and p_6 increases, and after Δt exceeds $\sqrt{3}\Delta t_c$, p_6 begins to decrease, until $\Delta t = 2\Delta t_c$ p_6 has no definition. Clearly, the shapes and trends of the curves $\frac{p_5}{\sigma_y} \sim \Delta t$ and $\frac{p_6}{\sigma_y} \sim \Delta t$ are affected by the value of k. Therefore, when k increases to a certain extent, the curves of $\frac{p_5}{\sigma_y} \sim \Delta t$ and $\frac{p_6}{\sigma_y} \sim \Delta t$ will have no intersection. As has been noted, when $\Delta t = \sqrt{3}\Delta t_c = \Delta t_m$, p_6 reaches the maximum $\frac{p_6}{\sigma_y} = \frac{p_{6m}}{\sigma_y} = 2\frac{k^2-1}{\sqrt{3}k^2} = 2\frac{p_e}{\sigma_y}$. Substituting $\Delta t = \sqrt{3}\Delta t_c = \Delta t_m$ into $\frac{p_5}{\sigma_y}$ and setting $\frac{p_5}{\sigma_y} = \frac{p_{6m}}{\sigma_y} = 2\frac{k^2-1}{\sqrt{3}k^2}$ obtains $k_m = 3.181854$, which is the maximum radius ratio with $p_5 = p_{6m} = p_u$. When $\Delta t = 2\Delta t_c = \Delta t_u$, $\frac{p_6}{\sigma_y} = \frac{k^2-1}{k^2} = \frac{p_{6c}}{\sigma_y}$. Substituting $\Delta t = 2\Delta t_c = \Delta t_u$ into $\frac{p_5}{\sigma_y}$ and setting $\frac{p_5}{\sigma_y} = \frac{p_{6c}}{\sigma_y} = \frac{k^2-1}{k^2}$ obtains $k_u = 3.613643$, which is the maximum radius ratio with

$p_5 = p_{6\text{c}}$. Curves of $\frac{p_5}{\sigma_\text{y}} \sim \Delta t$ and $\frac{p_6}{\sigma_\text{y}}$ are tangent at $\Delta t = \Delta t_\text{u}$. If $k > k_\text{u}$, the curves of $\frac{p_5}{\sigma_\text{y}} \sim \Delta t$ and $\frac{p_6}{\sigma_\text{y}} \sim \Delta t$ will have no intersection. However, this doesn't have an impact on design.

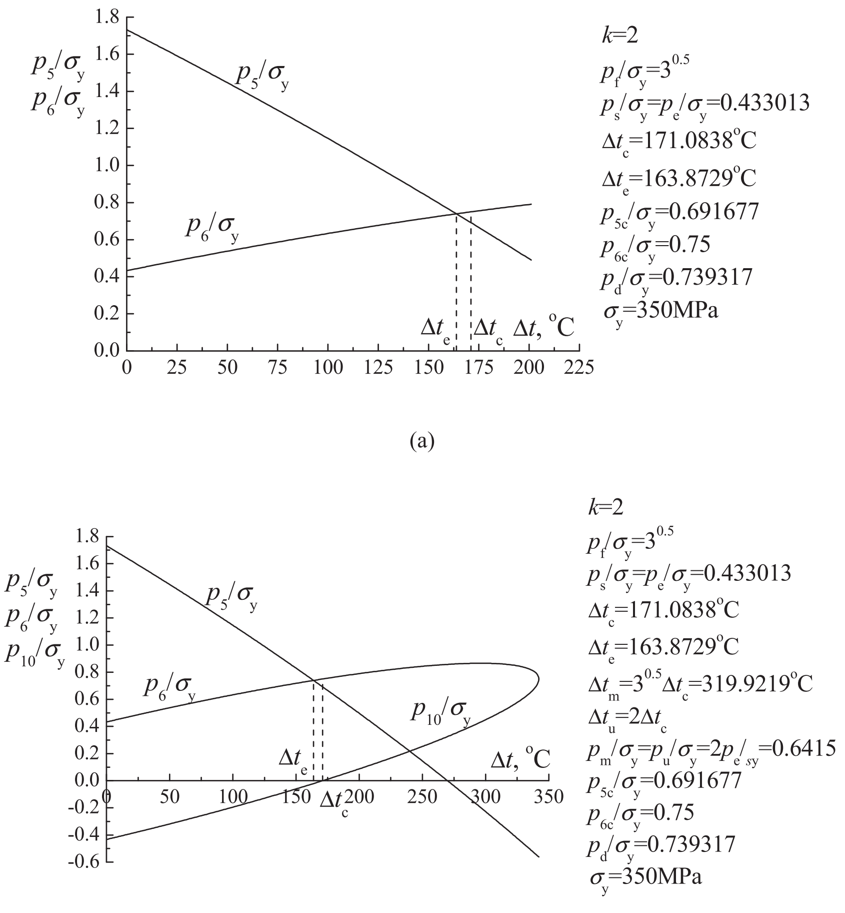
In order to confirm the above analysis intuitively, we demonstrate the main results of the analysis by the graphic method. For Elastic modulus $E = 1.95 \times 10^5$ MPa, Poisson ratio $\mu = 0.3$, thermal expansion coefficient of the materials $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, strength limit $\sigma_\text{y} = 350$ MPa and radius ratio $k = 1.5$, $k = 2$, $k = k_\text{i}$, $k = k_\text{c}$, $k = k_\text{m}$, $k = k_\text{u}$, $k = 3$ and $k = 4$, $\frac{p_5}{\sigma_\text{y}}$ and $\frac{p_6}{\sigma_\text{y}}$ are plotted in figures 8.4–8.11, respectively. For other parameters, the changing trends and morphologies of $\frac{p_5}{\sigma_\text{y}}$ and $\frac{p_6}{\sigma_\text{y}}$ are similar to those of figures 8.4–8.11, because for steel materials, the change of $\frac{E\alpha}{2(1-\mu)}$ is small, popularly $\frac{E\alpha}{2(1-\mu)} \approx 1.5 \sim 1.8$. For example, here $\frac{E\alpha}{2(1-\mu)} = 1.671429\ldots$



(a)

(b)

FIG. 8.4 – The graphs of p_5/σ_y and p_6/σ_y for $k = 1.5$.



(a)

(b)

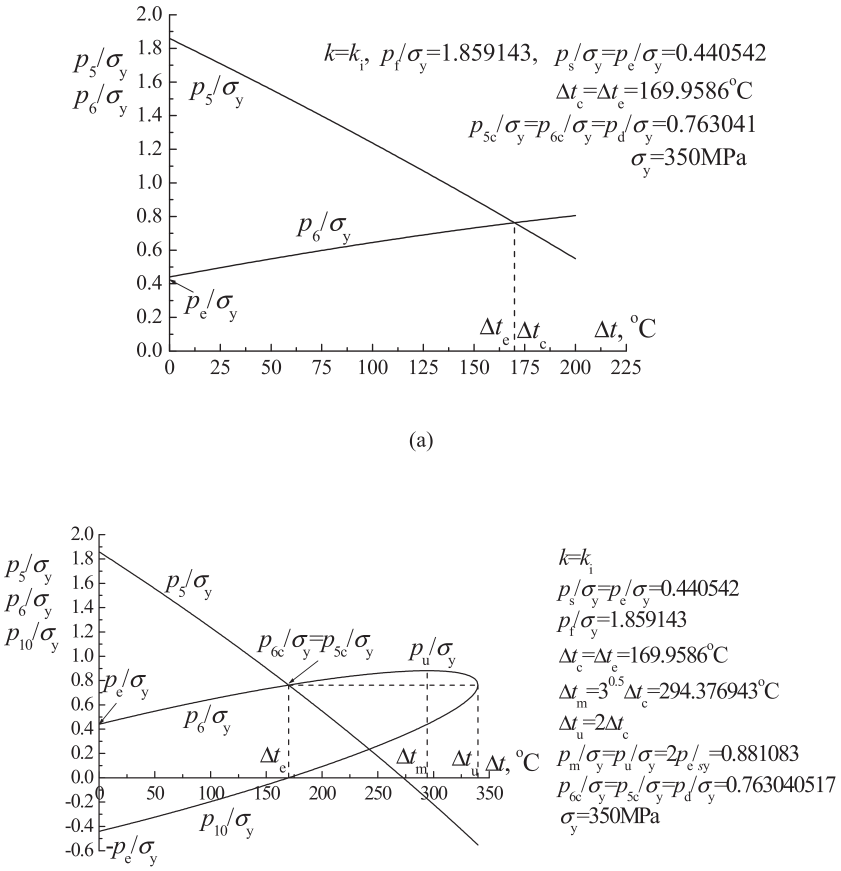
FIG. 8.5 – The graphs of p_5/σ_y and p_6/σ_y for $k = 2$.

The following figures can also be used as a reference for engineering design. In the following figures, the above are partial figures (figure (a)) and the below are corresponding panoramic views (figure (b)).

With the increase of k, the difference $p_5 - p_6$ gets greater and greater. In addition, when $\Delta t \to \Delta t_u = 2\Delta t_c$ (accordingly, $p_t \to 2p_c$), $\frac{p_6}{\sigma_y} \to \frac{p_{6c}}{\sigma_y} = \frac{k^2-1}{k^2}$, $\frac{p_{10}}{\sigma_y} \to \frac{p_{6c}}{\sigma_y} = \frac{k^2-1}{k^2}$.

Before plotting Δt_3, Δt_4, and Δt_0, we analyze them as follows,

$$\frac{d\left(\frac{\Delta t_3}{\sigma_y}\right)}{d\left(\frac{p}{\sigma_y}\right)} = -3\frac{b}{a}\frac{1-\mu}{E\alpha}\left(1+\frac{b\frac{p}{\sigma_y}}{\sqrt{4-3b^2\left(\frac{p}{\sigma_y}\right)^2}}\right) < 0.$$

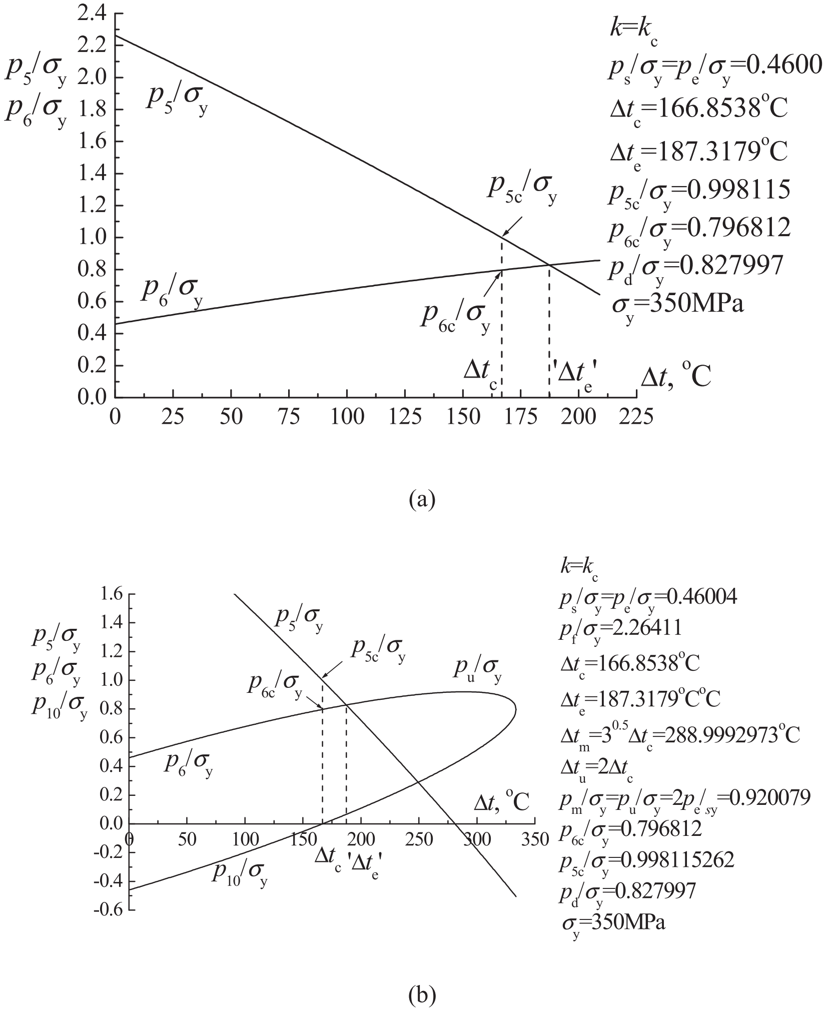


(b)

FIG. 8.6 – The graphs of p_5/σ_y and p_6/σ_y for $k = k_i$.

It is thus evident that Δt_3 monotonically decreases as p increases.

In order to show the changing trend of Δt_3, the correlation between Δt_3 and $\frac{p}{\sigma_y}$ is shown in figures 8.12–8.14. Figure 8.12 is for $k = 1.5 < k_i$, figure 8.13 is for $k = k_i$ and figure 8.14 is for $k = 4 > k_i$. They all confirm the above analysis. Since the greater k is, the smaller $\left|\frac{\mathrm{d}\left(\frac{\Delta t_3}{\sigma_y}\right)}{\mathrm{d}\left(\frac{p}{\sigma_y}\right)}\right|$ is, consequently, the greater k is, the slower Δt_3 decreases.



(a)

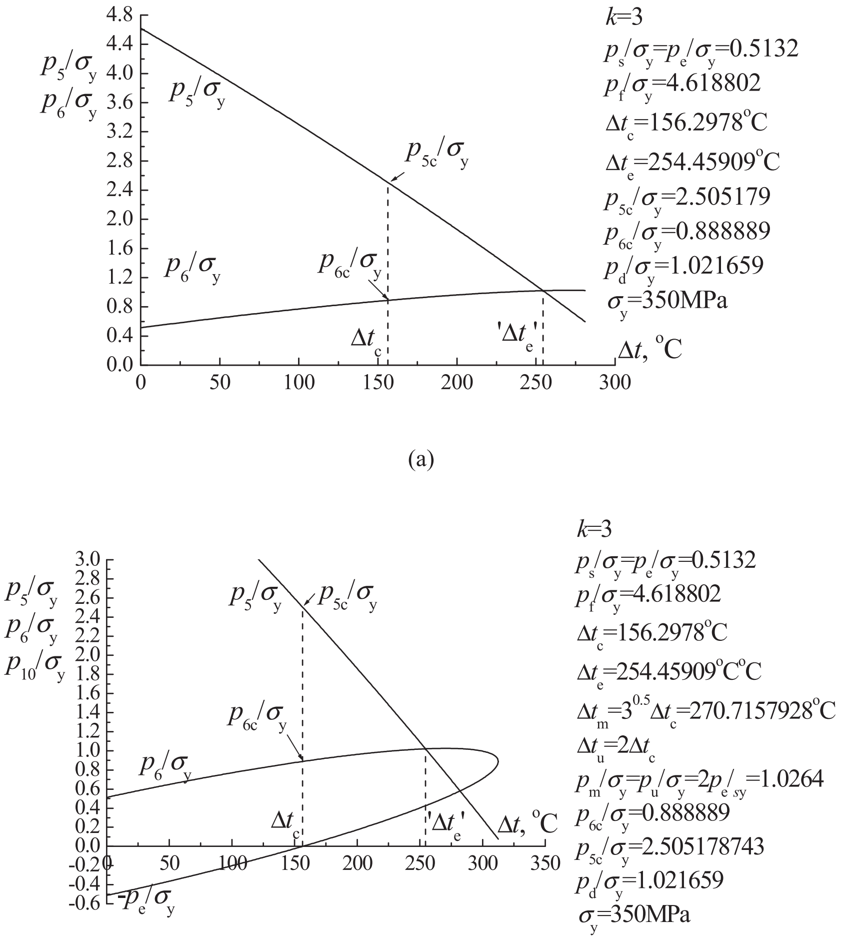
(b)

FIG. 8.7 – The graphs of p_5/σ_y and p_6/σ_y for $k = k_c$.

Clearly, $\Delta t_4 < \Delta t_0$.

$\frac{\mathrm{d}\left(\frac{\Delta t_4}{\sigma_y}\right)}{\mathrm{d}\left(\frac{p}{\sigma_y}\right)} = \frac{1-\mu}{E\alpha c}\left(3d + \frac{3d^2\frac{p}{\sigma_y}}{\sqrt{4-3d^2\left(\frac{p}{\sigma_y}\right)^2}}\right) > 0$. Therefore, Δt_4 monotonically increases as p increases, until $p = p_u = 2p_e = 2\frac{k^2-1}{\sqrt{3}k^2}\sigma_y$, it begins to be imaginary number. When $p = p_u$,

$$\Delta t_4 = \sqrt{3}\Delta t_c = \Delta t_m = \frac{\sqrt{3}(k^2-1)\ln k}{k^2\ln k^2 - k^2 + 1}\frac{2(1-\mu)\sigma_y}{E\alpha}.$$



(a)

(b)

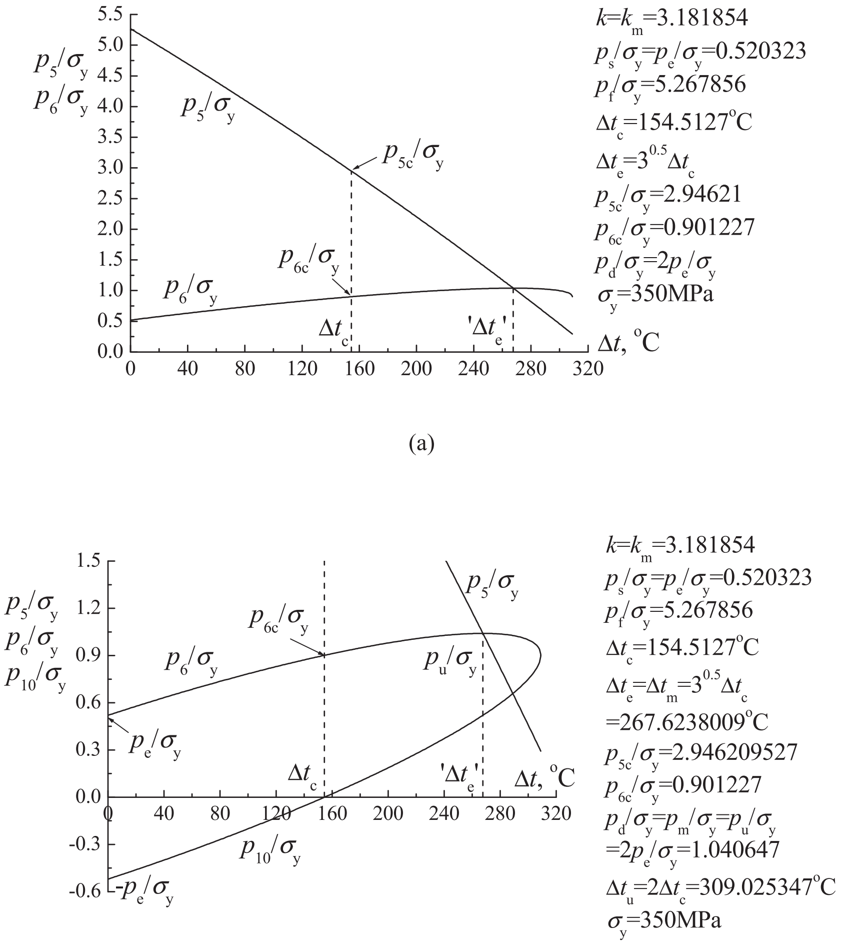
FIG. 8.8 – The graphs of p_5/σ_y and p_6/σ_y for $k = 3$.

$\frac{\mathrm{d}\left(\frac{\Delta t_0}{\sigma_y}\right)}{\mathrm{d}\left(\frac{p}{\sigma_y}\right)} = \frac{1-\mu}{E\alpha c}\left(3d - \frac{3d^2\frac{p}{\sigma_y}}{\sqrt{4-3d^2\left(\frac{p}{\sigma_y}\right)^2}}\right)$. Setting $\frac{\mathrm{d}\left(\frac{\Delta t_0}{\sigma_y}\right)}{\mathrm{d}\left(\frac{p}{\sigma_y}\right)} \geq 0$ gives

$$\frac{p}{\sigma_y} \leq \frac{k^2-1}{k^2} = \sqrt{3}\,\frac{p_e}{\sigma_y} = \frac{p_{6c}}{\sigma_y}$$

Therefore, when $p = p_{6c}$, Δt_0 gets the maximum, which is

$$\Delta t_0 = 2\Delta t_c = \Delta t_u = \frac{(k^2-1)\ln k^2}{k^2\ln k^2 - k^2 + 1}\frac{2(1-\mu)\sigma_y}{E\alpha}$$



(a)

(b)

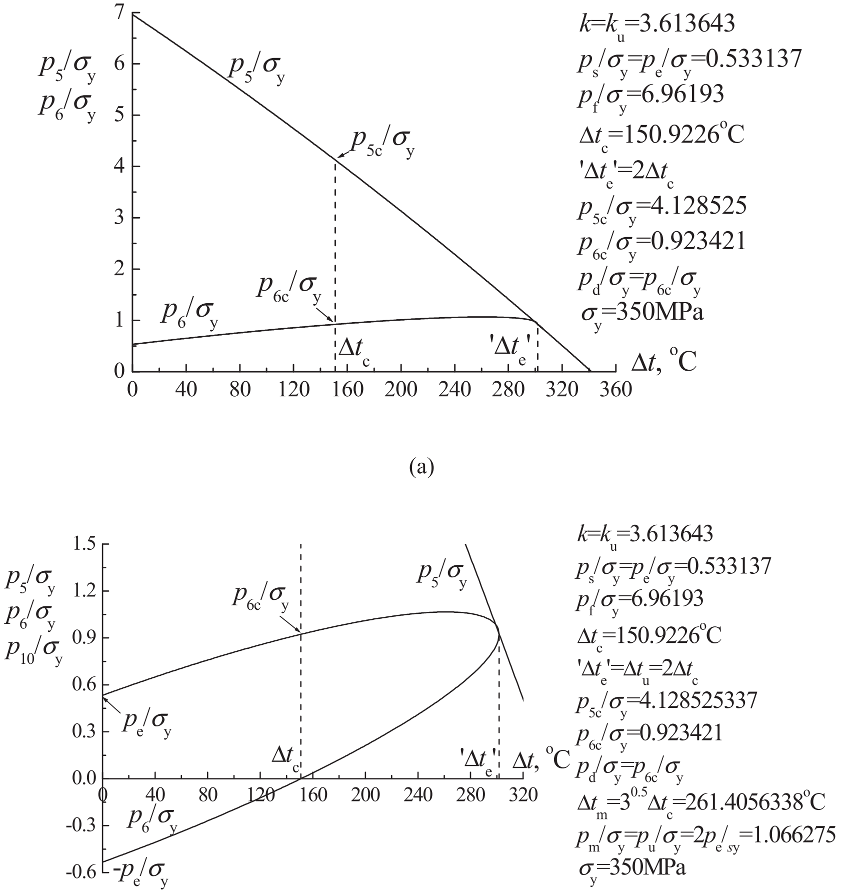
FIG. 8.9 – The graphs of p_5/σ_y and p_6/σ_y for $k = k_m$.

After $p = p_{6c}$ and $\Delta t_0 = 2\Delta t_c = \Delta t_u$, Δt_0 decreases as p increases, until $p = p_u = 2p_e = 2\frac{k^2-1}{\sqrt{3}k^2}\sigma_y$, then, it begins to be an imaginary number. When $p = p_u$, $\Delta t_0 = \sqrt{3}\Delta t_c = \Delta t_m = \frac{\sqrt{3}(k^2-1)\ln k}{k^2\ln k^2-k^2+1}\frac{2(1-\mu)\sigma_y}{E\alpha}$.

It is shown that when $p = p_u$, $\Delta t_0 = \Delta t_4 = \sqrt{3}\Delta t_c = \Delta t_m$. In consideration of that $\Delta t_0 > \Delta t_4$, thus, curves $\Delta t_0 \sim \frac{p}{\sigma_y}$ and $\Delta t_4 \sim \frac{p}{\sigma_y}$ are actually the same curve within $0 \leq p \leq p_u$.

When $\frac{p}{\sigma_y} = 0$,

$$\Delta t_4 = -\Delta t_c \text{ and } \Delta t_0 = -\Delta t_4 = \Delta t_c.$$



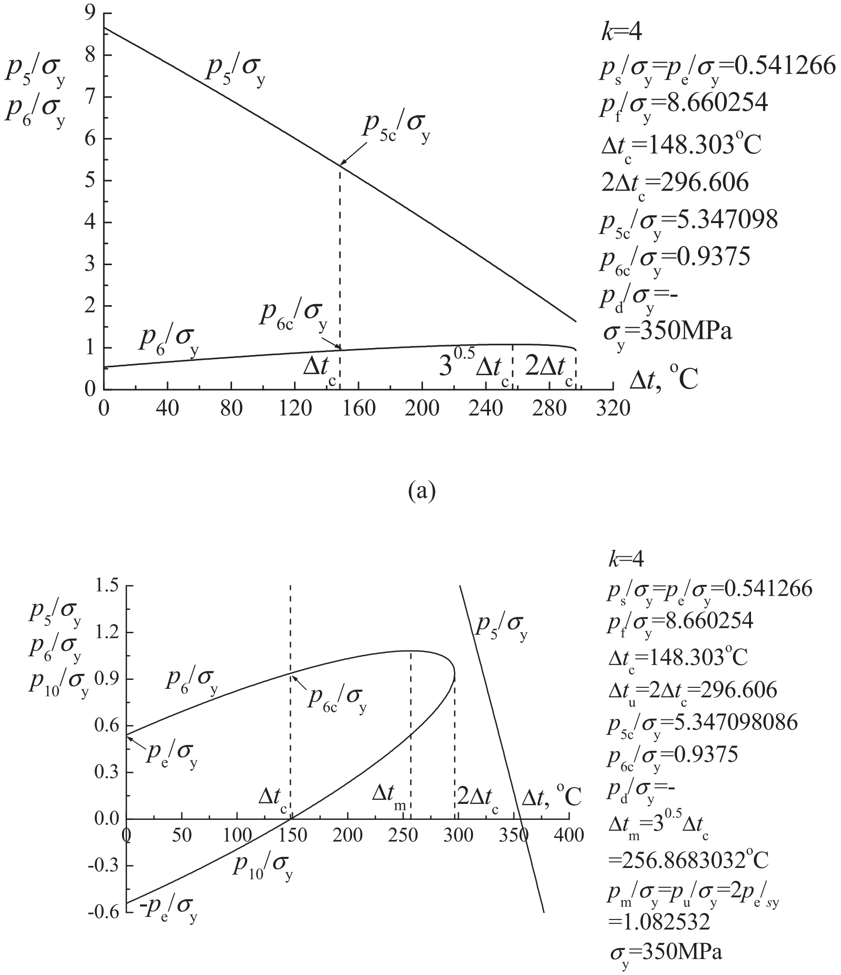
(a)

(b)

FIG. 8.10 – The graphs of p_5/σ_y and p_6/σ_y for $k = k_u$.

As $\frac{p}{\sigma_y}$ changes from 0 to $\frac{p_u}{\sigma_y}$, Δt_4 monotonically increases from $\Delta t_4 = -\Delta t_c$ to $\Delta t = \sqrt{3}\Delta t_c = \Delta t_m$. As $\frac{p}{\sigma_y}$ changes from 0 to $\frac{p_u}{\sigma_y}$, Δt_0 ($>\Delta t_4$) monotonically increases from $\Delta t_0 = \Delta t_c$ until $\frac{p}{\sigma_y} = \frac{p_{6c}}{\sigma_y}$, where it reaches the maximum $\Delta t_0 = 2\Delta t_c$ ($=\Delta t_u$), and then Δt_0 ($>\Delta t_4$) start to decrease monotonically from $\Delta t_0 = 2\Delta t_c$ (where $\frac{p}{\sigma_y} = \frac{p_{6c}}{\sigma_y}$) until $\Delta t_0 = \Delta t_4 = \sqrt{3}\Delta t_c = \Delta t_m$ (where $\frac{p}{\sigma_y} = \frac{p_u}{\sigma_y}$). Within $0 \leq p \leq p_u$, there always is that $\Delta t_0 > \Delta t_c$.

For example, when $k = 2 < k_i$, $k = k_i$ and $k = 4 > k_i$, $\Delta t_0 \sim \frac{p}{\sigma_y}$ and $\Delta t_4 \sim \frac{p}{\sigma_y}$ are illustrated in figures 8.15–8.17, respectively. They all confirm the above analysis.



(a)

(b)

FIG. 8.11 – The graphs of p_5/σ_y and p_6/σ_y for $k = 4$.

Ordinarily, curves $\Delta t_0 \sim \frac{p}{\sigma_y}$ and $\Delta t_3 \sim \frac{p}{\sigma_y}$ have a crossover point A and curves $\Delta t_4 \sim \frac{p}{\sigma_y}$ and $\Delta t_3 \sim \frac{p}{\sigma_y}$ have a crossover point B. Setting $\Delta t_0 = \Delta t_3$ results in

$$\left(3d\frac{p}{\sigma_y}+\sqrt{4-3d^2\left(\frac{p}{\sigma_y}\right)^2}\right)\frac{1}{c}=\left(-3b\frac{p}{\sigma_y}+\sqrt{4-3b^2\left(\frac{p}{\sigma_y}\right)^2}\right)\frac{1}{a} \tag{8.46}$$

Using the approximation method, for a certain k, we can find out the coordinates of intersection point A, which is A $(\frac{p}{\sigma_y}$, $\Delta t_0 = \Delta t_3)$. Setting $\Delta t_4 = \Delta t_3$ results in

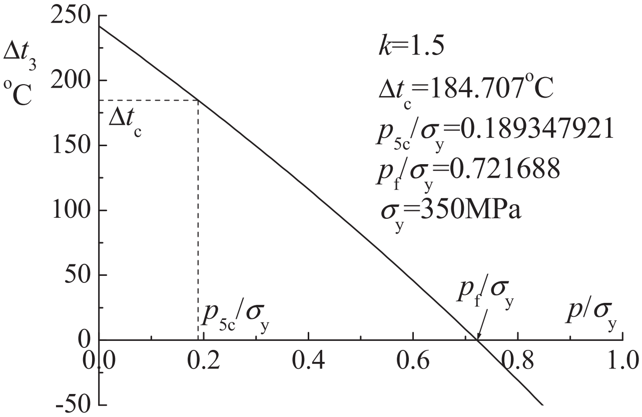


FIG. 8.12 – The correlation between Δt_3 and p/σ_y for $k = 1.5 < k_i$.

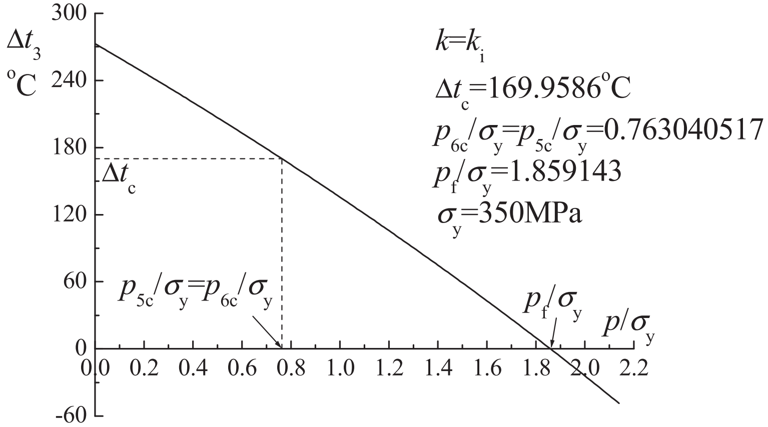


FIG. 8.13 – The correlation between Δt_3 and p/σ_y for $k = k_i$.

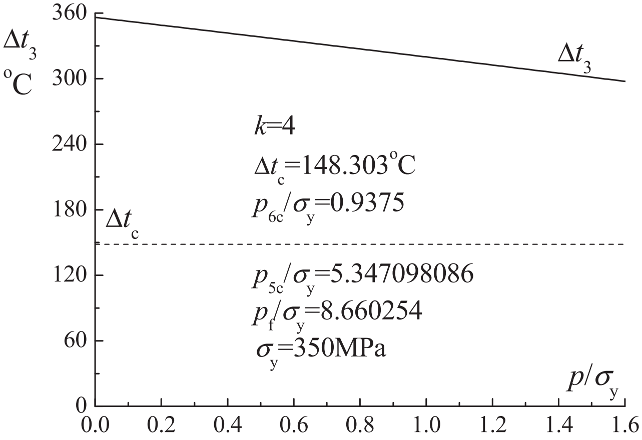


FIG. 8.14 – The correlation between Δt_3 and p/σ_y for $k = 4 > k_i$.

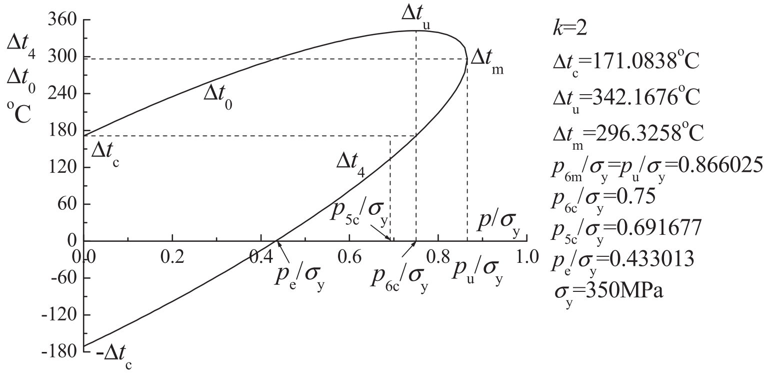


FIG. 8.15 – The graph of $\Delta t_0 \sim \frac{p}{\sigma_y}$ and $\Delta t_4 \sim p/\sigma_y$ with $k = 2 < k_i$.

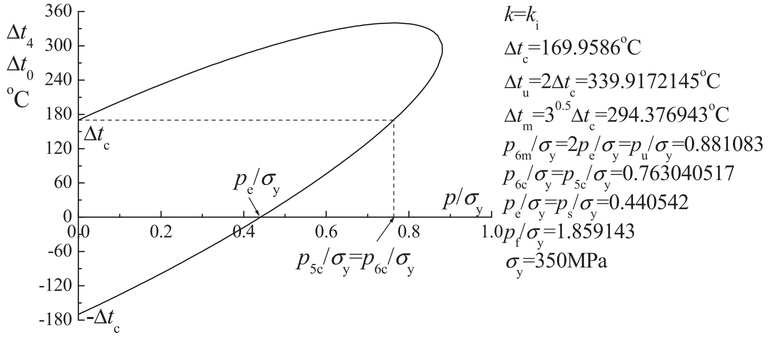


FIG. 8.16 – The graph of $\Delta t_0 \sim p/\sigma_y$ and $\Delta t_4 \sim p/\sigma_y$ with $k = k_i$.

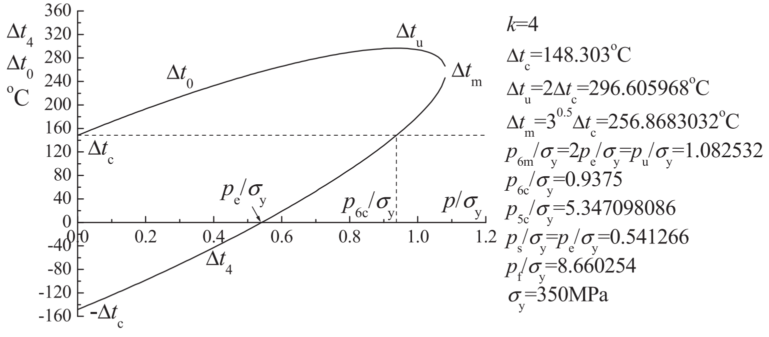


FIG. 8.17 – The graph of $\Delta t_0 \sim p/\sigma_y$ and $\Delta t_4 \sim p/\sigma_y$ with $k = 4 > k_i$.

$$\left(3d\frac{p}{\sigma_{\mathrm{y}}}-\sqrt{4-3d^2\left(\frac{p}{\sigma_{\mathrm{y}}}\right)^2}\right)\frac{1}{c}=\left(-3b\frac{p}{\sigma_{\mathrm{y}}}+\sqrt{4-3b^2\left(\frac{p}{\sigma_{\mathrm{y}}}\right)^2}\right)\frac{1}{a} \tag{8.47}$$

Similarly, using the approximation method, for a certain k, we can find out the coordinates of intersection point B, which is $B\ (\frac{p}{\sigma_{\mathrm{y}}},\ \Delta t_4=\Delta t_3)$.

Obviously, the position of the intersection depends on the value of the radius ratio k. In view of the shape of the curves $\Delta t_0\sim\frac{p}{\sigma_{\mathrm{y}}}$, $\Delta t_3\sim\frac{p}{\sigma_{\mathrm{y}}}$ and $\Delta t_4\sim\frac{p}{\sigma_{\mathrm{y}}}$ as shown typically in figures 8.12–8.17, when curve $\Delta t_3\sim\frac{p}{\sigma_{\mathrm{y}}}$ and curve $\Delta t_0\sim\frac{p}{\sigma_{\mathrm{y}}}$ or curve $\Delta t_4\sim\frac{p}{\sigma_{\mathrm{y}}}$ are tangent, the intersection is about to disappear. The point of tangency must be the highest point of curves $\Delta t_0\sim\frac{p}{\sigma_{\mathrm{y}}}$ in the light of the shapes of the curves $\Delta t_0\sim\frac{p}{\sigma_{\mathrm{y}}}$ and $\Delta t_3\sim\frac{p}{\sigma_{\mathrm{y}}}$ as shown typically in figures 8.12–8.17. Therefore, the point of tangency of the curves $\Delta t_0\sim\frac{p}{\sigma_{\mathrm{y}}}$ and $\Delta t_3\sim\frac{p}{\sigma_{\mathrm{y}}}$ can be obtained as follows.

Substituting $\frac{p}{\sigma_{\mathrm{y}}}=\frac{p_{6\mathrm{c}}}{\sigma_{\mathrm{y}}}=\frac{k^2-1}{k^2}$ into equation (8.29) and then setting $\Delta t=\Delta t_{\mathrm{u}}=2\Delta t_{\mathrm{c}}$ to obtain

$$\frac{\sqrt{4k^4-3}-3}{k^2(k^2-1-\ln k^2)}=\frac{4}{k^2\ln k^2-k^2+1} \tag{8.48}$$

The solution of equation (8.38) is

$$k_{\mathrm{u}}=3.613643$$

It is the same as above.

In consideration of the shapes of the curves $\Delta t_0\sim\frac{p}{\sigma_{\mathrm{y}}}$, $\Delta t_4\sim\frac{p}{\sigma_{\mathrm{y}}}$ and $\Delta t_3\sim\frac{p}{\sigma_{\mathrm{y}}}$, intersection point A is certainly above Δt_{c} because $\Delta t_0>\Delta t_{\mathrm{c}}$. As for intersection point B, its position depends on the value of radius ratio k. As mentioned above, when $\Delta t=\Delta t_{\mathrm{c}}$ and $p_5=p_6$, there is $k=k_{\mathrm{i}}=2.0542956825\ldots$. Therefore, when $k=k_{\mathrm{i}}$, the coordinates of intersection point B is $B\ (\frac{p_5}{\sigma_{\mathrm{y}}}=\frac{p_6}{\sigma_{\mathrm{y}}},\ \Delta t_4=\Delta t_3=\Delta t_{\mathrm{c}})$. When $k<k_{\mathrm{i}}$, the ordinate of intersection point B is $\Delta t_4=\Delta t_3<\Delta t_{\mathrm{c}}$; When $k>k_{\mathrm{i}}$, the ordinate of intersection point B is $\Delta t_4=\Delta t_3>\Delta t_{\mathrm{c}}$.

Now, we can plot curves Δt_0, Δt_3, and Δt_4 to intuitively observe their distributions as a function of $\frac{p}{\sigma_{\mathrm{y}}}$. The following figures may serve as a reference for engineering design.

For the above parameters of material characteristics, or Elastic modulus $E=1.95\times10^5$ MPa, Poisson ratio $\mu=0.3$, thermal expansion coefficient of the materials $\alpha=1.2\times10^{-5}$ °C^{-1} and strength limit $\sigma_{\mathrm{y}}=350$ MPa and for radius ratio $k=1.5$, $k=2$, $k=k_{\mathrm{i}}$, $k=k_{\mathrm{c}}$, $k=3$, $k=k_{\mathrm{u}}$ and $k=4$, $\Delta t_3\sim\frac{p}{\sigma_{\mathrm{y}}}$, $\Delta t_0\sim\frac{p}{\sigma_{\mathrm{y}}}$ and $\Delta t_4\sim\frac{p}{\sigma_{\mathrm{y}}}$ are plotted in figures 8.18–8.24, respectively. For other parameters, the changing trends and morphologies of $\Delta t_3\sim\frac{p}{\sigma_{\mathrm{y}}}$, $\Delta t_0\sim\frac{p}{\sigma_{\mathrm{y}}}$ and $\Delta t_4\sim\frac{p}{\sigma_{\mathrm{y}}}$ are similar to those of figures 8.18–8.24, the reason is also that the change of $\frac{E\alpha}{2(1-\mu)}$ is small for steel materials.

In fact, if the project application is not considered and only the mathematical relationship is concerned within the definition domain and the co-domain, the

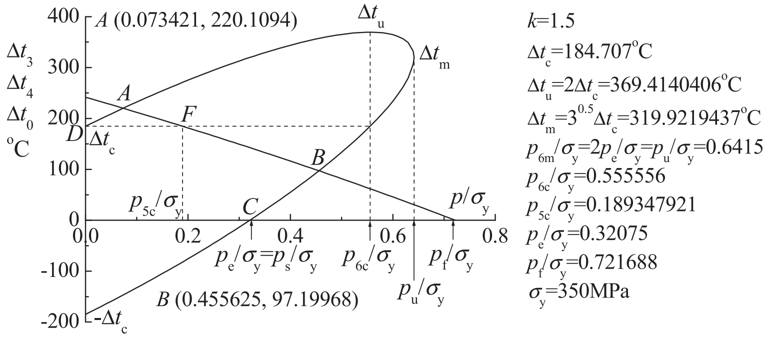


FIG. 8.18 – The graphs of $\Delta t_3 \sim p/\sigma_y$, $\Delta t_0 \sim p/\sigma_y$ and $\Delta t_4 \sim p/\sigma_y$ for $k = 1.5$.

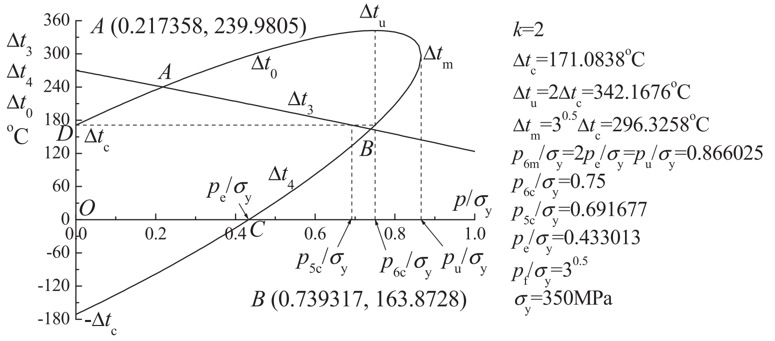


FIG. 8.19 – The graphs of $\Delta t_3 \sim p/\sigma_y$, $\Delta t_0 \sim p/\sigma_y$ and $\Delta t_4 \sim p/\sigma_y$ for $k = 2$.

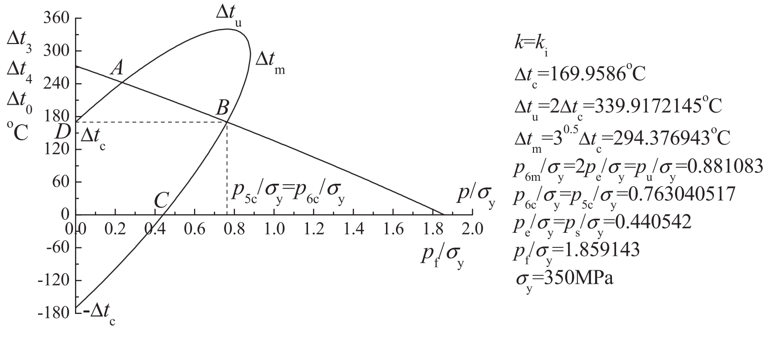


FIG. 8.20 – The graphs of $\Delta t_3 \sim p/\sigma_y$, $\Delta t_0 \sim p/\sigma_y$ and $\Delta t_4 \sim p/\sigma_y$ for $k = k_i$.

graphs of $\Delta t_0 \sim \frac{p}{\sigma_y}$ plus $\Delta t_4 \sim \frac{p}{\sigma_y}$ and $\frac{p_6}{\sigma_y} \sim \Delta t$ plus $\frac{p_{10}}{\sigma_y} \sim \Delta t$ are complete ellipses, because equation (8.40) plus equation (8.41) is directly a complete elliptic equation. Figures 8.25 and 8.26 are the graphs of $\Delta t_0 \sim \frac{p}{\sigma_y}$ plus $\Delta t_4 \sim \frac{p}{\sigma_y}$ for $k = 2$ and $k = 4$,

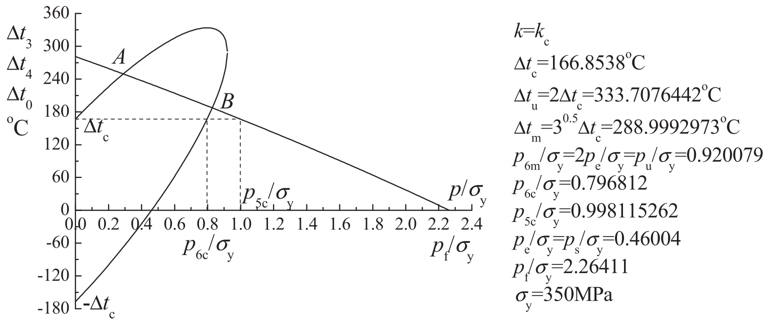


FIG. 8.21 – The graphs of $\Delta t_3 \sim p/\sigma_{\mathrm{y}}$, $\Delta t_0 \sim p/\sigma_{\mathrm{y}}$ and $\Delta t_4 \sim p/\sigma_{\mathrm{y}}$ for $k = k_{\mathrm{c}}$.

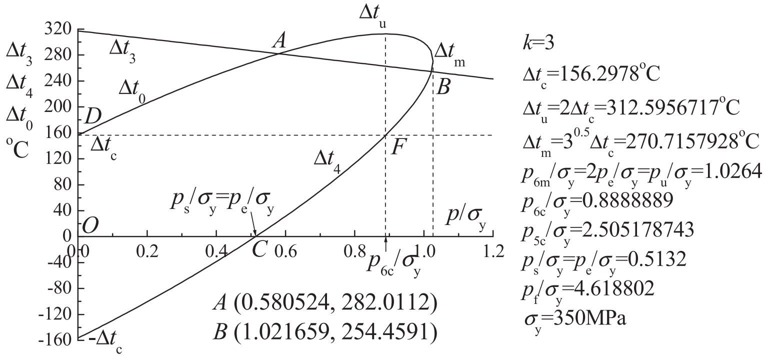


FIG. 8.22 – The graphs of $\Delta t_3 \sim p/\sigma_{\mathrm{y}}$, $\Delta t_0 \sim p/\sigma_{\mathrm{y}}$ and $\Delta t_4 \sim p/\sigma_{\mathrm{y}}$ for $k = 3$.

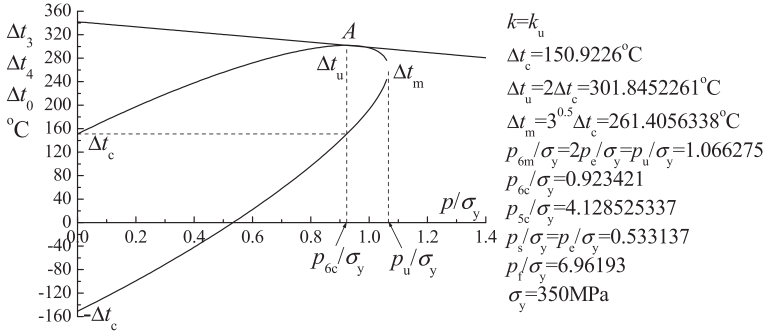


FIG. 8.23 – The graphs of $\Delta t_3 \sim p/\sigma_{\mathrm{y}}$, $\Delta t_0 \sim p/\sigma_{\mathrm{y}}$ and $\Delta t_4 \sim p/\sigma_{\mathrm{y}}$ for $k = k_{\mathrm{u}}$.

respectively. Figures 8.27 and 8.28 are the graphs of $\frac{p_6}{\sigma_{\mathrm{y}}} \sim \Delta t$ plus $\frac{p_{10}}{\sigma_{\mathrm{y}}} \sim \Delta t$ for $k = 2$ and $k = 4$, respectively. The left side of the vertical axis and the lower side of the lateral axis correspond to "outer heating".

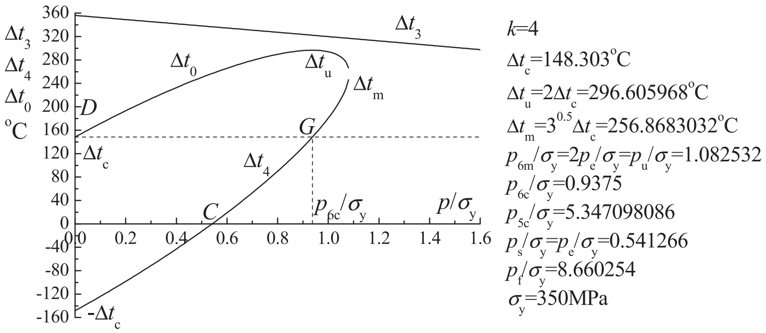


FIG. 8.24 – The graphs of $\Delta t_3 \sim p/\sigma_y$, $\Delta t_0 \sim p/\sigma_y$ and $\Delta t_4 \sim p/\sigma_y$ for $k = 4$.

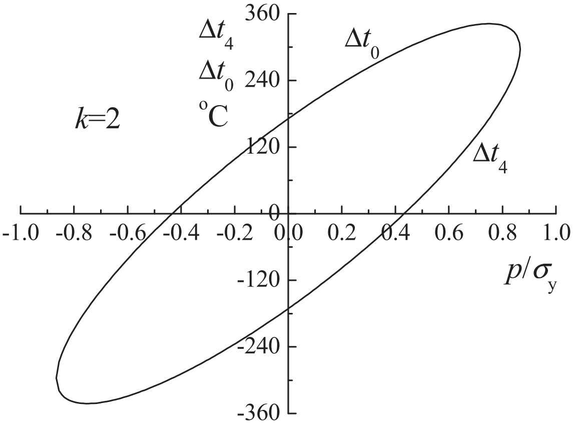


FIG. 8.25 – The graphs of $\Delta t_0 \sim p/\sigma_y$ and $\Delta t_4 \sim p/\sigma_y$ for $k = 2$.

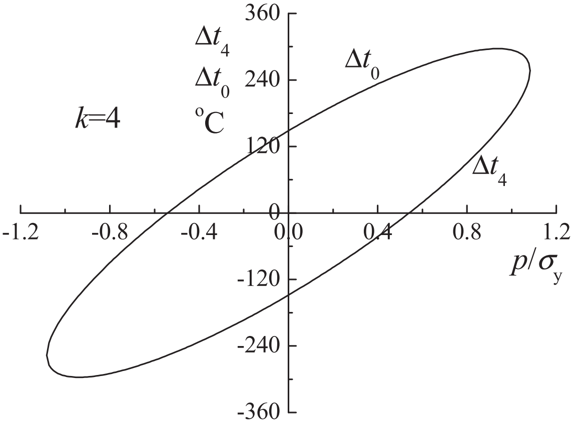


FIG. 8.26 – The graphs of $\Delta t_0 \sim p/\sigma_y$ and $\Delta t_4 \sim p/\sigma_y$ for $k = 4$.

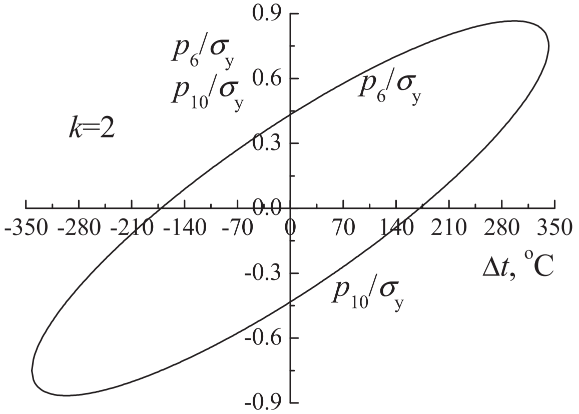


FIG. 8.27 – The graphs of $p_6/\sigma_\mathrm{y} \sim \Delta t$ plus $p_{10}/\sigma_\mathrm{y} \sim \Delta t$ for $k = 2$.

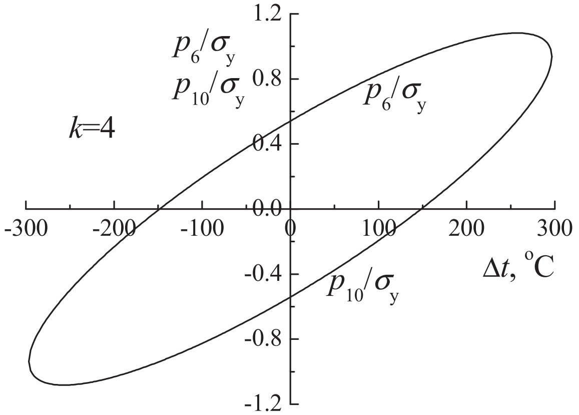


FIG. 8.28 – The graphs of $p_6/\sigma_\mathrm{y} \sim \Delta t$ plus $p_{10}/\sigma_\mathrm{y} \sim \Delta t$ for $k = 4$.

For figures 8.25 and 8.26, the definition domain of $\frac{p}{\sigma_\mathrm{y}}$ is $\frac{p}{\sigma_\mathrm{y}} = \frac{p_\mathrm{u}}{\sigma_\mathrm{y}} = 2\frac{p_\mathrm{e}}{\sigma_\mathrm{y}} = 2\frac{k^2-1}{\sqrt{3}k^2}$, and the codomain of Δt_0 (Δt_4) is $\Delta t = \Delta t_\mathrm{u} = 2\Delta t_\mathrm{c} = \frac{(k^2-1)\ln k^2}{k^2 \ln k^2 - k^2 + 1}\frac{2(1-\mu)\sigma_\mathrm{y}}{E\alpha}$.

For figures 8.27 and 8.28, the definition domain of Δt_0 (Δt_4) is $\Delta t = \Delta t_\mathrm{u} = 2\Delta t_\mathrm{c} = \frac{(k^2-1)\ln k^2}{k^2 \ln k^2 - k^2 + 1}\frac{2(1-\mu)\sigma_\mathrm{y}}{E\alpha}$, and the codomain of $\frac{p}{\sigma_\mathrm{y}}$ is $\frac{p}{\sigma_\mathrm{y}} = \frac{p_\mathrm{u}}{\sigma_\mathrm{y}} = 2\frac{p_\mathrm{e}}{\sigma_\mathrm{y}} = 2\frac{k^2-1}{\sqrt{3}k^2}$.

8.3 Examples

Example 1. Elastic modulus $E = 1.95 \times 10^5$ MPa, Poisson's ratio $\mu = 0.3$, thermal expansion coefficient $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, strength limit $\sigma_\mathrm{y} = 300$ MPa, $k = 3 > k_\mathrm{i}$.

If the temperature difference is not exerted, the load-bearing capacity of the pressure vessel with $k = 3$ is only $p = p_\mathrm{e} = \frac{k^2-1}{\sqrt{3}k^2}\sigma_\mathrm{y} = 153.9601$ MPa.

From equation (8.12), $\Delta t_{\rm c} = 133.9696$ °C; when $\Delta t = \Delta t_{\rm c} = 133.9696$ °C, from equation (8.38) or equation (8.45), $p = p_6 = 266.6667$ MPa; from equation (8.26), $p_5 = 751.5536$ MPa $> p_6$, then, take $p_6 = 266.6667$ MPa as the load; from equation (8.27), $p_{\rm t3} = 376.5421$ MPa $(p = p_6)$; from equation (8.39), $p_{\rm t4} = 223.9206$ MPa ($p = p_6$ and $\Delta t_4 = \Delta t_{\rm c}$), $p_{\rm t0} = 447.8411$ MPa ($p = p_6$ and $\Delta t_4 = \Delta t_{\rm c}$). From equation (8.42), $\Delta t_{\rm e} = 218.1078$ °C $> \Delta t_{\rm c}$. Therefore, for a pressure vessel with $k = 3$, the optimum operation parameters are $p = p_6 = 266.6667$ MPa, $\Delta t = \Delta t_{\rm c} = 133.9696$ °C, under these parameters, the total stresses along with $\sigma_{\rm e}^{\rm Td}$ and $\sigma_{\rm e}^{\rm Ts}$ are shown in figure 8.29. The meaning of each curve and the main parameters are marked in the figure.

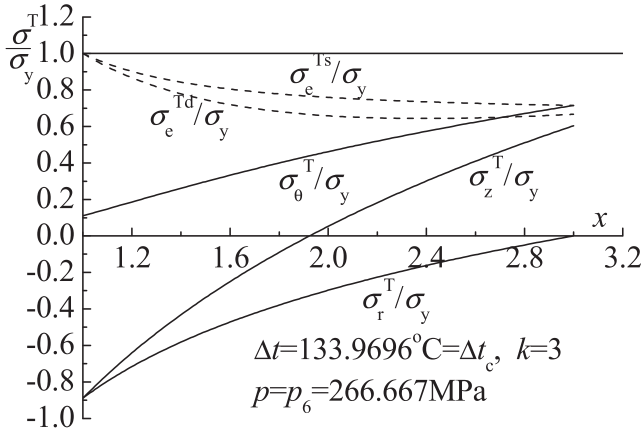


FIG. 8.29 – The distribution of total stresses with $k = 3$, $\Delta t = \Delta t_{\rm c}$ and $p = p_6$.

If based on the maximum shear stress theory (Tresca yield criterion), the optimum operation parameters are the same.

Example 2. $E = 1.95 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, $\sigma_{\rm y} = 300$ MPa, $k = 1.6 < k_{\rm i}$.

If the temperature difference is not exerted, the load-bearing capacity of the pressure vessel with $k = 1.6$ is $p = p_{\rm e} = \frac{k^2-1}{\sqrt{3}k^2}\sigma_{\rm y} = 105.5468$ MPa.

From equation (8.12), $\Delta t_{\rm c} = 155.4798$ °C; From equation (8.42), $\Delta t_{\rm e} = 96.31247$ °C $< \Delta t_{\rm c}$. Substituting $\Delta t = \Delta t_{\rm e} = 96.31247$ °C into equation (8.26) or equation (8.37) obtains $p = p_5 = p_6 = 156.9785$ MPa, $\sigma_{\rm y}\ln k = 141.0011$ MPa $< p < \frac{2}{\sqrt{3}}\sigma_{\rm y}\ln k = 162.814$ MPa.

When $\Delta t = \Delta t_{\rm e}$, $p_{\rm t} = 160.9794$ MPa. Then, from equations (8.20) to (8.22), $\sigma_{\rm ro}^{\rm T} - \sigma_{\rm \theta o}^{\rm T} = 337.38$ MPa; $\sigma_{\rm \theta o}^{\rm T} - \sigma_{\rm zo}^{\rm T} = 100.6$ MPa; $\sigma_{\rm zo}^{\rm T} - \sigma_{\rm ro}^{\rm T} = 236.8$ MPa; $\sigma_{\rm eo}^{\rm d} = \sqrt{\frac{1}{2}\left[(\sigma_{\rm \theta o} - \sigma_{\rm zo})^2 + (\sigma_{\rm zo} - \sigma_{\rm ro})^2 + (\sigma_{\rm ro} - \sigma_{\rm \theta o})^2\right]} = 300$ MPa. From equations (8.30) to (8.32), $\sigma_{\rm ri}^{\rm T} - \sigma_{\rm \theta i}^{\rm T} = 329.4$ MPa; $\sigma_{\rm \theta i}^{\rm T} - \sigma_{\rm zi}^{\rm T} = 257.6$ MPa; $\sigma_{\rm zi}^{\rm T} - \sigma_{\rm ri}^{\rm T} = 71.8$ MPa; $\sigma_{\rm ei}^{\rm Td} = \sqrt{\frac{1}{2}\left[(\sigma_{\rm \theta i}^{\rm T} - \sigma_{\rm zi}^{\rm T})^2 + (\sigma_{\rm zi}^{\rm T} - \sigma_{\rm ri}^{\rm T})^2 + (\sigma_{\rm ri}^{\rm T} - \sigma_{\rm \theta i}^{\rm T})^2\right]} = 300$ MPa.

From equations (8.28) and (8.40), $\Delta t_3 = \Delta t_4 = 96.31247$ °C $= \Delta t_e$, $\Delta t_0 = 304.2126$ °C.

If based on the maximum shear stress theory (Tresca criterion), this is the case of $k < k_c$, from equation (8.40) of the last chapter, $\Delta t_x = 84.35963$ °C $< \Delta t_c = 155.4798$ °C obtained from equation (8.27). When $\Delta t = \Delta t_x = 84.35963$ °C, equations (8.44) and (8.45) both lead to $\frac{p}{\sigma_y} = 0.47004 = \ln k = \ln 1.6$, or $p = 141.0011$ MPa $< p_5$ or p_6.

Substituting $\Delta t_c = 155.4798$ °C into equation (8.26) obtains $p_5 = 80.0235$ MPa; substituting $\Delta t_c = 155.4798$ °C into equation (8.38) obtains $p_6 = 182.8125$ MPa. Substituting $p_5 = 80.0235$ MPa into equation (8.28) obtains $\Delta t_3 = \Delta t_c$; Substituting $p_6 = 182.8125$ MPa into equation (8.40) obtains $\Delta t_4 = \Delta t_c$ and $\Delta t_0 = 310.9597$ °C.

When $\Delta t = \Delta t_e = 96.31247$ °C, $p = p_5 = p_6 = 156.9785$ MPa, the total stresses along with σ_e^{Td} and σ_e^{Ts} are shown in figure 8.30. The meaning of each curve and the main parameters are marked in the figure. According to the maximum shear stress theory (Tresca criterion), $\sigma_e^{\mathrm{Ts}} > 1$.

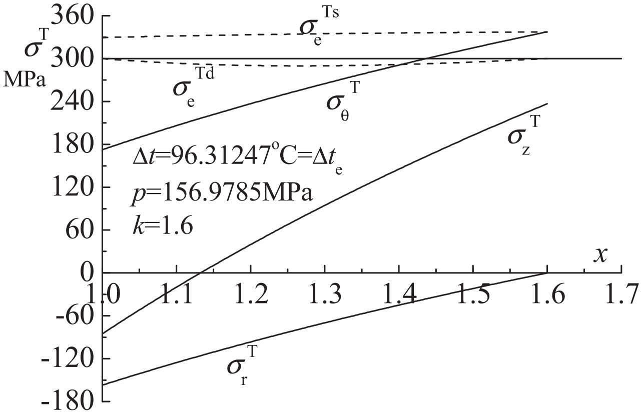


FIG. 8.30 – The distribution of total stresses with $k = 1.6$, $\Delta t = \Delta t_e$ and $p = p_5 = p_6$.

When $\Delta t = \Delta t_c = 155.4798$ °C, $p = p_5 = 80.0235$ MPa, the total stresses along with σ_e^{Td} and σ_e^{Ts} are shown in figure 8.31. The meaning of each curve and the main parameters are marked in the figure.

When p and k are given, the key is to determine the appropriate temperature difference Δt among Δt_0, Δt_4, Δt_3 and Δt_c by equations (8.28), (8.40), and (8.12). It is better to select a smaller temperature difference among Δt_0, Δt_4, Δt_3, and Δt_c. See example 2.

From figures 8.30 and 8.31, it is seen that $\sigma_{\theta o}^{\mathrm{T}} > \sigma_y$. This is because the pressure is too great. If it is necessary, the pressure should be controlled to control $\sigma_{\theta o}^{\mathrm{T}}$. From equation (8.15), we know that with x increasing, $\sigma_\theta^{\mathrm{T}}$ increases, or $\sigma_{\theta o}^{\mathrm{T}}$ is the maximum along x.

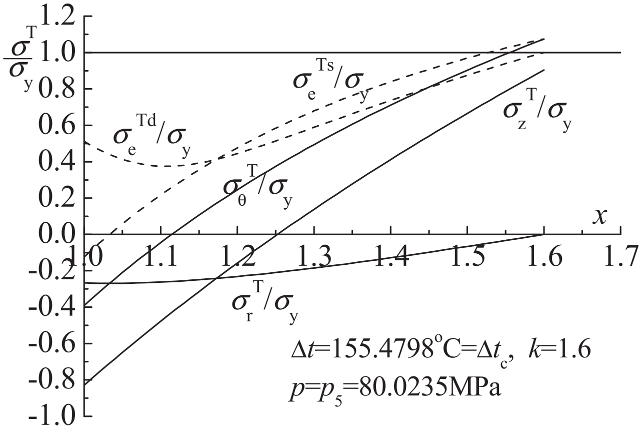


FIG. 8.31 – The distribution of total stresses with $k = 1.6$, $\Delta t = \Delta t_c$ and $p = p_5 << p_6$.

$\sigma_{\theta o} = \sigma_\theta^t + \sigma_\theta^p = p_t\left(\frac{1}{\ln k} - \frac{2}{k^2-1}\right) + \frac{2p}{k^2-1}$. Since $\sigma_{ro}^T = 0$, letting $\sigma_{\theta o}^T \le \sigma_y$ is equivalent to letting $\sigma_{\theta o}^T - \sigma_{ro}^T \le \sigma_y$. In the last chapter, letting $\sigma_{\theta o}^T - \sigma_{ro}^T \le \sigma_y$ obtains equation (8.44), or

$$\frac{p}{\sigma_y} \le \frac{k^2-1}{2} - \frac{p_t}{\sigma_y}\frac{k^2-1-\ln k^2}{\ln k^2} = \frac{k^2-1}{2} - \frac{E\alpha}{2(1-\mu)\sigma_y}\frac{k^2-1-\ln k^2}{\ln k^2}\Delta t = \frac{p_2}{\sigma_y} \quad (8.49)$$

Thus, when the pressure $p \le p_2$, $\sigma_{\theta o}^T \le \sigma_y$.

When is k fixed, if $p_2 = p_5$, usually, $\Delta t > \Delta t_c$; When is Δt fixed, if $p_2 = p_5$, usually, $k < 1$. Therefore, usually, p_2 is always greater than p_5.

When $\Delta t = \Delta t_e = 96.31247$ °C, $p = p_2 = 127.8242$ MPa, the total stresses along with σ_e^{Td} and σ_e^{Ts} are shown in figure 8.32. The meaning of each curve and the main parameters are marked in the figure. $\sigma_{\theta o}^T \le \sigma_y$.

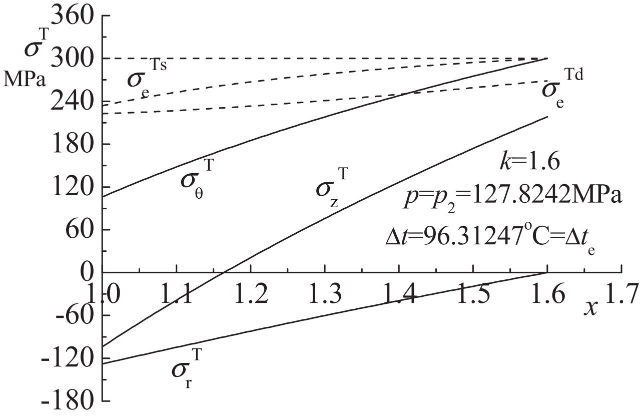


FIG. 8.32 – The distribution of total stresses with $k = 1.6$, $\Delta t = \Delta t_e$ and $p = p_2$.

When $\Delta t = \Delta t_c = 155.4798$ °C, $p = p_2 = 62.59748$ MPa, the total stresses along with σ_e^{Td} and σ_e^{Ts} are shown in figure 8.33. The meaning of each curve and the main parameters are marked in the figure. $\sigma_{\theta o}^{T} \le \sigma_y$.

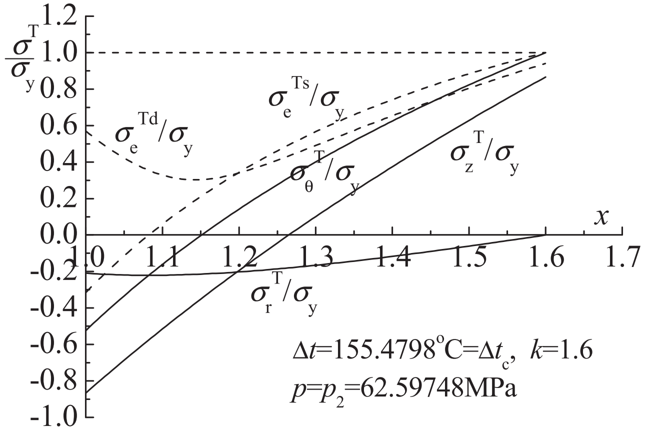


FIG. 8.33 – The distribution of total stresses with $k = 1.6$, $\Delta t = \Delta t_c$ and $p = p_2$.

Example 3. $E = 1.95 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, $\sigma_y = 300$ MPa, $k = 1.6 < k_i$, $p = 130$ MPa.

From equation (8.12), $\Delta t_c = 155.4798$ °C; from equation (8.40), $\Delta t_0 = 288.3432$ °C and $\Delta t_4 = 43.3471$ °C; from equation (8.28), $\Delta t_3 = 117.5862$ °C. Therefore, Δt should be selected between Δt_4 and Δt_3, or $\Delta t_4 \le \Delta t \le \Delta t_3$. Now we take $\Delta t = 70$ °C, the total stresses along with σ_e^{Td} and σ_e^{Ts} are shown in figure 8.34, where the meaning of each curve and the main parameters are marked. According to the maximum shear stress theory (Tresca criterion), $\sigma_e^{Ts} < 1$.

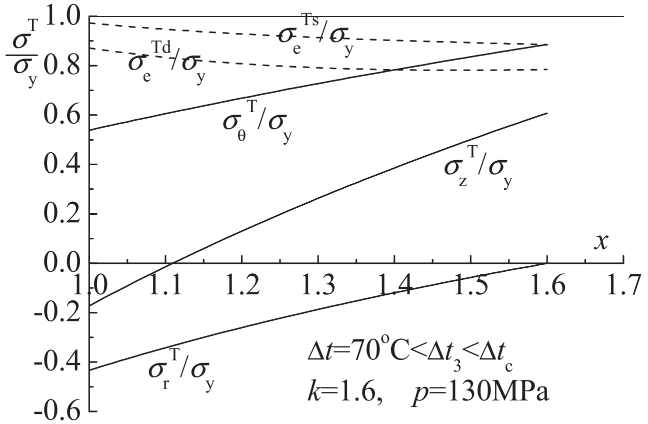


FIG. 8.34 – The distribution of total stresses with $k = 1.6$, $\Delta t = 70$ °C $< \Delta t_3 < \Delta t_c$, $p = 130$ MPa and $k = 1.6$.

If $\Delta t = 119\ ^\circ\mathrm{C} > \Delta t_3$, the total stresses along with $\sigma_\mathrm{e}^\mathrm{Td}$ and $\sigma_\mathrm{e}^\mathrm{Ts}$ are shown in figure 8.35, in which the meaning of each curve and the main parameters are marked. Because $\Delta t > \Delta t_3$, $\sigma_\mathrm{eo}^\mathrm{Td} > 1$.

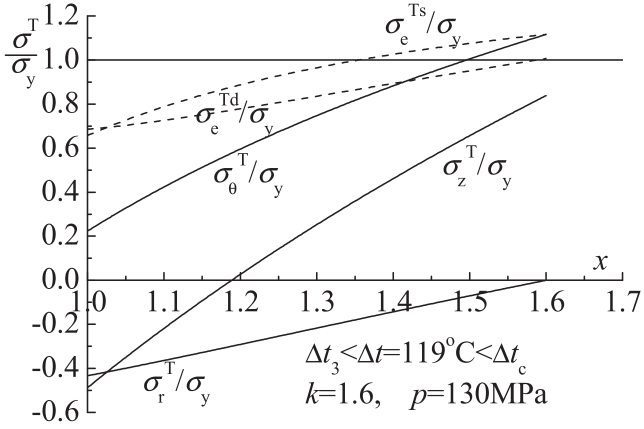


FIG. 8.35 – The distribution of total stresses with $k = 1.6$, $\Delta t_3 \Delta t < \Delta t_\mathrm{c}$ and $p = 130$ MPa.

When $\Delta t = 119\ ^\circ\mathrm{C}$, $p = p_2 = 102.8132$ MPa, the total stresses along with $\sigma_\mathrm{e}^\mathrm{Td}$ and $\sigma_\mathrm{e}^\mathrm{Ts}$ are shown in figure 8.36. The meaning of each curve and the main parameters are marked in the figure. $\sigma_{\theta\mathrm{o}}^\mathrm{T} \leq \sigma_\mathrm{y}$.

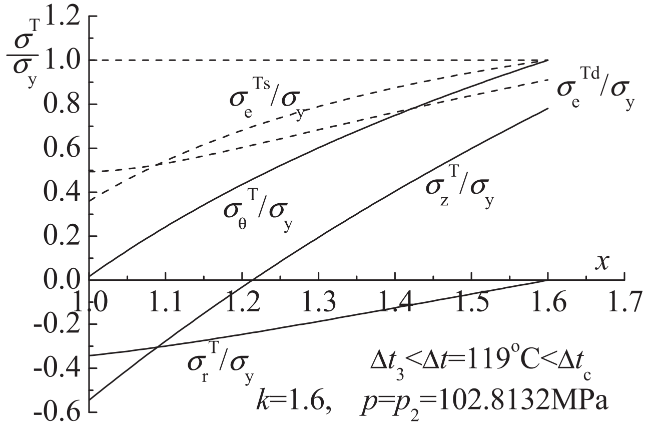


FIG. 8.36 – The distribution of total stresses with $k = 1.6$, $\Delta t = 119\ ^\circ\mathrm{C}$ and $p = p_2$.

Example 4. $E = 1.95 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}\ ^\circ\mathrm{C}^{-1}$, $\sigma_\mathrm{y} = 300$ MPa, $k = 2.1 > k_\mathrm{i}$, $p = 220$ MPa.

From equation (8.12), $\Delta t_\mathrm{c} = 144.9011\ ^\circ\mathrm{C}$; from equation (8.40), $\Delta t_0 = 288.7934\ ^\circ\mathrm{C}$ and $\Delta t_4 = 123.4736\ ^\circ\mathrm{C}$; from equation (8.28), $\Delta t_3 = 155.5836\ ^\circ\mathrm{C}$.

Therefore, Δt should be selected between Δt_4 and Δt_c, or $\Delta t_4 \leq \Delta t \leq \Delta t_c$. Now we take $\Delta t = 130$ °C, the total stresses along with σ_e^{Td} and σ_e^{Ts} are shown in figure 8.37, where the meaning of each curve and the main parameters are marked. According to the maximum shear stress theory (Tresca criterion), $\sigma_e^{Ts} < 1$.

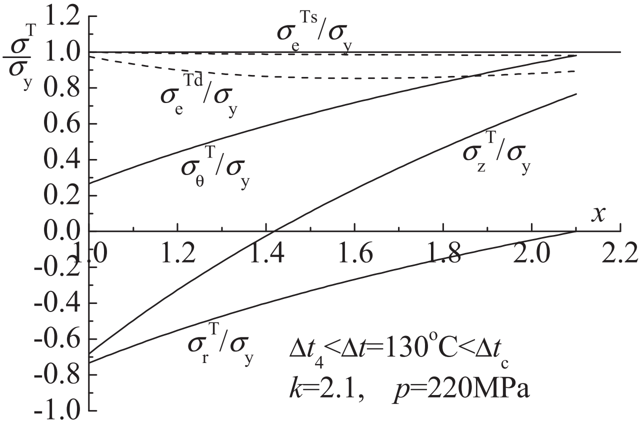


FIG. 8.37 – The distribution of total stresses with $k = 2.1$, $\Delta t_4 < \Delta t < \Delta t_c$ and $p = 220$ MPa.

If based on the maximum shear stress theory (Tresca criterion), this is the case of $k < k_c$, from equation (8.40) of last chapter, $\Delta t_x = 133.1682$ °C $< \Delta t_c = 144.9011$ °C obtained from equation (8.27) of the last chapter. When $\Delta t = \Delta t_x = 133.1682$ °C, equations (8.44) and (8.45) both lead to $\frac{p}{\sigma_y} = 0.741937 = \ln k = \ln 2.1$, or $p = 222.5812$ MPa > 220 MPa. When $\Delta t = \Delta t_x = 133.1682$ °C, $p = 222.5812$ MPa, the total stresses along with σ_e^{Td} and σ_e^{Ts} are shown in figure 8.38, in which the meaning of each curve and the main parameters are marked. In this case, $\sigma_e^{Ts} \equiv 1$.

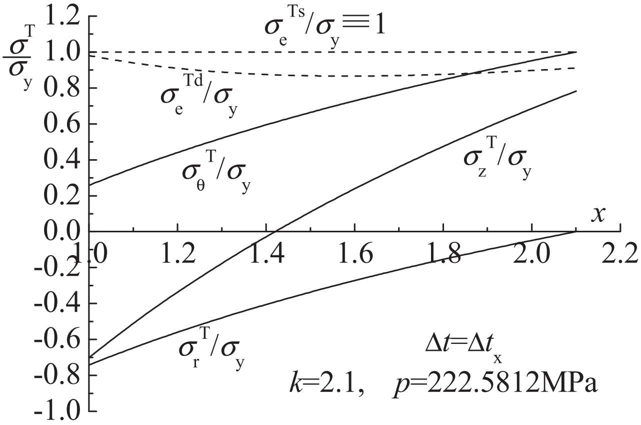


FIG. 8.38 – The distribution of total stresses with $k = 2.1$, $\Delta t = \Delta t_x$ and $p = 222.5812$ MPa.

If p and Δt are given, the key is to determine the appropriate radius ratio k. Set $p_5 = p$ to solve k by numerical approximation method from equation (8.26) and set $p_6 = p$ to solve k by numerical approximation method from equation (8.38), taking the greater k as real radius ratio.

Another way: If the solved k is greater than k_{i} and Δt is not limited or $\Delta t > \Delta t_{\mathrm{c}}$, solve k from equation (8.45) and take Δt_{c} as the real temperature difference. See example 5.

Example 5. $E = 1.95 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, $\sigma_{\mathrm{y}} = 300$ MPa, $p = 130$ Mpa, $\Delta t = 150$ °C.

Setting $p_5 = p = 130$ Mpa, from equation (8.26), $k = 1.770509 < k_{\mathrm{i}}$; Setting $p_6 = p = 130$ Mpa, from equation (8.38), $k = 1.343435 < k_{\mathrm{i}}$. Take $k = 1.770509$ as the real radius ratio. When $k = 1.770509$, $\Delta t_{\mathrm{c}} = 151.2867$ °C, $\Delta t_0 = 270.6439$ °C, $\Delta t_4 = 18.16004$ °C, $\Delta t_3 = \Delta t = 150$ °C, $p_5 = p = 130$ MPa, $p_6 = 203.7163$ MPa.

When $p = 130$ Mpa, $\Delta t = 150$ °C, and $k = 1.770509$, the total stresses along with $\sigma_{\mathrm{e}}^{\mathrm{Td}}$ and $\sigma_{\mathrm{e}}^{\mathrm{Ts}}$ are shown in figure 8.39. The meaning of each curve and the main parameters are marked in the figure.

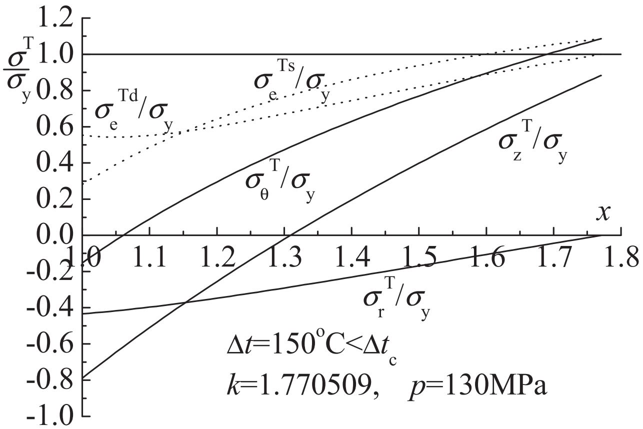


FIG. 8.39 – The distribution of total stresses with $k = 1.770509$, $\Delta t = 150$ °C and $p = 130$ MPa.

When $\Delta t = 150$ °C $< \Delta t_{\mathrm{c}}$, $k = 1.770509$, $p = p_2 = 102.4869$ MPa, the total stresses along with $\sigma_{\mathrm{e}}^{\mathrm{Td}}$ and $\sigma_{\mathrm{e}}^{\mathrm{Ts}}$ are shown in figure 8.40. The meaning of each curve and the main parameters are marked in the figure. $\sigma_{\theta\mathrm{o}}^{\mathrm{T}} \le \sigma_{\mathrm{y}}$.

If based on the maximum shear stress theory (Tresca criterion), set $p_1 = p$ to solve k by numerical approximation method from equation (8.42) of the last chapter and set $p_2 = p$ to solve k by numerical approximation method from equation (8.44) of last chapter, take the greater k as real radius ratio.

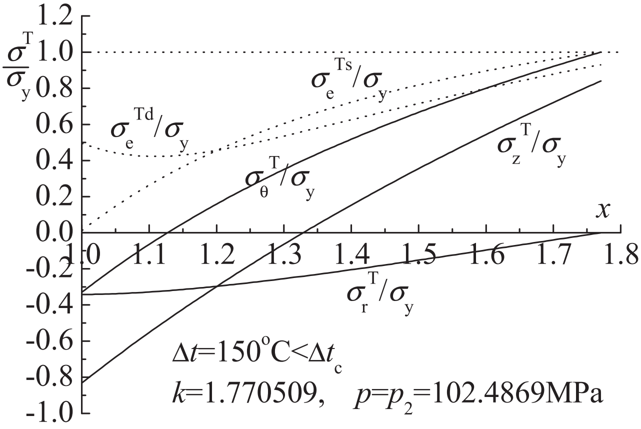


FIG. 8.40 – The distribution of total stresses with $k = 1.770509$, $\Delta t = 150$ °C and $p = p_2$.

Another way: If the solved k is greater than k_c and Δt is not limited or $\Delta t > \Delta t_c$, solve k from $k = \sqrt{\frac{\sigma_y}{\sigma_y - p}}$, which is just equation (8.45), take Δt_c as real temperature difference. If the solved k is smaller than k_c and Δt is not limited or $\Delta t > \Delta t_x$, solve k from $\frac{p}{\sigma_y} = \ln k$ and take Δt_x as the real temperature difference.

Setting $p_1 = p = 130$ MPa, we obtain $k = 1.350423$; Setting $p_2 = p = 130$ MPa, we obtain $k = 1.889912 < k_c$. Take $k = 1.889912$ as the real radius ratio. When $k = 1.889912$, $\Delta t_c = 148.7473358$ °C obtained from equation (8.27) of the last chapter or equation (8.9) of this chapter. $\Delta t = 150$ °C is a little bigger than Δt_c. When $p = 130$ Mpa, $\Delta t = 150$ °C and $k = 1.889912$, the total stresses along with σ_e^{Td} and σ_e^{Ts} are shown in figure 8.41. The meaning of each curve and the main parameters are marked in the figure.

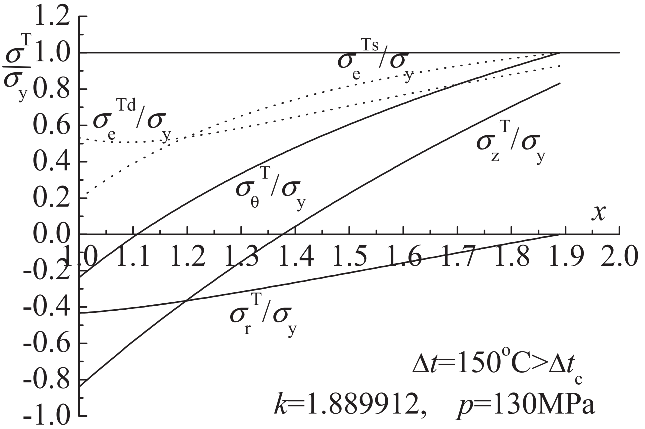


FIG. 8.41 – The distribution of total stresses with $k = 1.889912$, $\Delta t = 150$ °C and $p = 130$ MPa.

In view of $k = 1.889912 < k_{\mathrm{c}}$, from $\frac{p}{\sigma_{\mathrm{y}}} = \ln k$, we obtain $k = 1.54239$. When $k = 1.54239$, $\Delta t_{\mathrm{c}} = 157.077292$ °C obtained from equation (8.27) of the last chapter or equation (8.9) of this chapter; from equation (8.40) of the last chapter, $\Delta t_{\mathrm{x}} = 77.77775$ °C $< \Delta t_{\mathrm{c}}$. When $p = 130$ Mpa, $\Delta t = \Delta t_{\mathrm{x}} = 77.77775$ °C and $k = 1.54239$, the total stresses along with $\sigma_{\mathrm{e}}^{\mathrm{Td}}$ and $\sigma_{\mathrm{e}}^{\mathrm{Ts}}$ are shown in figure 8.42. The meaning of each curve and the main parameters are marked in the figure.

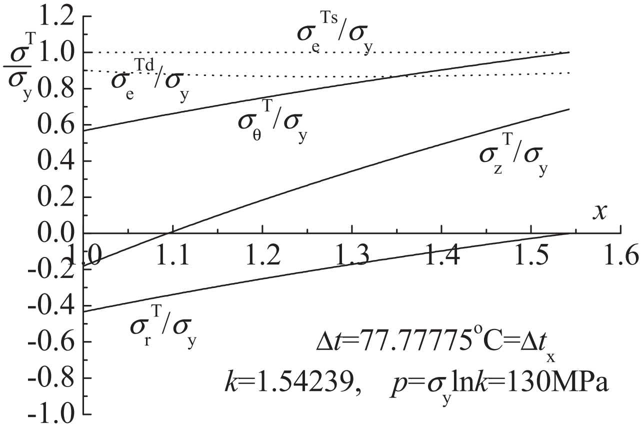


FIG. 8.42 – The distribution of total stresses with $\Delta t = \Delta t_{\mathrm{x}}$ and $p = \sigma_{\mathrm{y}}\ln k$.

Example 6. $E = 1.95 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, $\sigma_{\mathrm{y}} = 300$ MPa, $p = 250$ Mpa, $\Delta t = 110$ °C.

Setting $p_5 = p = 250$ Mpa, from equation (8.26), $k = 1.903431 < k_{\mathrm{i}}$; Setting $p_6 = p = 250$ Mpa, from equation (8.38), $k = 3.005172 > k_{\mathrm{i}}$. Take $k = 3.005172$ as the real radius ratio. When $k = 3.005172$, $\Delta t_{\mathrm{c}} = 133.9237816$ °C, $\Delta t_0 = 266.4988$ °C, $\Delta t_4 = 110$ °C $= \Delta t$, $\Delta t_3 = 228.5979$ °C, $p_5 = p = 130$ MPa, $p_6 = 203.7163$ MPa.

When $p = 250$ Mpa, $\Delta t = 110$ °C, and $k = 3.005172$, the total stresses along with $\sigma_{\mathrm{e}}^{\mathrm{Td}}$ and $\sigma_{\mathrm{e}}^{\mathrm{Ts}}$ are shown in figure 8.43. The meaning of each curve and the main parameters are marked in the figure.

When $p = 250$ MPa, $\Delta t = \Delta t_{\mathrm{c}}$ is allowed, from equation (8.45), we obtain $k = 2.44949 > k_{\mathrm{i}}$. When $k = 2.44949$, $\Delta t_{\mathrm{c}} = 139.8116$ °C, the total stresses along with $\sigma_{\mathrm{e}}^{\mathrm{Td}}$ and $\sigma_{\mathrm{e}}^{\mathrm{Ts}}$ are shown in figure 8.44, where the meaning of each curve and the main parameters are marked in the figure. This plan is better.

If based on the maximum shear stress theory (Tresca criterion), setting $p_1 = p = 250$ MPa, we obtain $k = 3.292849 > k_{\mathrm{c}}$ by numerical approximation method from equation (8.42) of the last chapter; Setting $p_2 = p = 250$ MPa, we obtain $k = 2.038807$ by numerical approximation method from equation (8.44) of last chapter. Take $k = 3.292849$ as the real radius ratio. When $k = 3.292849$, $\Delta t_{\mathrm{c}} = 131.5798562$ °C obtained from equation (8.27) of the last chapter or equation (8.9) of this chapter. $\Delta t = 110$ °C is smaller than Δt_{c}. When $p = 250$ Mpa, $\Delta t = 110$ °C

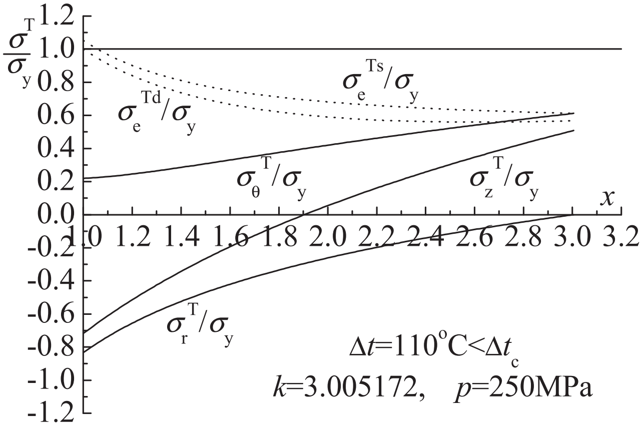


FIG. 8.43 – The distribution of total stresses with $k = 3.005172$, $\Delta t = 110$ °C and $p = 250$ MPa.

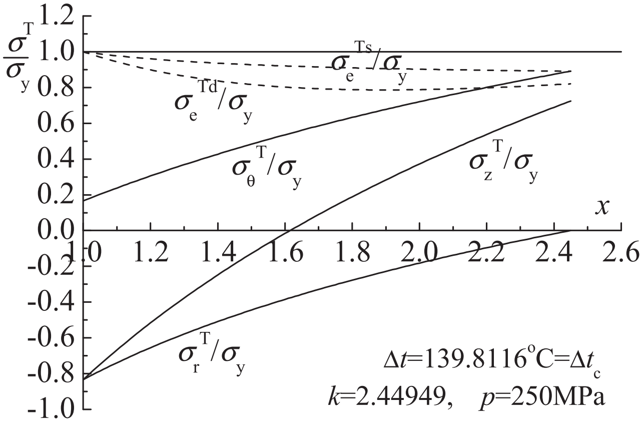


FIG. 8.44 – The distribution of total stresses with $k = 2.44949$, $\Delta t = \Delta t_c$ and $p = 250$ MPa.

and $k = 3.292849$, the total stresses along with σ_e^{Td} and σ_e^{Ts} are shown in figure 8.45. The meaning of each curve and the main parameters are marked in the figure.

In view of $k = 3.292849 > k_c$, from $k = \sqrt{\frac{\sigma_y}{\sigma_y - p}}$, we obtain the same results as figure 8.45.

Example 7. A process can supply a temperature difference $\Delta t = 120$ °C for a pressure vessel, study the size and the load-bearing capacity of the pressure vessel, $E = 1.95 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}$ °C^{-1}, $\sigma_y = 350$ MPa.

(1) If the load-bearing capacity is not required, there are many schemes. Such as,

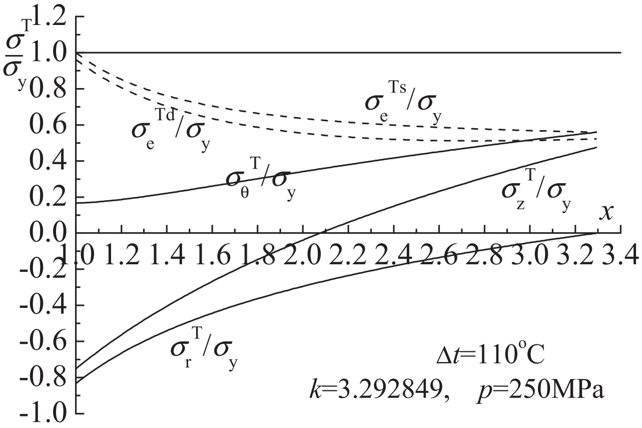


FIG. 8.45 – The distribution of total stresses with $k = 3.292849$, $\Delta t = 110$ °C and $p = 250$ MPa.

I. Taking $k = 2.5$, from equation (8.12), $\Delta t_c = 162.3777958 > \Delta t$, from equation (8.38) $\frac{p_6}{\sigma_y} = 0.761038552$, from equation (8.26) $\frac{p_5}{\sigma_y} = 1.898979$. Take $\frac{p}{\sigma_y} = \frac{p_6}{\sigma_y} = 0.761038552$ as real loading. The total stresses along with σ_e^{Td} and σ_e^{Ts} are shown in figure 8.46. The meaning of each curve and the main parameters are marked in the figure.

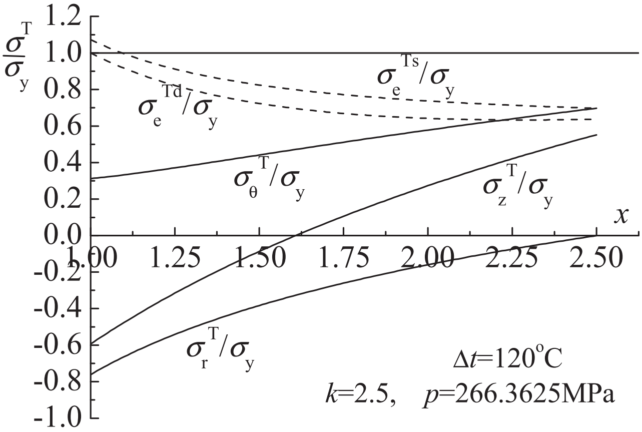


FIG. 8.46 – The distribution of total stresses with $k = 2.5$, $\Delta t = 120$ °C and $\frac{p}{\sigma_y} = \frac{p_6}{\sigma_y} = 0.761038552$.

II. Taking $k = 1.6$, from equation (8.12), $\Delta t_c = 181.3931336 > \Delta t$, from equation (8.38) $\frac{p_6}{\sigma_y} = 0.533583765$, from equation (8.26) $\frac{p_5}{\sigma_y} = 0.495861$. Take

$\frac{p}{\sigma_y} = 0.495861$ as real loading. The total stresses along with σ_e^{Td} and σ_e^{Ts} are shown in figure 8.47. The meaning of each curve and the main parameters are marked in the figure.

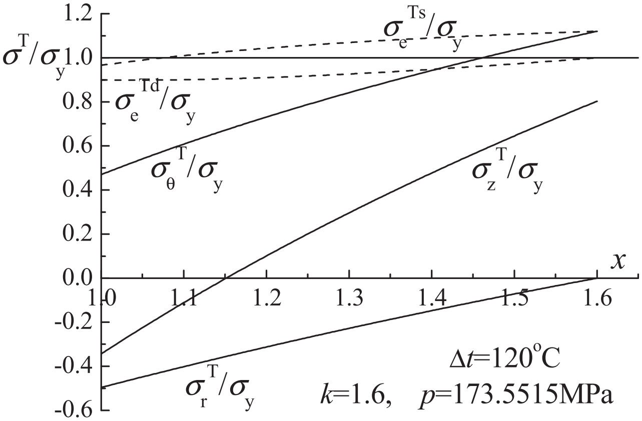


FIG. 8.47 – The distribution of total stresses with $k = 1.6$, $\Delta t = 120$ °C and $\frac{p}{\sigma_y} = \frac{p_5}{\sigma_y} = 0.495861$.

Such instances are legion.

When $\Delta t = 120$ °C $< \Delta t_c$, $k = 1.6$, $p = p_2 = 140.7108$ MPa, the total stresses along with σ_e^{Td} and σ_e^{Ts} are shown in figure 8.48. The meaning of each curve and the main parameters are marked in the figure. $\sigma_{\theta o}^T \leq \sigma_y$.

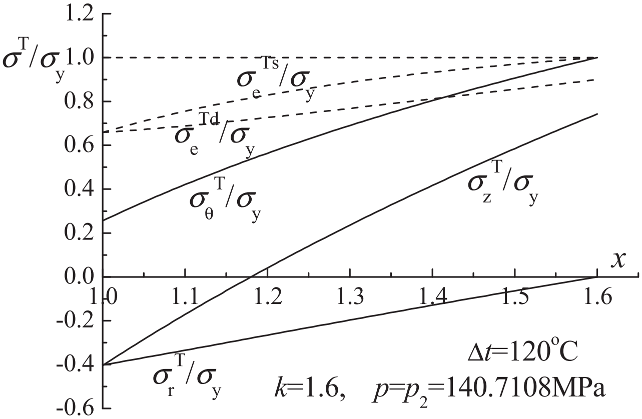


FIG. 8.48 – The distribution of total stresses with $k = 1.6$, $\Delta t = 120$ °C and $p = p_2$.

(2) Supposing $\Delta t = 120$ °C $= \Delta t_e$, then, from equation (8.43) by approximation, $k = 1.653158 < k_c$. When $k = 1.653158$, $\Delta t_c = 179.7739059 > \Delta t$. When

$k = 1.653158$, $\Delta t = 120\ ^\circ\mathrm{C} = \Delta t_\mathrm{e}$, from equations (8.26) and (8.38), we obtain $\frac{p}{\sigma_\mathrm{y}} = \frac{p_5}{\sigma_\mathrm{y}} = \frac{p_6}{\sigma_\mathrm{y}} = 0.556731958 < \frac{2}{\sqrt{3}} \ln k = 0.580453$.

(3) Supposing $\Delta t = 120\ ^\circ\mathrm{C} = \Delta t_\mathrm{c}$, then, from equation (8.27) $k = 49.872$, it is unrealistic, rejected. When $E = 1.95 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}\ ^\circ\mathrm{C}^{-1}$, $\sigma_\mathrm{y} = 350$ MPa and $k = k_\mathrm{c}$, $\Delta t_\mathrm{c} = 166.8538\ ^\circ\mathrm{C}$. The greater the k is, the greater the Δt_c is. If $\Delta t_\mathrm{c} = 120\ ^\circ\mathrm{C}$, k is of course quite great. When $E = 1.95 \times 10^5$ MPa, $\mu = 0.3$, $\alpha = 1.2 \times 10^{-5}\ ^\circ\mathrm{C}^{-1}$, $\sigma_\mathrm{y} = 300$ MPa and $k = k_\mathrm{c}$, $\Delta t_\mathrm{c} = 143.0175618\ ^\circ\mathrm{C}$.

(4) I. Supposing the pressure vessel is required to contain 200 MPa, or $\frac{p}{\sigma_\mathrm{y}} = \frac{200}{350} = 0.571429$. From equation (8.26) by approximation, $k = 1.665644$; from equation (8.38) by approximation, $k = 1.689516$. Take $k = 1.689516$. When $k = 1.689516$, $\Delta t_\mathrm{c} = 178.7176168\ ^\circ\mathrm{C} > \Delta t$. When $p = 200$ MPa, $\Delta t = 120\ ^\circ\mathrm{C}$ and $k = 1.689516$, the total stresses along with $\sigma_\mathrm{e}^\mathrm{Td}$ and $\sigma_\mathrm{e}^\mathrm{Ts}$ are shown in figure 8.49. The meaning of each curve and the main parameters are marked in the figure.

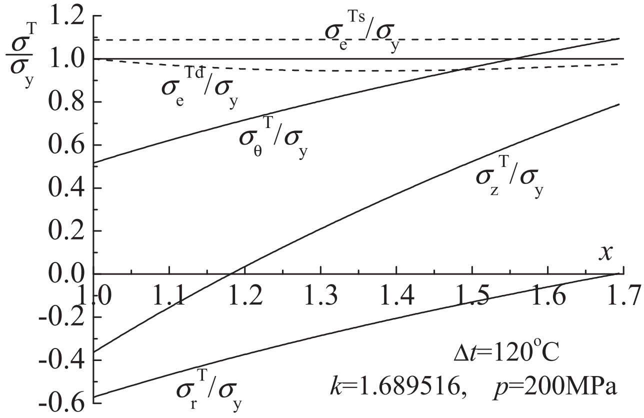


FIG. 8.49 – The distribution of total stresses with $k = 1.689516$, $\Delta t = 120\ ^\circ\mathrm{C}$ and $p = 200$ MPa.

When $\Delta t = 120\ ^\circ\mathrm{C} < \Delta t_\mathrm{c}$, $k = 1.689516$ $p = p_2 = 170.4853$ MPa, the total stresses along with $\sigma_\mathrm{e}^\mathrm{Td}$ and $\sigma_\mathrm{e}^\mathrm{Ts}$ are shown in figure 8.50. The meaning of each curve and the main parameters are marked in the figure. $\sigma_{\theta\mathrm{o}}^\mathrm{T} \leq \sigma_\mathrm{y}$.

II. Supposing the pressure vessel is required to contain 300 MPa, or $\frac{p}{\sigma_\mathrm{y}} = \frac{300}{350} = 0.857143$. From equation (8.26) by approximation, $k = 1.88692$; from equation (8.38) by approximation, $k = 3.629574$. Take $k = 3.629574$. When $k = 3.629574$, $\Delta t_\mathrm{c} = 150.804797\ ^\circ\mathrm{C} > \Delta t$. When $p = 300$ Mpa, $\Delta t = 120\ ^\circ\mathrm{C}$ and $k = 3.629574$, the total stresses along with $\sigma_\mathrm{e}^\mathrm{Td}$ and $\sigma_\mathrm{e}^\mathrm{Ts}$ are shown in figure 8.51. The meaning of each curve and the main parameters are marked in the figure.

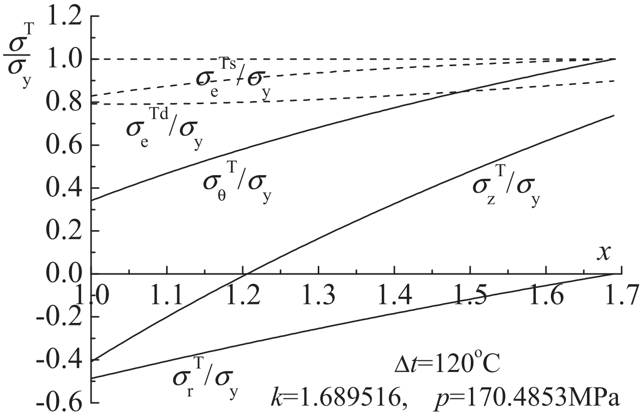


FIG. 8.50 – The distribution of total stresses with $k = 1.689516$, $\Delta t = 120$ °C and $p = p_2 = 170.4853$ MPa.

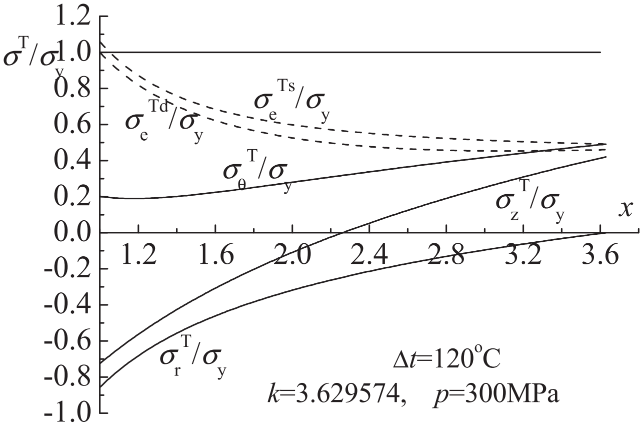


FIG. 8.51 – The distribution of total stresses with $k = 3.629574$, $\Delta t = 120$ °C and $p = 300$ MPa.

8.4 Chapter Summary

The main equations and conclusion are listed in table 8.1.

TAB. 8.1 – The main equations and conclusion of this chapter.

Thermal stresses

$$\sigma_{\mathrm{r}}^{\mathrm{t}} = p_{\mathrm{t}}\left(-\frac{\ln(k/x)}{\ln k} + \frac{k^2 - x^2}{x^2 k^2 - x^2}\right)$$

$$\sigma_{\theta}^{\mathrm{t}} = p_{\mathrm{t}}\left(\frac{1 - \ln(k/x)}{\ln k} - \frac{k^2 + x^2}{x^2 k^2 - x^2}\right)$$

$$\sigma_{\mathrm{z}}^{\mathrm{t}} = p_{\mathrm{t}}\left(\frac{1 - 2\ln(k/x)}{\ln k} - \frac{2}{k^2 - 1}\right)$$

$$p_{\mathrm{t}} = \frac{E\alpha\Delta t}{2(1-\mu)}$$

The thermal stresses at the inside and outside walls

$$\sigma_{\mathrm{ri}}^{\mathrm{t}} = \sigma_{\mathrm{ro}}^{\mathrm{t}} = 0$$

$$\sigma_{\theta\mathrm{i}}^{\mathrm{t}} = \sigma_{\mathrm{zi}}^{\mathrm{t}} = -p_{\mathrm{t}}\left(\frac{2k^2}{k^2 - 1} - \frac{1}{\ln k}\right)$$

$$\sigma_{\theta\mathrm{o}}^{\mathrm{t}} = \sigma_{\mathrm{zo}}^{\mathrm{t}} = p_{\mathrm{t}}\left(\frac{1}{\ln k} - \frac{2}{k^2 - 1}\right)$$

The equivalent thermal stress at the inside wall

$$\sigma_{\mathrm{ei}}^{\mathrm{ts}} = \sigma_{\mathrm{ri}}^{\mathrm{t}} - \sigma_{\mathrm{zi}}^{\mathrm{t}} = -\sigma_{\mathrm{zi}}^{\mathrm{t}} = -\sigma_{\theta\mathrm{i}}^{\mathrm{t}} = p_{\mathrm{t}}\left(\frac{2k^2}{k^2 - 1} - \frac{1}{\ln k}\right) = p_{\mathrm{t}}\frac{k^2\ln k^2 - k^2 + 1}{(k^2 - 1)\ln k}$$

Critical temperature difference, Δt_{c}

$$\Delta t_{\mathrm{c}} = \frac{(k^2 - 1)\ln k}{k^2\ln k^2 - k^2 + 1}\frac{2(1-\mu)\sigma_{\mathrm{y}}}{E\alpha}$$

Critical thermal loading p_{tc}

$$p_{\mathrm{tc}} = \frac{(k^2 - 1)\ln k}{k^2\ln k^2 - k^2 + 1}\sigma_{\mathrm{y}}$$

Total stresses (thermal stresses plus operation stresses)

$$\sigma_{\mathrm{r}}^{\mathrm{T}} = \sigma_{\mathrm{r}}^{\mathrm{t}} + \sigma_{\mathrm{r}}^{\mathrm{p}} = p_{\mathrm{t}}\left(-\frac{\ln(k/x)}{\ln k} + \frac{k^2 - x^2}{x^2 k^2 - x^2}\right) + \frac{p}{k^2 - 1}\left(1 - \frac{k^2}{x^2}\right)$$

$$\sigma_{\theta}^{\mathrm{T}} = \sigma_{\theta}^{\mathrm{t}} + \sigma_{\theta}^{\mathrm{p}} = p_{\mathrm{t}}\left(\frac{1 - \ln(k/x)}{\ln k} - \frac{k^2 + x^2}{x^2 k^2 - x^2}\right) + \frac{p}{k^2 - 1}\left(1 + \frac{k^2}{x^2}\right)$$

$$\sigma_{\mathrm{z}}^{\mathrm{T}} = \sigma_{\mathrm{z}}^{\mathrm{t}} + \sigma_{\mathrm{z}}^{\mathrm{p}} = p_{\mathrm{t}}\left(\frac{1 - 2\ln(k/x)}{\ln k} - \frac{2}{k^2 - 1}\right) + \frac{p}{k^2 - 1}$$

On the outer surface, $x = k$, then,

$$\sigma_{\mathrm{ro}}^{\mathrm{T}} - \sigma_{\theta\mathrm{o}}^{\mathrm{T}} = p_{\mathrm{t}}\left(\frac{1}{\ln k} - \frac{2}{k^2 - 1}\right) + \frac{2p}{k^2 - 1} = ap_{\mathrm{t}} + 2bp$$

$$\sigma_{\theta\mathrm{o}}^{\mathrm{T}} - \sigma_{\mathrm{zo}}^{\mathrm{T}} = \frac{p}{k^2 - 1} = bp$$

$$\sigma_{\mathrm{zo}}^{\mathrm{T}} - \sigma_{\mathrm{ro}}^{\mathrm{T}} = p_{\mathrm{t}}\left(\frac{1}{\ln k} - \frac{2}{k^2 - 1}\right) + \frac{p}{k^2 - 1} = ap_{\mathrm{t}} + bp$$

where $a = \frac{1}{\ln k} - \frac{2}{k^2 - 1} = \frac{k^2 - 1 - \ln k^2}{(k^2 - 1)\ln k}$, $b = \frac{1}{k^2 - 1}$

On the inner surface, $x = 1$, then,

$$\sigma_{\theta\mathrm{i}}^{\mathrm{T}} - \sigma_{\mathrm{ri}}^{\mathrm{T}} = -p_{\mathrm{t}}\left(\frac{2k^2}{k^2 - 1} - \frac{1}{\ln k}\right) + \frac{2k^2 p}{k^2 - 1} = -cp_{\mathrm{t}} + 2dp$$

TAB. 8.1 – (continued).

$$\sigma_{\theta\mathrm{i}}^{\mathrm{T}} - \sigma_{\mathrm{zi}}^{\mathrm{T}} = \frac{k^2 p}{k^2 - 1} = dp$$

$$\sigma_{\mathrm{zi}}^{\mathrm{T}} - \sigma_{\mathrm{ri}}^{\mathrm{T}} = -p_{\mathrm{t}}\left(\frac{2k^2}{k^2 - 1} - \frac{1}{\ln k}\right) + \frac{k^2 p}{k^2 - 1} = -cp_{\mathrm{t}} + dp$$

where $c = \frac{2k^2}{k^2 - 1} - \frac{1}{\ln k} = \frac{k^2 \ln k^2 - k^2 + 1}{(k^2 - 1)\ln k} = \frac{2}{k^2 - 1} - \frac{1}{\ln k} + 2 = 2 - a,\ d = \frac{k^2}{k^2 - 1} = 1 + b$

For $\sigma_{\mathrm{eo}}^{\mathrm{Td}} = \sqrt{\frac{1}{2}\left[\left(\sigma_{\theta\mathrm{o}}^{\mathrm{T}} - \sigma_{\mathrm{zo}}^{\mathrm{T}}\right)^2 + \left(\sigma_{\mathrm{zo}}^{\mathrm{T}} - \sigma_{\mathrm{ro}}^{\mathrm{T}}\right)^2 + \left(\sigma_{\mathrm{ro}}^{\mathrm{T}} - \sigma_{\theta\mathrm{o}}^{\mathrm{T}}\right)^2\right]} \le \sigma_{\mathrm{y}}$

$$3b^2p^2 + 3abp_{\mathrm{t}}p + a^2p_{\mathrm{t}}^2 - \sigma_{\mathrm{y}}^2 \le 0$$

or

$$p \le \frac{-3ap_{\mathrm{t}} + \sqrt{12\sigma_{\mathrm{y}}^2 - 3a^2p_{\mathrm{t}}^2}}{6b} = p_5$$

or

$$p_{\mathrm{t}} \le \frac{-3bp + \sqrt{4\sigma_{\mathrm{y}}^2 - 3b^2p^2}}{2a} = p_{\mathrm{t}3}$$

or

$$\Delta t \le \frac{-3bp + \sqrt{4\sigma_{\mathrm{y}}^2 - 3b^2p^2}}{2a}\frac{2(1-\mu)}{E\alpha} = \Delta t_3$$

For $\sigma_{\mathrm{ei}}^{\mathrm{Td}} = \sqrt{\frac{1}{2}\left[\left(\sigma_{\theta\mathrm{i}}^{\mathrm{T}} - \sigma_{\mathrm{zi}}^{\mathrm{T}}\right)^2 + \left(\sigma_{\mathrm{zi}}^{\mathrm{T}} - \sigma_{\mathrm{ri}}^{\mathrm{T}}\right)^2 + \left(\sigma_{\mathrm{ri}}^{\mathrm{T}} - \sigma_{\theta\mathrm{i}}^{\mathrm{T}}\right)^2\right]} \le \sigma_{\mathrm{y}}$

$$3d^2p^2 - 3cdp_{\mathrm{t}}p + c^2p_{\mathrm{t}}^2 - \sigma_{\mathrm{y}}^2 \le 0$$

or

$$p_{10} = \frac{3cp_{\mathrm{t}} - \sqrt{12\sigma_{\mathrm{y}}^2 - 3c^2p_{\mathrm{t}}^2}}{6d} \le p \le \frac{3cp_{\mathrm{t}} + \sqrt{12\sigma_{\mathrm{y}}^2 - 3c^2p_{\mathrm{t}}^2}}{6d} = p_6$$

or

$$p \le \frac{3cp_{\mathrm{t}} + \sqrt{12\sigma_{\mathrm{y}}^2 - 3c^2p_{\mathrm{t}}^2}}{6d} = p_6$$

or

$$\frac{3dp - \sqrt{4\sigma_{\mathrm{y}}^2 - 3d^2p^2}}{2c} = p_{\mathrm{t}4} \le p_{\mathrm{t}} \le \frac{3dp + \sqrt{4\sigma_{\mathrm{y}}^2 - 3d^2p^2}}{2c} = p_{\mathrm{t}0}$$

or

$$\frac{3dp - \sqrt{4\sigma_{\mathrm{y}}^2 - 3d^2p^2}}{2c}\frac{2(1-\mu)}{E\alpha} = \Delta t_4 \le \Delta t \le \frac{3dp + \sqrt{4\sigma_{\mathrm{y}}^2 - 3d^2p^2}}{2c}\frac{2(1-\mu)}{E\alpha} = \Delta t_0$$

The allowable load = Min $\{p_1,\ p_2\}$

When $3p_{\mathrm{t}}(ad + bc) + b\sqrt{12\sigma_{\mathrm{y}}^2 - 3c^2p_{\mathrm{t}}^2} - d\sqrt{12\sigma_{\mathrm{y}}^2 - 3a^2p_{\mathrm{t}}^2} = 0\ (k \le k_1)$,

$$p_6 = p_5$$

Tab. 8.1 – (continued).

The temperature difference solved from $3p_\mathrm{t}(ad+bc)+b\sqrt{12\sigma_\mathrm{y}^2-3c^2p_\mathrm{t}^2}-d\sqrt{12\sigma_\mathrm{y}^2-3a^2p_\mathrm{t}^2}=0$ is the optimal temperature difference. Δt_e

The boundary radius ratio k_i

When $p_\mathrm{t}=p_\mathrm{tc}$, or $\Delta t=\Delta t_\mathrm{c}$, $3p_\mathrm{t}(ad+bc)+b\sqrt{12\sigma_\mathrm{y}^2-3c^2p_\mathrm{t}^2}-d\sqrt{12\sigma_\mathrm{y}^2-3a^2p_\mathrm{t}^2}=0$ becomes

$$\frac{3(k^2-1)}{k^2\ln k^2-k^2+1}+\frac{3}{k^2-1}-\frac{k^2}{k^2-1}\sqrt{12-3\frac{\left(k^2-1-\ln k^2\right)^2}{\left(k^2\ln k^2-k^2+1\right)^2}}=0$$

The solution of $\frac{3(k^2-1)}{k^2\ln k^2-k^2+1}+\frac{3}{k^2-1}-\frac{k^2}{k^2-1}\sqrt{12-3\frac{\left(k^2-1-\ln k^2\right)^2}{\left(k^2\ln k^2-k^2+1\right)^2}}=0$ is the boundary radius ratio k_i

$$k_\mathrm{i}=2.0542956825\ldots$$

When $\Delta t=\Delta t_\mathrm{c}$, if $k=k_\mathrm{i}$, $p_5=p_6$; when $k>k_\mathrm{i}$, $p_5>p_6$; when $k<k_\mathrm{i}$, $p_5<p_6$

When $k\le k_\mathrm{i}$, $\Delta t_\mathrm{e}\le\Delta t_\mathrm{c}$; when $k\ge k_\mathrm{i}$, $\Delta t_\mathrm{e}\ge\Delta t_\mathrm{c}$

When $k\ge k_\mathrm{i}$, in the range of $\Delta t\le\Delta t_\mathrm{c}$, the allowable load is $p=p_6$; if $\Delta t=\Delta t_\mathrm{c}$, the allowable load is

$$p=p_6=\frac{k^2-1}{k^2}\sigma_\mathrm{y}\qquad\text{or}\qquad k=\sqrt{\frac{\sigma_\mathrm{y}}{\sigma_\mathrm{y}-p}}\qquad(k\ge k_\mathrm{i})$$

$$\frac{p_2}{\sigma_\mathrm{y}}=\frac{k^2-1}{2}-\frac{p_\mathrm{t}}{\sigma_\mathrm{y}}\frac{k^2-1-\ln k^2}{\ln k^2}=\frac{k^2-1}{2}-\frac{E\alpha}{2(1-\mu)\sigma_\mathrm{y}}\frac{k^2-1-\ln k^2}{\ln k^2}\Delta t$$

When $p\le p_2$, $\sigma_{\theta\mathrm{o}}^\mathrm{T}\le\sigma_\mathrm{y}$

References

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Nomenclature

r_{i}, r_{j}, r_{o}, r: Inside radius, radius of elastic-plastic juncture, outside radius and radius at a general location, respectively.

k: Radius ratio or ratio of outside to inside radius, $k = r_{\mathrm{o}}/r_{\mathrm{i}}$.

k_{j}: Depth of plastic zone, or plastic depth, $k_{\mathrm{j}} = r_{\mathrm{j}}/r_{\mathrm{i}}$.

$k_{\mathrm{j}*}$: The optimum plastic depth, $k_{\mathrm{j}*}$ is the value determined by $k^2 \ln k_{\mathrm{j}*}^2 - k^2 - k_{\mathrm{j}*}^2 + 2 = 0$.

k_{c}: Critical radius ratio, $k_{\mathrm{c}} = 2.218\ 457\ 489\ 916\ 7\ldots$.

x: Relative location, $x = r/r_{\mathrm{i}}$.

x_0: The relative location of the intersection of the three residual stress curves or the abscissa at which $\sigma'_{\mathrm{e}} = 0$, when $x = x_0$, $\sigma'_{\mathrm{z}} = \sigma'_{\mathrm{r}} = \sigma'_{\theta}$.

x_1: The relative location where $\sigma'_{\mathrm{z}} = 0$.

x_{a}: The relative location or the abscissa where $\sigma'_{\mathrm{e}} = \sigma'_{\theta}$ and/or $\sigma^{\mathrm{T}}_{\mathrm{e}} = \sigma^{\mathrm{T}}_{\theta}$ within the elastic region, $x_{\mathrm{a}} = \sqrt{\sqrt{3} - 1}k$.

x_0^{T}: The relative location of the intersection of the three total stress curves, when $x = x_0^{\mathrm{T}}$, $\sigma^{\mathrm{T}}_{\mathrm{z}} = \sigma^{\mathrm{T}}_{\mathrm{r}} = \sigma^{\mathrm{T}}_{\theta}$.

x_{z}: The relative location where equivalent elastic-plastic stress is equal to circumferential elastic-plastic, *i.e.* $\sigma_{\mathrm{e}} = \sigma_{\theta}$, $x_{\mathrm{z}} = \exp\left(\frac{\sqrt{3}}{2}\frac{p}{\sigma_{\mathrm{y}}} - \frac{2-\sqrt{3}}{2}\right)$.

p: Internal pressure.

p_{y}: Entire yield pressure.

p_{e}: Maximum elastic load-bearing capability or initial yield pressure of an unautofrettaged pressure vessel.

p_{a}: Autofrettage pressure.

p_{∞}: The internal pressure when $k \longrightarrow \infty$.

σ_{z}: Axial stress.

σ_{r}: Radial stress.

σ_θ: Circumferential stress.

σ_y: Yield strength.

σ_e: Equivalent stress.

ε: Overstrain.

λ: Load ratio, $\lambda = p/p_e$, λ is chosen between $1\sim2$, which indicates the multiple of initial yield pressure.

$k_{j\lambda}$: Plastic depth when $p= \lambda.p_e$, or the optimum plastic depth, or k_j corresponding to λ, when $\lambda = 2$, $k_{j\lambda}$ is marked as k_{j*}.

ε_λ: Overstrain when $p = \lambda.p_e$.

$k_{c\lambda}$: Critical radius ratio under a certain λ, or critical radius ratio corresponding to λ, determined by $k_{c\lambda}^2 \ln k_{c\lambda}^2 - \lambda(k_{c\lambda}^2 - 1) = 0$.

$k_{j\lambda\infty}$: $k_{j\lambda}$ when $k\rightarrow\infty$.

t: Temperature.

α : Thermal expansion coefficient.

μ: Poisson's ratio.

E: Young's elastic modules.

Δt: Temperature difference.

Δt_c: Critical temperature difference.

Superscripts:

$'$: Residual stresses.

p: Stresses caused by operation pressure p, or operation stresses.

T: Total stresses.

s: Quantity based on the maximum shear stress theory (Tresca yield criterion).

d: Quantity based on the maximum distortion strain energy (Mises yield criterion).

Subscripts:

j: Quantity at r_j.

i: Quantity at r_i, saying, σ'_{ei} represents the equivalent residual stress on the inside surface.

o: Quantity at r_o, saying, t_i represents the temperature on the outside surface

e: Equivalent stress.

z, r, θ: Axial, radial and circumferential direction, respectively.